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ОБ ОДНОЙ ОПЕРАЦИИ НА ФОРМАЦИЯХ КОНЕЧНЫХ ГРУПП

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ON ONE OPERATION ON THE FORMATIONS OF FINITE GROUPS

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Пусть π – множество простых чисел. В статье вводится операция w_π^* на формациях конечных групп. Если $\mathfrak{F}$ – непустая формация, то $\mathrm{w}_\pi^*\mathfrak{F}$ есть класс всех групп *G* таких, что $\pi(G) \subseteq \pi(\mathfrak{F})$ и каждая силовская *q*-подгруппа из *G* сильно K-$\mathfrak{F}$-субнормальна в *G* для $q \in \pi \cap \pi(G)$. Получены свойства w_π^*, в частности, $\mathrm{w}_\pi^*\mathfrak{F} = \mathrm{w}_\pi^*(\mathrm{w}_\pi^*\mathfrak{F})$ для наследственной формации $\mathfrak{F}$. Найдены наследственные насыщенные формации $\mathfrak{F}$, для которых $\mathrm{w}_\pi^*\mathfrak{F}$ совпадает с $\mathfrak{F}$.

Ключевые слова: *конечная группа*, *силовская подгруппа*, *нормализатор силовской подгруппы*, *наследственная формация*, $\mathfrak{F}$*-субнормальная подгруппа*, *сильно* K-$\mathfrak{F}$*-субнормальная подгруппа*.

Let π be a set of primes. In this article, the operation w_π^* on the formations of finite groups is introduced. If $\mathfrak{F}$ is a non-empty formation, then $\mathrm{w}_\pi^*\mathfrak{F}$ is the class of all groups *G* such that $\pi(G) \subseteq \pi(\mathfrak{F})$ and every Sylow *q*-subgroup of *G* is strongly K-$\mathfrak{F}$-subnormal in *G* for $q \in \pi \cap \pi(G)$. The properties of w_π^* are obtained, in particular, $\mathrm{w}_\pi^*\mathfrak{F} = \mathrm{w}_\pi^*(\mathrm{w}_\pi^*\mathfrak{F})$ for hereditary formations $\mathfrak{F}$. Hereditary saturated formations $\mathfrak{F}$ for which $\mathrm{w}_\pi^*\mathfrak{F}$ coincides with $\mathfrak{F}$ have been found.

Keywords: *finite group*, *Sylow subgroup*, *normalizer of Sylow subgroup*, *hereditary formation*, $\mathfrak{F}$*-subnormal subgroup*, *strongly* K-$\mathfrak{F}$*-subnormal subgroup*.

Introduction

We consider only finite groups. Let $\mathfrak{F}$ be a non-empty formation. Let *H* be a subgroup of a group *G*, and assume that

$$H = H_0 \le H_1 \le \cdots \le H_{n-1} \le H_n = G \qquad (0.1)$$

is a chain of subgroups of G. Then H is called:

$\mathfrak{F}$-subnormal in *G* [1], [2], if either $H = G$, or there exists a maximal chain (0.1) such that $H_i^{\mathfrak{F}} \le H_{i-1}$ for $i = 1, \ldots, n$;

K-$\mathfrak{F}$-subnormal in *G* [3], [4], if there is a chain of subgroups (0.1) such either $H_{i-1} \trianglelefteq H_i$, or $H_i^{\mathfrak{F}} \le H_{i-1}$ for $i = 1, ..., n$;

strongly K-$\mathfrak{F}$-subnormal in *G* [5], if $N_G(H)$ is a $\mathfrak{F}$-subnormal subgroup in *G*.

Denote by $sub_{\mathfrak{F}}(G)$ the set of all $\mathfrak{F}$-subnormal subgroups of a group *G*, by $sub_{\mathrm{K}-\mathfrak{F}}(G)$ the set of all K-$\mathfrak{F}$-subnormal subgroups of *G*, by $sub_{s\mathrm{K}-\mathfrak{F}}(G)$ the set of all strongly K-$\mathfrak{F}$-subnormal subgroups of *G*.

It is clear that $sub_{s\mathrm{K}-\mathfrak{F}}(G) \subseteq sub_{\mathrm{K}-\mathfrak{F}}(G)$. The example from [5] shows that the converse is not true.

***Definition* 0.1.** For a non-empty formation $\mathfrak{F}$ and a set of primes π we define the following class of groups: $\mathrm{w}_\pi^*\mathfrak{F} = (G \mid \pi(G) \subseteq \pi(\mathfrak{F})$ and $\{\mathrm{Syl}_q(G)\} \subseteq sub_{s\mathrm{K}-\mathfrak{F}}(G)$ for every $q \in \pi \cap \pi(G))$.

When $\pi = \mathbb{P}$ is the set of all primes, we denote $\mathrm{w}_{\mathbb{P}}^*\mathfrak{F} = \mathrm{w}^*\mathfrak{F}$.

The purpose of this article is to investigate the properties of $\mathrm{w}_\pi^*\mathfrak{F}$, in particular, their relation to the properties of $\mathfrak{F}$.

1 Preliminaries

We use standard notation and definitions. The appropriate information on groups theory and formations theory can be found in monographs [4] and [6]. We recall some concepts significant in the paper.

By $\mathbb{P}$ we denote the set of all primes. If $\pi \subseteq \mathbb{P}$, then $\pi' = \mathbb{P} \setminus \pi$. Let G be a group and p be a prime. Given a subgroup M of G we write $M \le G$; if $M \ne G$, then $M < G$ and if M is a maximal in *G*, then $M < \cdot G$. By symbol $|G|$ we denote the order of *G*; $\pi(G)$ is the set of all prime divisors of $|G|$; $\mathrm{Syl}_p(G)$ is the set of all Sylow p-subgroups of G; $\mathrm{Syl}(G)$ is the set of all Sylow subgroups of G; $S(G)$ is the set of all subgroups of G; Z_p is the cyclic group of order p; 1 is the identity subgroup (group).

A maximal chain of subgroups of a group G is the chain (0.1) such that $H_{i-1} <\cdot H_i$ for $i = 1, \ldots, n$.

In the next lemma, the some familiar properties of Sylow subgroups are collected.

***Lemma* 1.1.** *Let G be a group,* $P \in \mathrm{Syl}_p(G)$, *and* N, $K \trianglelefteq G$. *Then*

(a) $P \cap N \in \mathrm{Syl}_p(N)$ *and* $PN/N \in \mathrm{Syl}_p(G/N)$; *moreover*, $N_G(PN/N) = N_G(P)N/N$,

(b) $H/N = PN/N$ *for some* $P \in \mathrm{Syl}_p(G)$ *whenever* $H/N \in \mathrm{Syl}_p(G/N)$,

(c) $P \cap NK = (P \cap N)(P \cap K)$ *and*

$$PN \cap PK = P(N \cap K),$$

(d) $G = \langle P_1, \ldots, P_r \rangle$ *for* $\pi(G) = \{p_1, \ldots, p_r\}$ *and* $P_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, \ldots, r$.

***Lemma* 1.2** [6, lemma A.1.2]. *Let U, V and W be subgroups of a group G. Then*

$$U \cap VW = (U \cap V)(U \cap W)$$

if and only if $UV \cap UW = U(V \cap W)$.

A class of groups $\mathfrak{F}$ is called a formation, if

1) $\mathfrak{F}$ is a homomorph, i. e., from $G \in \mathfrak{F}$ and $N \trianglelefteq G$ it follows that $G/N \in \mathfrak{F}$ and

2) from $N_i \trianglelefteq G$ and $G/N_i \in \mathfrak{F}$ $(i = 1, 2)$ it follows that $G/N_1 \cap N_2 \in \mathfrak{F}$.

In the sequel, $\mathfrak{F}$ will denote a non-empty formation. By $\pi(\mathfrak{F})$ we denote the set of all prime divisors of orders of groups belonging to $\mathfrak{F}$; $\mathfrak{F}_\pi$ is the class of all π-groups belonging to $\mathfrak{F}$; $\mathfrak{F}_p = \mathfrak{F}_\pi$ for $\pi = \{p\}$.

A formation $\mathfrak{F}$ is called hereditary if, together with each group, $\mathfrak{F}$ contains all its subgroups. A formation $\mathfrak{F}$ is called saturated, if from $G/\Phi(G) \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$. By symbol $G^{\mathfrak{F}}$ we denote the $\mathfrak{F}$-residual of G; i. e., the least normal subgroup of G for which $G/G^{\mathfrak{F}} \in \mathfrak{F}$.

A minimal non-$\mathfrak{F}$-group is a group G such that $G \notin \mathfrak{F}$, and any proper subgroup of G belongs to $\mathfrak{F}$.

We will use the following notation:

$\mathfrak{E}$ is the class of all groups,

$\mathfrak{N}$ is the class of all nilpotent groups,

$\mathfrak{N}^2$ is the class of all metanilpotent groups,

$\mathfrak{F} = \mathfrak{N}^3$ is the class of all soluble groups whose nilpotent length is ≤ 3,

$\mathfrak{N}\mathfrak{A}$ is the class of all groups G with the nilpotent commutator subgroup G'.

We give some known properties of $\mathfrak{F}$-subnormal subgroups.

***Lemma* 1.3.** *Let* $\mathfrak{F}$ *be a formation*, *H and M are subgroups of a group G*, *and* $N \trianglelefteq G$.

(1) *If* $H \in sub_{\mathfrak{F}}(G)$ *then* $HN/N \in sub_{\mathfrak{F}}(G/N)$.

(2) *If* $N \leq H$ *and* $H/N \in sub_{\mathfrak{F}}(G/N)$ *then* $H \in sub_{\mathfrak{F}}(G)$.

(3) *If* $H \in sub_{\mathfrak{F}}(G)$ *then* $HN \in sub_{\mathfrak{F}}(G)$.

(4) *If* $H \in sub_{\mathfrak{F}}(M)$ *and* $M \in sub_{\mathfrak{F}}(G)$ *then* $H \in sub_{\mathfrak{F}}(G)$.

(5) *If all composition factors of G belong to* $\mathfrak{F}$ *and* $H \trianglelefteq\trianglelefteq G$ *then* $H \in sub_{\mathfrak{F}}(G)$.

(6) *Let* p *be a prime and let G be a p-group. If* $Z_p \in \mathfrak{F}$ *then* $S(G) \subseteq sub_{\mathfrak{F}}(G)$.

***Lemma* 1.4.** *Let* $\mathfrak{F}$ *be a hereditary formation*, $H \leq G$ *and* $M \leq G$.

(1) *If* $H \in sub_{\mathfrak{F}}(G)$ *then* $H \cap M \in sub_{\mathfrak{F}}(M)$.

(2) *If* $H \in sub_{\mathfrak{F}}(G)$ *and* $M \in sub_{\mathfrak{F}}(G)$ *then* $H \cap M \in sub_{\mathfrak{F}}(G)$.

(3) *If* $G^{\mathfrak{F}} \leq H$ *then* $H \in sub_{\mathfrak{F}}(G)$.

(4) *If* $H \in sub_{\mathfrak{F}}(G)$ *then* $H^x \in sub_{\mathfrak{F}}(G)$ *for all* $x \in G$.

Recall that a subgroup H of a group G is called: pronormal in G if, for each $g \in G$, the subgroups H and H^g are conjugate in their join $\langle H, H^g \rangle$; abnormal in G if $g \in \langle H, H^g \rangle$ for all $g \in G$.

***Lemma* 1.5** [6, Lemma I.6.20]. *Let* H *be an abnormal subgroup of a group G. Then*

(a) H *is pronormal in G*,

(b) $H = N_G(H)$, *and*

(c) *if* $H \leq L \leq G$, *then* H *is abnormal in* L *and* L *is abnormal in G.*

***Lemma* 1.6** [6, Lemma I.6.21]. *Let* H *be a subgroup of a group G.*

(a) *If* H *is pronormal in G*, *then* $N_G(H)$ *is abnormal in G*;

(b) H *is abnormal in G if and only if* H *is pronormal in G and* $H = N_G(H)$.

2 Main Results

The class of groups $\mathrm{w}_\pi^*\mathfrak{F}$ is defined as follows:

$$\mathrm{w}_\pi^*\mathfrak{F} = (G \mid \pi(G) \subseteq \pi(\mathfrak{F})$$

and

$$\{\mathrm{Syl}_q(G)\} \subseteq sub_{s\mathrm{K}-\mathfrak{F}}(G)$$

for every $q \in \pi \cap \pi(G)$), i. e.

$$\mathrm{w}_\pi^*\mathfrak{F} = (G \mid \pi(G) \subseteq \pi(\mathfrak{F})$$

and $\{N_G(Q)\} \subseteq sub_{\mathfrak{F}}(G)$ for every $Q \in \mathrm{Syl}_q(G)$ and $q \in \pi \cap \pi(G)$).

***Example* 2.1.** Let $M = S_4$ be a symmetric group of degree 4. From [6, theorem B. 10.9] it follows that there exists an irreducible and faithful M-module U over the field F_3 of 3 elements. Let $G = [U]M$ be a semidirect product U and M, and note that the nilpotent length of G is 4 and $\pi(G) = \{2,3\}$. Since M is a minimal non-$\mathfrak{N}^2$-subgroup, we deduced that G is minimal non-$\mathfrak{N}^3$-group. It is easy to see that $\{N_G(P)\} \subseteq sub_{\mathfrak{F}}(G)$ for all $P \in \mathrm{Syl}(G)$. This means that G belongs to $\mathrm{w}^*(\mathfrak{N}^3)$.

This example shows that $\mathrm{w}^*_\pi\mathfrak{F} \neq \mathfrak{F}$ in the general case.

***Proposition* 2.2.** *Let G be a group,* $P \in \mathrm{Syl}_p(G)$. *If* $L \trianglelefteq G$ *and* $K \trianglelefteq G$, *then*

$$N_G(P) \cap LK = (N_G(P) \cap L)(N_G(P) \cap K)$$

and

$$N_G(P)L \cap N_G(P)K = N_G(P)(L \cap K).$$

Proof. We proceed by induction on $|G|$. Let L and K be normal subgroups of G and $P \in \mathrm{Syl}_p(G)$. If $L \cap K \neq 1$, then there exists a minimal normal subgroup N of G, contained in $L \cap K$. By induction

$$N_{G/N}(PN/N) \cap L/N \cdot K/N = (N_{G/N}(PN/N) \cap L/N)(N_{G/N}(PN/N) \cap K/N).$$

By Lemma 1.1 (1)

$$N_{G/N}(PN/N) = N_G(P)N/N.$$

By the Dedekind identity, we have

$$N_G(P)N/N \cap LK/N = (N_G(P) \cap LK)N/N$$

and

$$N_G(P)N/N \cap L/N = (N_G(P) \cap L)N/N,$$

$$N_G(P)N/N \cap K/N = (N_G(P) \cap K)N/N.$$

Then

$$N_G(P) \cap LK = N_G(P) \cap (N_G(P)N \cap LK) = N_G(P) \cap (N_G(P) \cap L)N \cdot (N_G(P) \cap K)N = (N_G(P) \cap L)(N_G(P) \cap K)(N_G(P) \cap N) = (N_G(P) \cap L)(N_G(P) \cap K).$$

Let $L \cap K = 1$. Let $T = N_G(P)L \cap N_G(P)K$. Since $PL \trianglelefteq N_G(P)L$ and $PK \trianglelefteq N_G(P)K$ we have $PL \cap PK \trianglelefteq T$. From $L \cap K = 1$ and lemma 1.1 (3) it follows that $PL \cap PK = P(L \cap K) = P$. Therefore, $P \trianglelefteq T$ and $T = N_G(P)$. Then

$$N_G(P)(L \cap K) = N_G(P) = N_G(P)L \cap N_G(P)K.$$

By lemma 1.2

$$N_G(P) \cap LK = (N_G(P) \cap L)(N_G(P) \cap K). \quad \square$$

***Proposition* 2.3.** *Let* $\mathfrak{F}$ *be a formation. Then*

(1) $\mathrm{w}^*\mathfrak{F} \subseteq \mathrm{w}^*_{\pi_1}\mathfrak{F} \subseteq \mathrm{w}^*_\pi\mathfrak{F}$ *for* $\pi \subseteq \pi_1 \subseteq \mathbb{P}$,

(2) $\mathfrak{N}_{\pi\cap\pi(\mathfrak{F})} \subseteq \mathrm{w}^*_\pi\mathfrak{F} = \mathrm{w}^*_{\pi\cap\pi(\mathfrak{F})}\mathfrak{F}$,

(3) $\mathrm{w}^*_\pi\mathfrak{F}$ *is a homomorph,*

(4) $\mathrm{w}^*_\pi\mathfrak{F}_1 \subseteq \mathrm{w}^*_\pi\mathfrak{F}$ *for every formation* $\mathfrak{F}_1 \subseteq \mathfrak{F}$.

Proof. (1): Let $G \in \mathrm{w}^*_{\pi_1}\mathfrak{F}$, and assume that Q is any Sylow q-subgroup of G with $q \in \pi \cap \pi(G)$. Since $q \in \pi_1 \cap \pi(G)$, we have $N_G(Q) \in sub_{\mathfrak{F}}(G)$. Hence $\mathrm{w}^*_{\pi_1}\mathfrak{F} \subseteq \mathrm{w}^*_\pi\mathfrak{F}$. From $\pi_1 \subseteq \mathbb{P}$ we conclude that $\mathrm{w}^*\mathfrak{F} \subseteq \mathrm{w}^*_{\pi_1}\mathfrak{F}$.

(2): Let $G \in \mathfrak{N}_{\pi\cap\pi(\mathfrak{F})}$. Then

$$\pi(G) \subseteq (\pi \cap \pi(\mathfrak{F})) \subseteq \pi(\mathfrak{F}).$$

Since $N_G(P) = G$ for every $P \in \mathrm{Syl}(G)$, it follows that $G \in \mathrm{w}^*_\pi\mathfrak{F}$ and $\mathfrak{N}_{\pi\cap\pi(\mathfrak{F})} \subseteq \mathrm{w}^*_\pi\mathfrak{F}$.

From (1) it follows that $\mathrm{w}^*_\pi\mathfrak{F} \subseteq \mathrm{w}^*_{\pi\cap\pi(\mathfrak{F})}\mathfrak{F}$. Let $G \in \mathrm{w}^*_{\pi\cap\pi(\mathfrak{F})}\mathfrak{F}$. Since $\pi(G) \subseteq \pi(\mathfrak{F})$, we have

$$\pi \cap \pi(\mathfrak{F}) \cap \pi(G) = \pi \cap \pi(G).$$

Consequently, if $q \in \pi \cap \pi(G)$, then

$$\{N_G(P)\} \subseteq sub_{\mathfrak{F}}(G)$$

for all $P \in \mathrm{Syl}_q(G)$. So $G \in \mathrm{w}^*_\pi\mathfrak{F}$ and

$$\mathrm{w}^*_\pi\mathfrak{F} = \mathrm{w}^*_{\pi\cap\pi(\mathfrak{F})}\mathfrak{F}.$$

(3): To prove that $\mathrm{w}^*_\pi\mathfrak{F}$ is a homomorph, let $G \in \mathrm{w}^*_\pi\mathfrak{F}$, $N \trianglelefteq G$ and $p \in \pi \cap \pi(G/N)$. Consider $H/N \in \mathrm{Syl}_p(G/N)$. By lemma 1.1 (2) $H/N = PN/N$ for some Sylow p-subgroup P of G. From $G \in \mathrm{w}^*_\pi\mathfrak{F}$ it follows that $N_G(P) \in sub_{\mathfrak{F}}(G)$. Then by lemma 1.1 (1) and lemma 1.3 (1)

$$N_{G/N}(H/N) = N_G(P)N/N \in sub_{\mathfrak{F}}(G/N).$$

From here and $\pi(G/N) \subseteq \pi(G) \subseteq \pi(\mathfrak{F})$ we have that $G/N \in \mathrm{w}^*_\pi\mathfrak{F}$. So $\mathrm{w}^*_\pi\mathfrak{F}$ is a homomorph.

(4): Let $G \in \mathrm{w}^*_\pi\mathfrak{F}_1$. Then $\pi(G) \subseteq \pi(\mathfrak{F}_1) \subseteq \pi(\mathfrak{F})$. From $q \in \pi \cap \pi(G)$ it follows that every $Q \in \mathrm{Syl}_q(G)$ is strongly K-$\mathfrak{F}_1$-subnormal in G. Suppose that $N_G(Q) \neq G$. Then a maximal chain of subgroups $N_G(Q) = H_0 < H_1 < \ldots < H_n = G$ exists and $H_i^{\mathfrak{F}_1} \leq H_{i-1}$ for $i = 1, \ldots, n$. From $H_i / H_i^{\mathfrak{F}_1} \in \mathfrak{F}_1 \subseteq \mathfrak{F}$ we have $H_i^{\mathfrak{F}} \leq H_i^{\mathfrak{F}_1} \leq H_{i-1}$. Hence $N_G(Q) \in sub_{\mathfrak{F}}(G)$. For $N_G(Q) = G$ it is obviously that $N_G(Q) \in sub_{\mathfrak{F}}(G)$. So $\mathrm{w}^*_\pi\mathfrak{F}_1 \subseteq \mathrm{w}^*_\pi\mathfrak{F}$. $\square$

***Lemma* 2.4.** *Let* $\mathfrak{F}$ *be a hereditary formation and let* $\mathfrak{X} = \mathrm{w}^*_\pi\mathfrak{F}$. *If G is a group,* $P \in \mathrm{Syl}_p(G)$ *for* $p \in \pi \cap \pi(G)$ *and* $G^{\mathfrak{X}} \leq N_G(P)$, *then*

$$N_G(P) \in sub_{\mathfrak{F}}(G).$$

Proof. Since P is pronormal in G it follows by Lemma 1.6 $N_G(P)$ is abnormal in G. Therefore we have $G^{\mathfrak{X}} \neq N_G(P)$. From $G^{\mathfrak{X}} \leq N_G(P)$ and $G/G^{\mathfrak{X}} \in \mathfrak{X}$ it follows, that

$$N_G(P)/G^{\mathfrak{X}} = N_{G/G^{\mathfrak{X}}}(PG^{\mathfrak{X}}/G^{\mathfrak{X}}) \in sub_{\mathfrak{F}}(G/G^{\mathfrak{X}}).$$

By Lemma 1.3 (2) $N_G(P) \in sub_{\mathfrak{F}}(G)$. □

***Lemma* 2.5.** *Let $\mathfrak{F}$ be a hereditary formation, and let $\mathfrak{X} = \mathrm{w}_\pi^*\mathfrak{F}$. Let G be a group, and assume that $M \in sub_{\mathfrak{X}}(G)$, $N_G(P) \in sub_{\mathfrak{X}}(M)$ for some $P \in \mathrm{Syl}_p(G)$ and $p \in \pi \cap \pi(G)$. If $N_G(P) <\cdot M <\cdot G$, then $N_G(P) \in sub_{\mathfrak{F}}(G)$.*

Proof. Since $P \in \mathrm{Syl}_p(M)$, $p \in \pi \cap \pi(M)$ and $M^{\mathfrak{X}} \leq N_M(P) = N_G(P)$ we have by Lemma 2.4 $N_G(P) \in sub_{\mathfrak{F}}(M)$. From $M \in sub_{\mathfrak{X}}(G)$ and $M <\cdot G$ it follows $G^{\mathfrak{X}} \leq M$.

If $G^{\mathfrak{X}} \leq N_G(P)$, then by Lemma 2.4

$$N_G(P) \in sub_{\mathfrak{F}}(G).$$

Suppose that $G^{\mathfrak{X}} \not\leq N_G(P)$. The subgroup $N_G(P)$ is maximal in M. Therefore $M = N_G(P)G^{\mathfrak{X}}$. From $G/G^{\mathfrak{X}} \in \mathfrak{X}$ we have

$$M/G^{\mathfrak{X}} = N_G(P)G^{\mathfrak{X}}/G^{\mathfrak{X}} =$$

$$= N_{G/G^{\mathfrak{X}}}(PG^{\mathfrak{X}}/G^{\mathfrak{X}}) \in sub_{\mathfrak{F}}(G/G^{\mathfrak{X}}).$$

By Lemma 1.3 (2) $M \in sub_{\mathfrak{F}}(G)$. Since $N_G(P) \in sub_{\mathfrak{F}}(M)$, we have $N_G(P) \in sub_{\mathfrak{F}}(G)$. □

***Definition* 2.6.** *A class of groups $\mathfrak{F}$ is called S_H-closed, if from $G \in \mathfrak{F}$ it follows that every Hall subgroup belongs to $\mathfrak{F}$.*

***Theorem* 2.7.** If $\mathfrak{F}$ is a hereditary formation, then $\mathfrak{F} \subseteq \mathrm{w}^*\mathfrak{F} \subseteq \mathrm{w}_\pi^*\mathfrak{F} = \mathrm{w}_\pi^*(\mathrm{w}_\pi^*\mathfrak{F})$ and $\mathrm{w}_\pi^*\mathfrak{F}$ is an S_H-closed formation.

Proof. If a group $G \in \mathfrak{F}$, then $G^{\mathfrak{F}} = 1 \leq N_G(P)$ for every $P \in \mathrm{Syl}(G)$. By Lemma 1.4 (3) it follows that $N_G(P) \in sub_{\mathfrak{F}}(G)$ and $\mathfrak{F} \subseteq \mathrm{w}^*\mathfrak{F}$. From Proposition 2.3 (1) we have $\mathrm{w}^*\mathfrak{F} \subseteq \mathrm{w}_\pi^*\mathfrak{F}$.

We will show that $\mathrm{w}_\pi^*\mathfrak{F}$ is a formation. By Proposition 2.3 (3) $\mathrm{w}_\pi^*\mathfrak{F}$ is a homomorph.

Let us prove that $\mathrm{w}_\pi^*\mathfrak{F}$ is closed under subdirect products. Suppose that is false, and let G be a counterexample with $|G|$ as small as possible. Then there exists a subgroup $N_i \trianglelefteq G$ such that $G/N_i \in \mathrm{w}_\pi^*\mathfrak{F}$, $i = 1,2$, but $G/N_1 \cap N_2 \notin \mathrm{w}_\pi^*\mathfrak{F}$. We note that from $\pi(G/N_i) \subseteq \pi(\mathfrak{F})$, $i = 1,2$, it follows that $\pi(G/N_1 \cap N_2) \subseteq \pi(\mathfrak{F})$. By the choice of G we can assume that $N_1 \cap N_2 = 1$. Let $p \in \pi \cap \pi(G)$ and $R \in \mathrm{Syl}_p(G)$. Since RN_i/N_i is a Sylow p-subgroup of G/N_i and $G/N_i \in \mathrm{w}_\pi^*\mathfrak{F}$, we have

$$N_{G/N_i}(RN_i/N_i) \in sub_{\mathfrak{F}}(G/N_i), \ \ i = 1,2.$$

By Lemmas 1.1 (1) and 1.3 (2) $N_G(R)N_i \in sub_{\mathfrak{F}}(G)$, $i = 1,2$. From Lemma 1.4 (2) it follows

$$N_G(R)N_1 \cap N_G(R)N_2 \in sub_{\mathfrak{F}}(G).$$

From Proposition 1.3 we conclude that

$$N_G(R)N_1 \cap N_G(R)N_2 =$$

$$= N_G(R)(N_1 \cap N_2) = N_G(R) \in sub_{\mathfrak{F}}(G).$$

We have the contradiction to the choice of G. So $\mathrm{w}_\pi^*\mathfrak{F}$ is closed under subdirect products.

To prove S_H-closure of $\mathrm{w}_\pi^*\mathfrak{F}$, let $G \in \mathrm{w}_\pi^*\mathfrak{F}$, and let H be a Hall subgroup of G. Then $\pi(H) \subseteq \pi(G) \subseteq \pi(\mathfrak{F})$. Let $q \in \pi \cap \pi(H)$, and let S be a Sylow q-subgroup of H. Since $S \in \mathrm{Syl}_q(G)$, it follows that $N_G(S) \in sub_{\mathfrak{F}}(G)$. By Lemma 1.4 (1)

$$N_H(S) = N_G(S) \cap H \in sub_{\mathfrak{F}}(H).$$

Therefore $H \in \mathrm{w}_\pi^*\mathfrak{F}$ and $\mathrm{w}_\pi^*\mathfrak{F}$ is S_H-closed.

Now we will show that $\mathrm{w}_\pi^*\mathfrak{F} = \mathrm{w}_\pi^*(\mathrm{w}_\pi^*\mathfrak{F})$. Denote $\mathfrak{X} = \mathrm{w}_\pi^*\mathfrak{F}$. Let $G \in \mathfrak{X}$. Then $\pi(G) \subseteq \pi(\mathfrak{F})$. By what was proved above, we have that $\mathfrak{F} \subseteq \mathfrak{X}$. Therefore $\pi(G) \subseteq \pi(\mathfrak{X})$. Let $q \in \pi \cap \pi(G)$ and $Q \in \mathrm{Syl}_q(G)$. From $G \in \mathfrak{X}$ it follows that $N_G(Q) \in sub_{\mathfrak{F}}(G)$. Assume that $N_G(Q) \neq G$. Then there is a maximal chain of subgroups

$$N_G(Q) = H_0 < H_1 < \ldots < H_n = G$$

such that $H_i^{\mathfrak{F}} \leq H_{i-1}$ for $i = 1,\ldots,n$. We have proved above that $\mathfrak{X}$ is a formation. Therefore from $H_i/H_i^{\mathfrak{F}} \in \mathfrak{F} \subseteq \mathfrak{X}$ it follows that $H_i^{\mathfrak{X}} \leq H_i^{\mathfrak{F}} \leq H_{i-1}$. This means that $N_G(Q) \in sub_{\mathfrak{X}}(G)$. If $N_G(Q) = G$, then $N_G(Q) \in sub_{\mathfrak{X}}(G)$. So $G \in \mathrm{w}_\pi^*\mathfrak{X}$ and $\mathfrak{X} \subseteq \mathrm{w}_\pi^*\mathfrak{X}$ is proved.

Suppose that $\mathfrak{X} \neq \mathrm{w}_\pi^*\mathfrak{X}$. Let G be a group of minimal order in $\mathrm{w}_\pi^*\mathfrak{X} \backslash \mathfrak{X}$. Then $\pi(G) \subseteq \pi(\mathfrak{X}) \subseteq \pi(\mathfrak{F})$. Since $G \notin \mathfrak{X}$, there exists $P \in \mathrm{Syl}_p(G)$ such that $p \in \pi \cap \pi(G)$ and $N_G(P)$ is not $\mathfrak{F}$-subnormal in G. We note that $N_G(P) \in sub_{\mathfrak{X}}(G)$. Then $N_G(P) \neq G$ and there exists a maximal chain of subgroups $N_G(P) = H_0 < H_1 < \ldots < H_{n-1} < H_n = G$ such that $H_i^{\mathfrak{X}} \leq H_{i-1}$ for $i = 1,\ldots,n$. Since $N_G(P) = N_{H_i}(P)$, $N_{H_i}(P)H_i^{\mathfrak{X}} \leq H_{i-1}$ and $H_i/H_i^{\mathfrak{X}} \in \mathfrak{X}$, we have

$$N_{H_i}(P)H_i^{\mathfrak{X}}/H_i^{\mathfrak{X}} =$$

$$= N_{H_i/H_i^{\mathfrak{X}}}(PH_i^{\mathfrak{X}}/H_i^{\mathfrak{X}}) \in sub_{\mathfrak{F}}(H_i/H_i^{\mathfrak{X}}).$$

By Lemma 1.3 (2) $N_{H_i}(P)H_i^{\mathfrak{X}} \in sub_{\mathfrak{F}}(H_i)$ for $i = 1,\ldots,n$. Therefore $H_n^{\mathfrak{X}} = G^{\mathfrak{X}} \not\subseteq N_G(P)$. From the maximality of $N_G(P)$ in H_1 it follows that $N_G(P) \in sub_{\mathfrak{F}}(H_1)$. So $n \neq 1$. Suppose that $n = 2$. Then by Lemma 2.5 $N_G(P) \in sub_{\mathfrak{F}}(H_2) = sub_{\mathfrak{F}}(G)$. This is the contradiction with the choice of *G*. So, we can assume that $n \geq 3$ and $N_G(P) \in sub_{\mathfrak{F}}(H_{n-1})$. Since $N_G(P)H_n^{\mathfrak{X}} \leq H_{n-1}$, by Lemma 1.4 (1) we have

$$N_G(P) = N_G(P) \cap N_G(P)H_n^{\mathfrak{X}} \in sub_{\mathfrak{F}}(N_G(P)H_n^{\mathfrak{X}}).$$

From $N_G(P)H_n^{\mathfrak{X}} \in sub_{\mathfrak{F}}(G)$ it follows that $N_G(P) \in sub_{\mathfrak{F}}(G)$. This contradicts the choice of *G*. So $\mathfrak{X} = w_\pi^*\mathfrak{X}$. □

By $l_p(G)$ we denote the *p*-length of the *p*-soluble group *G*; an arithmetic length of the soluble group *G* is $al(G) = \mathrm{Max}\, l_p(G)$, where *p* runs through all primes $p \in \pi(G)$; $\mathfrak{L}_a(n)$ is the class of all soluble groups *G* with $al(G) < n$; $\mathfrak{L}_a(1)$ is the class of all soluble groups *G* with $al(G) \leq 1$.

From $\mathfrak{L}_a(1) = \cap \mathfrak{E}_{p'}\mathfrak{N}_p\mathfrak{E}_{p'}$ for all $p \in \mathbb{P}$ it follows that $\mathfrak{L}_a(1)$ is a hereditary saturated Fitting formation.

***Lemma* 2.8** [8, Lemma 4.1]. *Let G be a soluble group, and let $\Phi(G) = 1$. Then G is a minimal non-$\mathfrak{L}_a(1)$-group if and only if the following statements hold*:

(1) $|G| = p^\alpha q^\beta$, $l_p(G) = 1$, $l_q(G) = 2$, $l(G) = 3$,

(2) *G has precisely three conjugate classes of maximal subgroups, whose representatives have the following structure*: $G_p \rtimes G_q^*$ *is the Schmidt group*, $F(G) \rtimes G_p$ *and* $G_q \rtimes \Phi(G_p)$, *where*

$$G_q = F(G) \rtimes G_q^*.$$

***Lemma* 2.9.** *Let G be a biprimary group and let $G \in \mathfrak{L}_a(1)$. Then G is metanilpotent.*

***Lemma* 2.10.** *Let $\mathfrak{F}$ be a hereditary formation, and let G be a solvable group. If $G \in \mathfrak{L}_a(1)$, $G \neq N_G(P)$ and $N_G(P) \in \mathfrak{F}$ for all $P \in \mathrm{Syl}(G)$, then $G \in \mathfrak{F}$.*

***Theorem* 2.11.** *Let $\mathfrak{F}$ be a hereditary saturated formation, and let $\mathfrak{F} \subseteq \mathfrak{L}_a(1)$. A group $G \in \mathfrak{F}$ if and only if $G \in w^*\mathfrak{F}$.*

A subgroup *H* of a group *G* is called $\mathbb{P}$-subnormal in *G* [9], [10] if either $H = G$ or there exists a chain (0.1) such that $|H_i : H_{i-1}|$ is a prime for any $i = 1, \ldots, n$.

***Corollary* 2.12** [7]. *If the normalizers of all Sylow subgroup of a group G are $\mathbb{P}$-subnormal subgroups in G, then G is supersolvable.*

***Corollary* 2.13** [5]. *A group $G \in \mathfrak{N}^2$ if and only if $G \in w^*(\mathfrak{N}^2)$.*

***Corollary* 2.14** [5]. *A group $G \in \mathfrak{N}\mathfrak{A}$ if and only if $G \in w^*(\mathfrak{N}\mathfrak{A})$.*

***Corollary* 2.15.** *A group $G \in \mathfrak{L}_a(1)$ if and only if $G \in w^*(\mathfrak{L}_a(1))$.*

Conclusions

The properties of the operation w_π^* on the formations of groups are found. In particular, if $\mathfrak{F}$ is a hereditary formations, then $w_\pi^*\mathfrak{F} = w_\pi^*(w_\pi^*\mathfrak{F})$ is the formation, and every Hall subgroup of a group *G* belongs to $w_\pi^*\mathfrak{F}$ whenever $G \in w_\pi^*\mathfrak{F}$. Hereditary saturated formations $\mathfrak{F}$ for which $w_\pi^*\mathfrak{F}$ coincides with $\mathfrak{F}$ have been found.

Example. We show that

$$\mathfrak{A} \subset w^*\mathfrak{A} = \mathfrak{N} = w^*\mathfrak{N}.$$

Since the class $\mathfrak{A}$ of all abelian groups is a hereditary formation, then by Theorem 2.8 $\mathfrak{A} \subseteq w^*\mathfrak{A}$.

Let $G \in w^*\mathfrak{A}$ and $P \in \mathrm{Syl}_p(G)$. If $N_G(P) \neq G$, then there is a maximal chain of subgroups

$$N_G(P) = H_0 < H_1 < \ldots < H_n = G$$

such that $H_{i'} \leq H_{i-1}$ for $i = 1, \ldots, n$. Then $H_{n-1}/G' \trianglelefteq G/G'$. But $N_G(P)$ is abnormal in *G*. We have obtained a contradiction with $N_G(P) \leq H_{n-1} \trianglelefteq G$. Therefore $N_G(P) = G$ and $G \in \mathfrak{N}$. Hence $w^*\mathfrak{A} \subseteq \mathfrak{N}$. Since be Proposition 2.3 $\mathfrak{N} \subseteq w^*\mathfrak{A}$, it follows $w^*\mathfrak{A} = \mathfrak{N}$. So $\mathfrak{A} \subset w^*\mathfrak{A}$.

By Theorem 2.7 $\mathfrak{N} \subseteq w^*\mathfrak{N}$. Suppose that $G \in w^*\mathfrak{N}$. If $N_G(Q) \neq G$ for some $Q \in \mathrm{Syl}_q(G)$, then *G* has a maximal subgroup *M* such that $N_G(Q) \leq M$ and $G^{\mathfrak{N}} \leq M$. Since $M/G^{\mathfrak{N}}$ is a maximal subgroup in $G/G^{\mathfrak{N}} \in \mathfrak{N}$, it follows that $M/G^{\mathfrak{N}} \trianglelefteq G/G^{\mathfrak{N}}$. Therefore $N_G(Q) \leq M \trianglelefteq G$. Since $N_G(Q)$ is abnormal in *G*, this is impossible. So $N_G(Q) = G$ for all $Q \in \mathrm{Syl}_q(G)$ and $G \in \mathfrak{N}$. Hence $w^*\mathfrak{N} = \mathfrak{N}$.

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