МАТЕМАТИКА

УДК 512.542

О РЕШЕТКАХ ОБОБЩЕННО СУБНОРМАЛЬНЫХ ПОДГРУПП В КОНЕЧНЫХ МЕТАНИЛЬПОТЕНТНЫХ ГРУППАХ

А.Ф. Васильев, И.Н. Халимончик

Гомельский государственный университет им. Ф. Скорины, Гомель, Беларусь

ON LATTICES OF GENERALIZED SUBNORMAL SUBGROUPS IN FINITE METANILPOTENT GROUPS

A.F. Vasil'ev, I.N. Khalimonchik

F. Scorina Gomel State University, Gomel, Belarus

Решена проблема Шеметкова об описании наследственных насыщенных решеточных формаций и решеточных подгрупповых функторов в классе всех конечных метанильпотентных групп.

Ключевые слова: *конечная группа, решеточная формация, решеточный подгрупповой функтор, $\mathfrak{F}$-субнормальная подгруппа.*

The Shemetkov problem about description of hereditary saturated lattice formations and lattice subgroup functors in the class of all finite metanilpotent groups has been solved.

Keywords: *finite group, lattice formation, lattice subgroup functor, $\mathfrak{F}$-subnormal subgroup.*

Introduction

All groups considered are finite. In 1939 H. Wielandt [1] showed that the set of all subnormal subgroups of a group G forms a sublattice of the lattice of all subgroups of G. Now there are various generalizations of the concept of subnormal subgroups. In 1969 T. Hawkes [2] introduced the definition of $\mathfrak{F}$-subnormal subgroup in the class of soluble groups. In 1978 L.A. Shemetkov [3] extended the concept of $\mathfrak{F}$-subnormal subgroups to arbitrary finite groups. He set up the problem at number 12 [3, p. 93]: in which cases the set of all $\mathfrak{F}$-subnormal subgroups of G forms a lattice?

Let $\mathfrak{X}$ be a class of groups. The formation $\mathfrak{F}$ is called a lattice formation in $\mathfrak{X}$ if the set of all $\mathfrak{F}$-subnormal subgroups forms a sublattice of the lattice of all subgroups in every $\mathfrak{X}$-group.

Noted above the Shemetkov problem can be formulated as follows:

***Problem* 1.** *Let $\mathfrak{X}$ be a hereditary saturated formation. Describe all saturated lattice formations in $\mathfrak{X}$.*

In the work [4] authors established hereditary saturated lattice formations in class of all soluble groups.

This result was extended for normally-hereditary saturated lattice formations in the work [5]. In [5] also described hereditary saturated lattice formations in class of all groups.

In [6] authors investigated non-saturated hereditary lattice formations in class of all soluble groups.

In [7], [8] the Problem 1 was solved in the class $\mathfrak{N}^k$ for $k \geq 3$ and in class $\mathfrak{N}\mathfrak{A}$ respectively.

In [9] A.F. Vasil′ev and S.F. Kamornikov developed a general functor method of studing subgroup lattices similar to the lattice of subnormal subgroups. They introduced the concept of NTL-functor (natural transitive lattice subgroup functor) and described all NTL-functors in the class of all soluble groups. They showed that the NTL-functor in the soluble universe is exactly the functor of all $\mathfrak{F}$-subnormal subgroups for some hereditary saturated lattice formation $\mathfrak{F}$.

In [10] S.F. Kamornikov constructed a continuum of many NTL-functors that do not correspond to any hereditary lattice formations in the class of all finite groups.

The above-noted results lead to the following problem:

***Problem* 2.** *Let $\mathfrak{X}$ be a hereditary saturated formation. To describe all lattice subgroup functors in $\mathfrak{X}$.*

In [7] the Problem 2 was solved in the class $\mathfrak{N}^k$ for $k \geq 3$.

In this paper Problem 1 and Problem 2 has been solved in the class $\mathfrak{X} = \mathfrak{N}^2$ of all metanilpotent groups.

1 Preliminary results

All definitions, notations and results correspond to [11], [12]. We denote by: $S(G)$ the set of all subgroups of group G; $[K]H$ the semidirect product of normal subgroup K and subgroup H;

Z_p the group of prime order p; $\mathfrak{N}$ the class of all nilpotent groups; $\mathfrak{S}$ the class of all soluble groups; $\mathfrak{N}_\pi$ the class of all nilpotent π-groups; $\mathfrak{N}^2$ the class of all metanilpotent groups; $\mathfrak{N}^n$ the class of all soluble groups nilpotent length of which does not exceed a given positive integer n; $\mathbb{P}$ the set of all prime numbers.

The class of groups is called a formation if it is closed under factorgroups and subdirect products. The formation is called hereditary if it is closed under subgroups. The formation is called normally hereditary if it is closed under normal subgroups. The formation is called saturated if $\mathfrak{F} = (G \mid$ if $N \lhd G, N \subseteq \Phi(G)$, then $G/N \in \mathfrak{F})$. Let h be a function which associates with each prime p a class $h(p)$ of finite groups. Recall [12] h a local function if $h(p)$ is a formation for all prime number p. Let h be a local function, and let $\mathfrak{X}$ be a local formation. The h is called: integrated if $h(p) \subseteq \mathfrak{X}$ for all $p \in \mathbb{P}$; full if $h(p) = \mathfrak{N}_p h(p)$ for all $p \in \mathbb{P}$. The uniquely determined full and integrated formation function defining a local formation $\mathfrak{F}$ was called [12] the canonical local definition of $\mathfrak{F}$.

Formation $\mathfrak{F}$ is called Shemetkov formation in class $\mathfrak{X}$ if every minimal non-$\mathfrak{F}$-group of $\mathfrak{X}$ is Schmidt group or group of prime order.

***Definition* 1.1** [13, p. 13]. *Let $\mathfrak{X}$ be a class of groups. And let θ be a function mapping each group $G \in \mathfrak{X}$ into a some non-empty system $\theta(G)$ of its subgroups. The map θ is called a subgroup $\mathfrak{X}$-functor (a subgroup functor in $\mathfrak{X}$) if the following condition is satisfied: $(\theta(G))^\phi = \theta(G^\phi)$ for any isomorphism ϕ of every group $G \in \mathfrak{X}$.*

***Definition* 1.2**. *Let $\mathfrak{X}$ be a class of groups. A subgroup functor θ in $\mathfrak{X}$ is called*:

1) *lowerlattice (briefly, L_*-functor), if from $G \in \mathfrak{X}$, $A \in \theta(G)$ and $B \in \theta(G)$ always implies $A \cap B \in \theta(G)$;*

2) *upperlattice (briefly, L^*-functor), if from $G \in \mathfrak{X}$, $A \in \theta(G)$ and $B \in \theta(G)$ always implies $\langle A, B \rangle \in \theta(G)$;*

3) *semilattice (briefly, L_0-functor), if θ is L_*-functor in $\mathfrak{X}$ and from $G \in \mathfrak{X}$, $A \in \theta(G)$, $B \in \theta(G)$ and $AB = BA$ always implies $AB \in \theta(G)$;*

4) *lattice (briefly, L-functor), if θ is the L_* and L^*-functor in $\mathfrak{X}$ at the same time.*

***Definition* 1.3** [13, p. 14]. *Let $\mathfrak{X}$ be a homomorph. A subgroup $\mathfrak{X}$-functor θ is called functor of Skiba in $\mathfrak{X}$, if the following conditions for any group $G \in \mathfrak{X}$ and $N \lhd G$ are satisfied*:

1) *$H \in \theta(G)$ then $HN/N \in \theta(G/N)$;*

2) *$H/N \in \theta(G/N)$ then $H \in \theta(G)$.*

Let H be a subgroup of G. Then $H \cap \theta(G) = \{H \cap L \mid L \in \theta(G)\}$.

***Definition* 1.4** [13, p. 15]. *Let $\mathfrak{X}$ be a class of groups. A subgroup $\mathfrak{X}$-functor θ is called*:

1) *hereditary, if $H \cap \theta(G) \subseteq \theta(H)$ for any $\mathfrak{X}$-subgroup H of $G \in \mathfrak{X}$;*

2) *transitive, if from $S \in \theta(H)$ and $H \in \theta(G) \cap \mathfrak{X}$, it follows that $S \in \theta(G)$ for any group $G \in \mathfrak{X}$.*

In this paper all considered subgroup functors we assume as hereditary transitive subgroup functors of Skiba in class of groups $\mathfrak{X}$. The example of such subgroup functors is $\theta = sn_{\mathfrak{F}}$, where $\mathfrak{F}$ is a hereditary formation and $\theta(G) = sn_{\mathfrak{F}}(G)$ is the set of all $\mathfrak{F}$-subnormal subgroups for any group G.

***Lemma* 1.1.** *Let X be a hereditary saturated formation and θ be a subgroup functor in $\mathfrak{X}$. Let $Z_p \in \mathfrak{X}$ and $1 \in \theta(Z_p)$. Then $\theta(P) = S(P)$ for any p-group P.*

The proof is similar to lemma 3.2.1 [13].

Let $\mathfrak{X}$ be a class of groups and θ be a subgroup $\mathfrak{X}$-functor. We denote by $\Lambda_{\mathfrak{X}}(\theta)$ the class $(G \in \mathfrak{X} \mid 1 \in \theta(G)$ and $P \in \theta(G)$ for any Sylow subgroup P from $G)$.

***Lemma* 1.2.** *Let $\mathfrak{X}$ be a hereditary formation and θ be a subgroup L_*-functor in $\mathfrak{X}$. Then $\Lambda_{\mathfrak{X}}(\theta)$ is a hereditary formation.*

Proof. Let $G \in \Lambda_{\mathfrak{X}}(\theta)$ and $N \lhd G$. Since $1 \in \theta(G)$, it follows that $N/N \in \theta(G/N)$. Let H/N be a Sylow subgroup of G/N. There is a Sylow subgroup P of group G such that $HN/N = PN/N$. From $P \in \theta(G)$ and 1) definition 1.3 we obtain $PN/N = H/N \in \theta(G/N)$. Therefore $G/N \in \Lambda_{\mathfrak{X}}(\theta)$.

Let $N_1, N_2 \lhd G$ and $N_1 \cap N_2 = 1$. Assume $G/N_i \in \Lambda_{\mathfrak{X}}(\theta)$, $i = 1, 2$. If P is Sylow subgroup of G, then $PN_i/N_i \in \theta(G/N_i)$, $i = 1, 2$. From 2) definition 1.3 we have $PN_i \in \theta(G)$, $i = 1, 2$. Since θ is L_*-functor,

$$PN_1 \cap PN_2 = P(N_1 \cap N_2) = P \in \theta(G).$$

From $N_i/N_i \in \theta(G/N_i)$ we conclude $N_i \in \theta(G)$, $i = 1, 2$. Then $N_1 \cap N_2 = 1 \in \theta(G)$. Hence $\Lambda_{\mathfrak{X}}(\theta)$ is a formation.

Let H be a subgroup of G. Suppose that P is arbitrary Sylow subgroup of H. Then $P \subseteq S$ where S is some Sylow subgroup of G. Since $S \in \theta(G)$ and by hereditary of functor θ, we have $P = S \cap H \in \theta(H)$. From $1 \in \theta(G)$ we conclude that

$1 \cap H = 1 \in \theta(H)$. Therefore $\Lambda_{\mathfrak{X}}(\theta)$ is a hereditary formation.

***Definition* 1.5.** Let $\mathfrak{X}$ be a class of groups. The set $\pi(\theta)$ of all primes p for which $1 \in \theta(Z_p)$ where $Z_p \in \mathfrak{X}$ is called a characteristic of subgroup $\mathfrak{X}$-functor θ.

***Lemma* 1.3.** *Let $\mathfrak{X}$ be a hereditary saturated formation and π is characteristic of subgroup L_*-functor in $\mathfrak{X}$. Then $\mathfrak{N}_{\pi(\theta)} \subseteq \Lambda_{\mathfrak{X}}(\theta)$.*

Proof. Let $p \in \pi$. By lemma 1.1, $\theta(P) = S(P)$ for any *p*-subgroup P. Hence $P \in \Lambda_{\mathfrak{X}}(\theta)$. Since $\Lambda_{\mathfrak{X}}(\theta)$ is formation, we have $\mathfrak{N}_{\pi} \subseteq \Lambda_{\mathfrak{X}}(\theta)$.

Let θ be a subgroup functor in class of groups $\mathfrak{X}$. We will denote the class $(G \in \mathfrak{X} \mid \theta(G) = S(G))$ by $S_{\mathfrak{X}}(\theta)$.

***Lemma* 1.4.** *Let $\mathfrak{X}$ be a hereditary formation of soluble groups and let θ be a subgroup L_0-functor in $\mathfrak{X}$. Then $\Lambda_{\mathfrak{X}}(\theta) = S_{\mathfrak{X}}(\theta)$.*

Proof. It is clear that $S_{\mathfrak{X}}(\theta) \subseteq \Lambda_{\mathfrak{X}}(\theta)$. Let $G \in \Lambda_{\mathfrak{X}}(\theta)$. Then $P \in \theta(G)$ for any Sylow subgroup P of G. From $1 \in \theta(G)$ and by the properties of the functor θ we conclude $1 = 1 \cap P \in \theta(P)$. By lemma 1.1 $\theta(P) = S(P)$ for any Sylow subgroup P of G. From the transitivity of θ we obtain $S(P) \subseteq \theta(G)$ for any Sylow subgroup P of G. Let H be any subgroup. Since H is soluble, according to Hall's theorem (see [12]) there is Sylow basis $P_1, \cdots, P_n$ in H. Since $H = P_1 P_2 \cdots P_n$ and θ is L_0-functor we have $H \in \theta(G)$. Therefore $G \in S_{\mathfrak{X}}(\theta)$.

***Lemma* 1.5** [13, p. 148]. *Let θ be a subgroup L-functor. Then and only then $G \in S_{\mathfrak{S}}(\theta)$ when $1 \in \theta(G)$ and $P \in \theta(G)$ for any Sylow subgroup P of G.*

***Lemma* 1.6** [13, p. 148]. *Let θ be a subgroup L-functor. Let $G = [P] < A, B >$ where P is p-subgroup, $< A, B >$ is q-group, p, q is primes, $p \neq q$. If $PA \in S_{\mathfrak{S}}(\theta)$ and $PB \in S_{\mathfrak{S}}(\theta)$, then $G \in S_{\mathfrak{S}}(\theta)$.*

***Definition* 1.6.** Formation $\mathfrak{F}$ is called Fitting formation in class of group $\mathfrak{X}$ if the following conditions are satisfied:

1) $\mathfrak{F}$ is normally hereditary formation;

2) if $G = AB$ where $G \in X$, $A \lhd G$, $B \lhd G$, $A \in \mathfrak{F}$, $B \in \mathfrak{F}$ then $G \in \mathfrak{F}$.

***Lemma* 1.7.** *Let $\mathfrak{X}$ be a hereditary formation of soluble groups and θ be a subgroup L_0-functor in $\mathfrak{X}$. Then $\Lambda_{\mathfrak{X}}(\theta)$ is a Fitting formation in $\mathfrak{X}$.*

Proof. By lemma 1.2 $\Lambda_{\mathfrak{X}}(\theta)$ is a hereditary formation. Let $G \in \mathfrak{X}$ be a group of minimal order such that $G = MN$, where $M, N \lhd G$ and $M, N \in \Lambda_{\mathfrak{X}}(\theta)$, but $G \notin \Lambda_{\mathfrak{X}}(\theta)$.

Since $\Lambda_{\mathfrak{X}}(\theta)$ is a formation, it follows that G has the unique minimal normal subgroup $D = G^{\Lambda_{\mathfrak{X}}(\theta)}$. Let P be an arbitrary Sylow subgroup of G. By theorem 6.4 [12],

$$P = P \cap MN = (P \cap M)(P \cap N),$$

where $P \cap M$ and $P \cap N$ are Sylow subgroups of M and N respectively. Since $M \in \Lambda_{\mathfrak{X}}(\theta)$, we have $P \cap M \in \theta(M)$. From $G / D \in \Lambda_{\mathfrak{X}}(\theta) = S_{\mathfrak{X}}(\theta)$ we conclude that $M / D \in \theta(G / D)$. Therefore $M \in \theta(G)$. By the transitivity of θ, $P \cap M \in \theta(G)$. Similarly, $P \cap N \in \theta(G)$. Since θ is L_0-functor in $\mathfrak{X}$, it follows that $P \in \theta(G)$. Hence $G \in \Lambda_{\mathfrak{X}}(\theta)$. A contradiction. Therefore $\Lambda_{\mathfrak{X}}(\theta)$ is Fitting formation in $\mathfrak{X}$.

***Lemma* 1.8.** *Let $\mathfrak{X} = \mathfrak{N}^2$ and θ be a subgroup L_0-functor in $\mathfrak{X}$. If $R \in \Lambda_{\mathfrak{X}}(\theta)$ be a p-closed $\{p, q\}$-group of Schmidt and $\Phi(R) = 1$, then $\Lambda_{\mathfrak{X}}(\theta)$ contains all extensions of p-groups by cyclic q-groups.*

Proof. By [3], there is the unique (up to isomorphism) *p*-closed $\{p, q\}$-group of Schmid $R = [N] Z_q$ where N is the unique minimal normal subgroup of R, $\Phi(R) = 1$. Assume that $R \in \Lambda_{\mathfrak{X}}(\theta)$. Let us first prove that $\Lambda_{\mathfrak{X}}(\theta)$ contains all extensions of *p*-groups by a group of prime order q.

Let $G = P Z_q$ where P is *p*-group and $P \lhd G$. We use induction on $|G|$ to prove $G \in \Lambda_{\mathfrak{X}}(\theta)$.

Let $Z_q \lhd G$. Then $G = P \times Z_q$. From $R \in \Lambda_{\mathfrak{X}}(\theta)$ and from hereditary class $\Lambda_{\mathfrak{X}}(\theta)$ it follows that $\{p, q\} \subseteq \pi(\Lambda_{\mathfrak{X}}(\theta))$. By lemma 1.3, $\mathfrak{N}_{\pi} \subseteq \Lambda_{\mathfrak{X}}(\theta)$ where $\pi = \pi(\Lambda_{\mathfrak{X}}(\theta))$. Hence $G \in \Lambda_{\mathfrak{X}}(\theta)$. Therefore we assume that Z_q is not normal in G. Let K is minimal normal subgroup of group G. Suppose that $K \neq P$. It is clear that $K \subseteq P$ and $K Z_q \neq G$. By induction $G / K \in \Lambda_{\mathfrak{X}}(\theta)$. Therefore

$$K Z_q / K \in \theta(G / K)$$

and $K Z_q \in \theta(G)$. As $|K Z_q| < |G|$ we have $K Z_q \in \Lambda_{\mathfrak{X}}(\theta)$. Hence $Z_q \in \theta(G)$. From $G / P \in \Lambda_{\mathfrak{X}}(\theta)$ we obtain that $P / P \in \Lambda_{\mathfrak{X}}(\theta)$. So $P \in \theta(G)$. Thus any Sylow subgroup from G contains in $\theta(G)$. From $P \in \theta(G)$ and $Z_q \in \theta(G)$ it follows that $1 = P \cap Z_q \in \theta(G)$. Therefore $G \in \Lambda_{\mathfrak{X}}(\theta)$.

Suppose that $K = P$. Then G is a Schmid group and $\Phi(G) = 1$. Therefore $G \in \Lambda_{\mathfrak{X}}(\theta)$. We proved that any extension *p*-group by the group of prime order q belongs to $\Lambda_{\mathfrak{X}}(\theta)$.

Let now $G = PZ_{q^n}$, where P is a p-group and Z_{q^n} is a cyclic q-group of order q^n. We prove that $G \in \Lambda_{\mathfrak{X}}(\theta)$ by induction on n. By the above, for $n = 1$ the statement is true. Let $n > 1$. Consider the group $E = Pwr(Z_q wr Z_{q^{n-1}})$. By theorem 18.9A [12], G is isomorphic to a subgroup from E. Note that $Z_q wr Z_{q^{n-1}} = [B]Z_{q^{n-1}}$, where B is the base of wreath $Z_q wr Z_{q^{n-1}}$. Denote by P^* the Sylow p-subgroup of E. Then $E = P^*([B]Z_{q^{n-1}})$. Since

$$E / P^* \simeq [B]Z_{q^{n-1}} \in \Lambda_{\mathfrak{X}}(\theta),$$

it follows that $P^* Z_{q^{n-1}} / P^* \in \Lambda_{\mathfrak{X}}(\theta)$. Hence $P^* Z_{q^{n-1}} \in \theta(E)$. By induction $P^* Z_{q^{n-1}} \in \Lambda_{\mathfrak{X}}(\theta)$. Then $Z_{q^{n-1}} \in \theta(P^* Z_{q^{n-1}})$ and $P^* \in \theta(P^* Z_{q^{n-1}})$. From the transitivity of θ we conclude that $Z_{q^{n-1}} \in \theta(E)$ and $P^* \in \theta(E)$.

Similarly $P^* B / P^* \in \theta(E / P^*)$. Hence $P^* B \in \theta(E)$. Since $B \simeq Z_q \times \cdots \times Z_q$, by induction (case $n = 1$), by lemmas 1.6 and 1.4, it follows that $P^* B \in \Lambda_{\mathfrak{X}}(\theta)$. Then $B \in \theta(P^* B)$. By the transitivity of θ, $B \in \theta(G)$. Since θ is L_0-functor, we obtain $< B, Z_{q^{n-1}} >= [B]Z_{q^{n-1}} \in \theta(E)$. By lemmas 1.5 and 1.4, $E \in \Lambda_{\mathfrak{X}}(\theta)$. Since $\Lambda_{\mathfrak{X}}(\theta)$ is a hereditary class, it follows that $G \in \Lambda_{\mathfrak{X}}(\theta)$.

***Lemma* 1.9** [14]. *Let $\mathfrak{F}$ be a hereditary local formation, h be the canonical local function of $\mathfrak{F}$. Let $\mathfrak{X}$ be a hereditary local formation, x be the canonical hereditary local function of $\mathfrak{X}$ and $\mathfrak{F} \subseteq \mathfrak{X}$. Then and only then formation $\mathfrak{F}$ is a Shemetkov formation in $\mathfrak{X}$ when it's canonical hereditary local x-function f:*

1) $f(p) = \mathfrak{S}_{\pi(f(p))} \cap x(p)$ *for any* $p \in \pi(\mathfrak{F})$;

2) $f(p) = \varnothing$ *for any* $p \in \pi'(\mathfrak{F})$.

***Lemma* 1.10.** *Let $\mathfrak{X} = \mathfrak{N}^2$. If θ is a subgroup L_0-functor in $\mathfrak{X}$, then $\Lambda_{\mathfrak{X}}(\theta)$ is a saturated formation.*

Proof. Denote $\mathfrak{F} = \Lambda_{\mathfrak{X}}(\theta)$. Let group G be a counterexample of minimal order. Then $\Phi(G) \neq 1$ and $G / \Phi(G) \in \mathfrak{F}$, but $G \notin \mathfrak{F}$.

Let N be a minimal normal subgroup of G. Then $\Phi(G)N / N \subseteq \Phi(G / N)$. Since $G / N / \Phi(G)N / N \simeq G / \Phi(G)N$ and $G / \Phi(G) \in \mathfrak{F}$ it follows that $G / N / \Phi(G)N / N \in \mathfrak{F}$. By the choice of G we get $G / N \in \mathfrak{F}$.

If K is a minimal normal subgroup of G and $K \neq N$, then $G / K \in \mathfrak{F}$. Since $\mathfrak{F}$ is a formation, we have $G / K \cap N \simeq G \in \mathfrak{F}$. The contradiction. Therefore G has the unique minimal normal subgroup N, where $N \subseteq \Phi(G)$ and N is a p-group for some prime p. It is easy to show that the Fitting subgroup $F(G)$ is a p-group.

Let G_q be a Sylow q-subgroup, where $q \neq p$. From $G \in \mathfrak{N}^2$, we conclude that $G_q F(G) \lhd G$.

Consider the subgroup $G_q F(G)$. Since $\mathfrak{F}$ is a hereditary formation, $G_q F(G) / N \in \mathfrak{F}$. Since $N \subseteq \Phi(G)$, we conclude $G_q F(G) / \Phi(G) \in \mathfrak{F}$.

If $G_q F(G) \in \mathfrak{N}$, then

$$G_q \subseteq C_{G_q F(G)}(F(G)) \subseteq C_G(F(G)) = F(G).$$

The contradiction. Therefore $G_q F(G)$ is a non-nilpotent p-closed group. Note that $G_q F(G) / N$ is a non-nilpotent p-closed $\{p, q\}$-group. Formation $\mathfrak{F}$ is hereditary. There is p-closed $\{p, q\}$-group of Schmidt R, $\Phi(R) = 1$ and $R \in \mathfrak{F}$. By lemmas 1.8 and 1.7 $G_q F(G) \in \mathfrak{F}$. Since this result holds for any Sylow subgroup of G, it follows that G is a product of their normal $\mathfrak{F}$-subgroups. Since $\mathfrak{F}$ is a Fitting formation in $\mathfrak{N}^2$, we obtain $G \in \mathfrak{F}$. The contradiction. Therefore $\mathfrak{F}$ is a saturated formation.

2 Main results

***Theorem* 2.1.** *Let θ be a subgroup L-functor in $\mathfrak{X} = \mathfrak{N}^2$. Then:*

1) *the class $\mathfrak{F} = \Lambda_{\mathfrak{X}}(\theta)$ is a hereditary saturated Shemetkov formation in $\mathfrak{N}^2$ and has the canonical local function $f : f(p) = \mathfrak{N}_p \mathfrak{N}_{\pi(f(p))}$ for any $p \in \pi(\mathfrak{F})$; $f(p) = \varnothing$ for any $p \in \pi'(\mathfrak{F})$;*

2) *$\theta(G) = sn_{\Lambda_{\mathfrak{X}}(\theta)}(G)$ for any group $G \in \mathfrak{X}$.*

***Theorem* 2.2**. *Let $\mathfrak{F}$ be a saturated formation, $\mathfrak{F} \subseteq \mathfrak{N}^2$. Then statements are equivalent:*

1) *$\mathfrak{F}$ is a lattice formation in $\mathfrak{N}^2$;*

2) *$\mathfrak{F}$ is a Shemetkov formation in $\mathfrak{N}^2$;*

3) *$\mathfrak{F}$ has the canonical local function f: $f(p) = \mathfrak{N}_p \mathfrak{N}_{\pi(f(p))}$ for any $p \in \pi(\mathfrak{F})$; $f(p) = \varnothing$ for any $p \in \pi'(\mathfrak{F})$.*

REFERENCES

1. *Wielandt*, *H.* Eine Verallgemeinerung der invarianten Untergruppen / H. Wielandt // Math. Z. – 1939. – Bd. 45. – S. 209–244.

2. *Hawkes*, *T.* On formations subgroups of a finite soluble groups / T. Hawkes // J. London Math. Soc. – 1969. – Vol. 44. – P. 243–250.

3. *Shemetkov*, *L.A.* Formations of finite groups / L.A. Shemetkov. – M.: Nauka, 1978.

4. *Ballester-Bolinches*, A. On the lattice on $\mathfrak{F}$-subnormal subgroups / A. Ballester-Bolinches, K. Doerk, M.D. Perez-Ramos // J. Algebra. – 1992. – Vol. 115. – P. 393–396.

5. *Vasil'ev*, *A.F.* On lattices of subgroups of finite groups / A.F. Vasil'ev, S.F. Kamornikov, V.N. Semenchuk // In N.S. Chernikov editor, Infinite groups and related algebraic structures. Institut Matematiki AN Ukrainy. – Kiev. – 1993. – P. 27–54.

6. *Vasil'ev*, *A.F.* On Kegel-Shemetkov problem on lattices of generalized subnormal subgroups of finite groups / A.F. Vasil'ev, S.F. Kamornikov // Algebra Logika. – 2002. – Vol. 41, № 4. – P. 411–428.

7. *Khalimonchik*, *I.N.* On lattices of subgroups subnormal type in finite groups / I.N. Khalimonchik, A.F. Vasil'ev // Proceedings of the Institute of Mathematics. – 2013. – Vol. 21, № 1. – P. 25–34.

8. *Khalimonchik*, *I.N.* Conditionally lattice formations of finite groups / I.N. Khalimonchik, A.F. Vasil'ev // Proceedings of the F.Scorina Gomel State University. – 2008. – Vol. 47, № 2. – P. 50–55.

9. *Vasil'ev*, *A.F.* On the functor method for studing of lattices of subgroups of finite groups / A.F. Vasil'ev, S.F. Kamornikov // Sibirsk. Math. Zh. – 2001. – Vol. 2. – P. 30–40.

10. *Kamornikov*, *S.F.* On a class of lattice subgroup functors / S.F. Kamornikov // Mathematical Notes. – 2011. – Vol. 89, № 3–4. – P. 340–348.

11. *Ballester-Bolinches*, *A*. Classes of Finite Groups / A. Ballester-Bolinches, L.M. Ezquerro. – Springer, 2006.

12. *Doerk*, *K*. Finite soluble groups / K. Doerk, T. Hawkes. – Berlin-New York: Walter de Gruyter, 1992.

13. *Kamornikov*, *S.F.* Subgroup functors and classes of finite groups / S.F. Kamornikov, M.V. Selkin. – Minsk: Belaruskaya nauka, 2003.

14. *Khalimonchik*, *I.N.* Shemetkov formation in class $\mathfrak{X}$ / I.N. Khalimnchik // Proceedings of the F. Scorina Gomel State University. – 2009. – Vol. 55, № 4. – P. 219–223.

Поступила в редакцию 11.02.15.