МАТЕМАТИКА

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ОБ УПОРЯДОЧЕННЫХ ГРУППОИДАХ АБЕЛЯ-ГРАССМАНА

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ON ORDERED ABEL-GRASSMANN'S GROUPOIDS

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Введено понятие (m,n)-идеалов упорядоченных $\mathcal{AG}$-группоидов и получены характеризации $(0,2)$-идеалов и $(1,2)$-идеалов упорядоченного $\mathcal{AG}$-группоида в терминах левых идеалов. Показано, что упорядоченный $\mathcal{AG}$-группоид S является $0-(0,2)$-бипростым в том и только в том случае, когда S является правым 0-простым. Результаты данной работы позволяют расширить концепцию $\mathcal{AG}$-группоида без введенного порядка. Получены характеризации внутренне-регулярного упорядоченного $\mathcal{AG}$-группоида в терминах левых и правых идеалов.

Ключевые слова: *упорядоченные $\mathcal{AG}$-группоиды, обратимое слева тождество, левая единица, (m,n)-идеал.*

The concept of (m,n)-ideals in ordered $\mathcal{AG}$-groupoids is introduced and the $(0,2)$-ideals and $(1,2)$-ideals of an ordered $\mathcal{AG}$-groupoid in terms of left ideals are characterised. It is shown that an ordered $\mathcal{AG}$-groupoid S is $0-(0,2)$-bisimple if and only if S is right 0-simple. The results of this paper extend the concept of an $\mathcal{AG}$-groupoid without order. Finally, we characterize an intra-regular ordered $\mathcal{AG}$-groupoid in terms of left and right ideals.

Keywords: *ordered $\mathcal{AG}$-groupoids, left invertive law, left identity, (m,n)-ideals.*

Mathematics Subject Classification (2010): 20D10, 20D20

Introduction

The concept of a left almost semigroup (*LA*-semigroup) [3] was first introduced by M.A. Kazim and M. Naseeruddin in 1972. In [1], the same structure is called a left invertive groupoid. P.V. Protić and N. Stevanović called it an Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid) [10].

An $\mathcal{AG}$-groupoid is a groupoid S satisfying the left invertive law $(ab)c=(cb)a$ for all $a,b,c\in S$. This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc=cba$. An $\mathcal{AG}$-groupoid satisfies the medial law $(ab)(cd)=(ac)(bd)$ for all $a,b,c,d\in S$. Since $\mathcal{AG}$-groupoids satisfy medial law, they belong to the class of entropic groupoids which are also called abelian quasigroups [12]. If an $\mathcal{AG}$-groupoid S contains a left identity, then it satisfies the paramedial law $(ab)(cd)=(dc)(ba)$ and the identity $a(bc)=b(ac)$ for all $a,b,c,d\in S$ [5].

An $\mathcal{AG}$-groupoid is a useful algebraic structure, midway between a groupoid and a commutative semigroup. An $\mathcal{AG}$-groupoid is non-associative and non-commutative in general, however, there is a close relationship with semigroup as well as with commutative structures. It has been investigated in [5] that if an $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative semigroup. The connection of a commutative inverse semigroup with an $\mathcal{AG}$-groupoid has been given by Yousafzai et al. in [14] as, a commutative inverse semigroup $(S,\ .)$ becomes an $\mathcal{AG}$-groupoid $(S,\ *)$ under $a*b=ba^{-1}r^{-1}$ for all $a,b,r\in S$. The $\mathcal{AG}$-groupoid S with left identity becomes a semigroup under the binary operation "$\circ_e$" defined as, $x\circ_e y=(xe)y$ for all x, $y\in S$ [15]. The $\mathcal{AG}$-groupoid is the generalization of a semigroup theory [5] and has vast applications in collaboration with semigroups like other branches of mathematics. Many interesting results on $\mathcal{AG}$-groupoids have been investigated in [7], [8], [9].

If S is an $\mathcal{AG}$-groupoid with product $\cdot: S\times S\to S$, then $ab\cdot c$ and $(ab)c$ both denote the product $(a\cdot b)\cdot c$.

Definition **0.1** [16]. *An $\mathcal{AG}$-groupoid $(S,\cdot)$ together with a partial order $\leq$ on S that is compatible with an $\mathcal{AG}$-groupoid operation, meaning that for $x,y,z\in S$,*

$$x\leq y\Rightarrow zx\leq zy \text{ and } xz\leq yz,$$

is called an ordered $\mathcal{AG}$-groupoid.

Let $(S,\cdot,\leq)$ be an ordered $\mathcal{AG}$-groupoid. If A and B are nonempty subsets of S, we let

$$AB=\{xy\in S\mid x\in A,y\in B\},$$

and $(A]=\{x\in S\mid x\leq a$ for some $a\in A\}$.

***Definition* 0.2** [16]. *Let $(S,\cdot,\leq)$ be an ordered $\mathcal{AG}$-groupoid. A nonempty subset A of S is called a left (resp. right) ideal of S if the followings hold:*

(i) $SA\subseteq A$ *(resp. $AS\subseteq A$);*

(ii) *for $x\in A$ and $y\in S$, $y\leq x$ implies $y\in A$.*

Equivalently $(SA]\subseteq A$ (resp. $(AS]\subseteq A$).

If A is both a left and a right ideal of S, then A is called a two-sided ideal or an ideal of S.

A nonempty subset A of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ is called $\mathcal{AG}$-subgroupoid of S if $xy\in A$ for all $x,y\in A$.

It is clear to see that every left and right ideals of an ordered $\mathcal{AG}$-groupoid is an $\mathcal{AG}$-subgroupoid.

Let $(S,\cdot,\leq)$ be an ordered $\mathcal{AG}$-groupoid and let A and B be nonempty subsets of S, then the following was proved in [13]:

(i) $A\subseteq(A]$;

(ii) If $A\subseteq B$, then $(A]\subseteq(B]$;

(iii) $(A](B]\subseteq(AB]$;

(iv) $(A]=((A]]$;

(vi) $((A](B]]=(AB]$.

Also for every left (resp. right) ideal T of S, $(T]=T$.

The concept of (m,n)-ideals in ordered semigroups were given by J. Sanborisoot and T. Changphas in [11]. It's natural to ask whether the concept of (m,n)-ideals in ordered $\mathcal{AG}$-groupoids is valid or not? The aim of this paper is to deal with (m,n)-ideals in ordered $\mathcal{AG}$-groupoids. We introduce the concept of (m,n)-ideals in ordered $\mathcal{AG}$-groupoids as follows:

***Definition* 0.3.** *Let $(S,\cdot,\leq)$ be an ordered $\mathcal{AG}$-groupoid and let m,n be non-negative integers. An $\mathcal{AG}$-subgroupoid A of S is called an (m,n)-ideal of S if the followings hold:*

(i) $A^mS\cdot A^n\subseteq A$;

(ii) *for $x\in A$ and $y\in S$, $y\leq x$ implies $y\in A$.*

Here, A^0 is defined as $A^0S\cdot A^n=SA^n$ and $A^mS\cdot A^0=A^mS$.

Equivalently an $\mathcal{AG}$-subgroupoid A of S is called an (m,n)-ideal of S if

$$(A^mS\cdot A^n]\subseteq A.$$

If $m=n=1$, then an (m,n)-ideal A of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ is called a bi-ideal of S.

1 0-*minimal* $(0,2)$-*bi-ideals in ordered* $\mathcal{AG}$-*groupoid*

In this section, we study and generalize the work of W. Jantanan and T. Changphas [2] by converting it from an associative ordered structure in to a non-associative ordered structure. We use the concept of (m,n)-ideals and investigate $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals in ordered $\mathcal{AG}$-groupoids. All the results of this section can be obtain for an $\mathcal{AG}$-groupoid without order.

***Defintion* 1.1.** *If there is an element 0 of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ such that $x\cdot 0=0\cdot x=x$ for all $x\in S$, we call 0 a zero element of S.*

***Example* 1.1.** Let $S=\{a,b,c,d,e\}$ with a left identity d. Then the following multiplication table and order shows that $(S,\cdot,\leq)$ is a unitary ordered $\mathcal{AG}$-groupoid with a zero element a.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

$$\leq:=\{(a,a),(a,b),(c,c),(a,c),(d,d),(a,e),(e,e),(b,b)\}.$$

If S is a unitary ordered $\mathcal{AG}$-groupoid, then it is easy to see that $(S^2]=S$, $(SA^2]=(A^2S]$ and $A\subseteq(SA]$ $\forall A\subseteq S$. Note that every right ideal of a unitary ordered $\mathcal{AG}$-groupoid S is a left ideal of S but the converse is not true in general. Example 1.1 shows that there exists a subset $\{a,b,e\}$ of S which is a left ideal of S but not a right ideal of S. It is easy to see that $(SA]$ and $(SA^2]$ are the left and right ideals of a unitary ordered $\mathcal{AG}$-groupoid S. Thus $(SA^2]$ is an ideal of a unitary ordered $\mathcal{AG}$-groupoid S.

We characterize of $(0,2)$-ideals of an ordered $\mathcal{AG}$-groupoid in terms of left ideals as follows:

***Lemma* 1.1.** *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then A is a $(0,2)$-ideal of S if and only if A is an ideal of some left ideal of S.*

Proof. Let A be a $(0,2)$-ideal of S, then

$$((SA]\cdot A]=(SA\cdot A]=(AA\cdot S]=(SA^2]\subseteq A,$$

and

$$(A\cdot(SA]]=(A\cdot SA]=(S\cdot AA]=(SA^2]\subseteq A.$$

Hence A is an ideal of a left ideal $(SA]$ of S.

Conversely, assume that A is a left ideal of some left ideal L of S, then

$$(SA^2]=(AA\cdot S]=(SA\cdot A]\subseteq$$
$$\subseteq(SL\cdot A]\subseteq((SL]\cdot A]\subseteq(LA]\subseteq A,$$

and clearly A is an $\mathcal{AG}$-subgroupoid of S, therefore A is a (0,2)-ideal of S.

***Corollary* 1.1.** *Let* $(S,\cdot,\leq)$ *be a unitary ordered* $\mathcal{AG}$*-groupoid. Then* A *is a* $(0,2)$*-ideal of* S *if and only if A is a left ideal of some left ideal of S.*

Now we characterize the $(0,2)$-bi-ideals of an ordered $\mathcal{AG}$-groupoid in terms of right ideals as follows:

***Lemma* 1.2.** *Let* $(S,\cdot,\leq)$ *be a unitary ordered* $\mathcal{AG}$*-groupoid. Then* A *is a* $(0,2)$*-bi-ideal of* S *if and only if* A *is an ideal of some right ideal of* S.

Proof. Let A be a $(0,2)$-bi-ideal of S, then

$$((SA^2]\cdot A] = (SA^2\cdot A] = (A^2S\cdot A] = \\ = (AS\cdot A^2] \subseteq (SA^2] \subseteq A,$$

and

$$(A\cdot(SA^2]] = (A\cdot SA^2] = \\ = (A\cdot(S^2]A^2] \subseteq ((A]\cdot(S^2](A^2]] \subseteq ((A\cdot S^2A^2]] = \\ = (A\cdot S^2A^2] = (SS\cdot AA^2] = \\ = (A^2A\cdot SS] = (SA\cdot A^2] \subseteq (SA^2] \subseteq A.$$

Hence A is an ideal of some right ideal $(SA^2]$ of S.

Conversely, assume that A is an ideal of some right ideal R of S, then

$$(SA^2] = (A\cdot SA] \subseteq ((A]\cdot(S^2](A]] \subseteq \\ \subseteq ((A\cdot S^2A]] = (A\cdot S^2A] = \\ = (A\cdot(AS)S] \subseteq (A\cdot(RS)R] \subseteq (A\cdot((RS])R] \\ \subseteq (A\cdot(RS]] \subseteq (AR] \subseteq A,$$

and $(AS\cdot A] \subseteq ((RS]\cdot A] \subseteq (RA] \subseteq A$, which shows that A is a $(0,2)$-ideal of S.

The following result gives some characterizations of $(1,2)$-ideals of an ordered $\mathcal{AG}$-groupoid.

***Theorem* 1.1.** *Let* $(S,\cdot,\leq)$ *be a unitary ordered* $\mathcal{AG}$*-groupoid. Then the following statements are equivalent.*

(i) *A is a* $(1,2)$*-ideal of* S;

(ii) *A is a left ideal of some bi-ideal of* S;

(iii) *A is a bi-ideal of some ideal of* S;

(iv) *A is a* $(0,2)$*-ideal of some right ideal of S*;

(v) *A is a left ideal of some* $(0,2)$*-ideal of* S.

Proof. $(i)\Rightarrow(ii)$: It is easy to see that $(SA^2\cdot S]$ is a bi-ideal of S. Let A be a $(1,2)$-ideal of S, then

$$(((SA^2\cdot S])A] \subseteq ((SA^2\cdot SS)A] = \\ = ((SS\cdot A^2S)A] \subseteq (((S^2]\cdot A^2S)A] = \\ = ((S\cdot A^2S)A] = ((A^2\cdot SS)A] \subseteq (A^2S\cdot A] = \\ = (AS\cdot A^2] \subseteq A,$$

which shows that A is a left ideal of some bi-ideal $(SA^2\cdot S]$ of S.

$(ii)\Rightarrow(iii)$: Let A be a left ideal of some bi-ideal B of S and e be a left identity of S, then

$$((A\cdot(SA^2])A] \subseteq ((A\cdot SA^2)A] = ((S\cdot AA^2)A] = \\ = e((S\cdot AA^2)A] \subseteq (S]((S\cdot AA^2)A] \subseteq \\ \subseteq ((S(SA\cdot AA))A] = \\ = ((S(AA\cdot AS))A] = ((AA\cdot S(AS))A] = \\ = (((S(AS)\cdot A)A)A] = (((A(SS)\cdot A)A)A] \subseteq \\ \subseteq (((AS\cdot A)A)A] \subseteq (((BS\cdot B)A)A] \subseteq \\ \subseteq (BA\cdot A] \subseteq A,$$

which shows that A is a bi-ideal of an ideal $(SA^2]$ of S.

$(iii)\Rightarrow(iv)$: Let A be a bi-ideal of some ideal I of S, then

$$((SA^2]\cdot A^2] = (SA^2\cdot A^2] = ((A^2\cdot AA)S] = \\ = ((A\cdot A^2A)S] \subseteq ((A\cdot((AI)A])S] \subseteq (AA\cdot S] = \\ = (SA\cdot A] \subseteq ((SI]\cdot S] \subseteq I,$$

which shows that A is a $(0,2)$-ideal of a right ideal $(SA^2]$ of S.

$(iv)\Rightarrow(v)$: It is easy to see that $(SA^3]$ is a $(0,2)$-ideal of S. Let A be a $(0,2)$-ideal of a right ideal R of S, then

$$(A\cdot(SA^3]] \subseteq (A(SS\cdot A^2A)] \subseteq \\ \subseteq (A(AA^2\cdot S)] \subseteq (A((SA\cdot AA)S)] \\ = (A((AA\cdot AS)S)] = ((AA)((A\cdot AS)S)] \\ = ((S\cdot A(AS))A^2] = ((A\cdot S(AS))A^2] \\ \subseteq ((RS]\cdot A^2] \subseteq (RA^2] \subseteq A,$$

which shows that A is a left ideal of a $(0,2)$-ideal $(SA^3]$ of S.

$(v)\Rightarrow(i)$: Let A be a left ideal of a $(0,2)$-ideal O of S, then

$$(AS\cdot A^2] \subseteq ((AA\cdot SS)A] \subseteq (SA^2\cdot A] \subseteq \\ \subseteq ((SO^2]\cdot A] \subseteq (OA] \subseteq A,$$

which shows that A is a $(1,2)$-ideal of S.

The following characterizes $(1,2)$-ideals in terms of left and right ideals of an ordered $\mathcal{AG}$-groupoid.

***Lemma* 1.3.** *Let* $(S,\cdot,\leq)$ *be a unitary ordered* $\mathcal{AG}$*-groupoid and* A *be an idempotent subset of* S. *Then* A *is a* $(1,2)$*-ideal of* S *if and only if there exist a left ideal* L *and a right ideal* R *of* S *such that* $(RL] \subseteq A \subseteq R\cap L$.

Proof. Assume that A is a $(1,2)$-ideal of S such that A is idempotent.

Setting L=(SA] and R=(SA ^{2}], then

$$(RL] = ((SA^2]\cdot(SA]] \subseteq (A^2S\cdot SA] \subseteq (A^2S^2\cdot SA] = \\ = ((SA\cdot SS)A^2] = \\ = ((SS\cdot AS)A^2] \subseteq ((S(AA\cdot SS))A^2] = \\ = ((S(SS\cdot AA))A^2] = \\ = ((S(A(SS\cdot A)))A^2] \subseteq ((A(S\cdot SA))A^2] \subseteq \\ \subseteq (AS\cdot A^2] \subseteq A.$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right ideal and L be a left ideal of S such that $(RL] \subseteq A \subseteq R \cap L$, then

$$(AS \cdot A^2] = (AS \cdot AA] \subseteq ((RS] \cdot (SL]] \subseteq (RL] \subseteq A.$$

***Definition* 1.2.** *A (0,2)-ideal A of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ with zero is said to be 0-minimal if $A \neq \{0\}$ and $\{0\}$ is the only $(0,2)$-ideal of S properly contained in A.*

***Remark* 1.1.** *Assume that $(S, \cdot, \leq)$ is a unitary ordered $\mathcal{AG}$-groupoid with zero. Then it is easy to see that every left (right) ideal of S is a $(0,2)$-ideal of S. Hence if O is a 0-minimal $(0,2)$-ideal of S and A is a left (right) ideal of S contained in O, then either $A = \{0\}$ or $A = O$.*

***Lemma* 1.4.** *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid with zero. Assume that A is a 0-minimal ideal of S and O is an $\mathcal{AG}$-subgroupoid of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Proof. Let O be a $(0,2)$-ideal of S contained in a 0-minimal ideal A of S. Then $(SO^2] \subseteq O \subseteq A$. Since $(SO^2]$ is an ideal of S, therefore by minimality of A, $(SO^2] = \{0\}$ or $(SO^2] = A$. If $(SO^2] = A$, then $A = (SO^2] \subseteq O$ and therefore $O = A$. Let $(SO^2] = \{0\}$, then

$$(O^2S] \subseteq (O^2S^2] = (S^2O^2] \subseteq (SO^2] = \{0\} \subseteq O^2,$$

which shows that O^2 is a right ideal of S, and hence an ideal of S contained in A, therefore by minimality of A, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$, then $O = A$.

Conversely, let $O^2 = \{0\}$, then

$$(SO^2] \subseteq (O^2S] = (\{0\}S] = \{0\} = (O].$$

Now if $O = A$, then

$$(SO^2] \subseteq (SS \cdot OO] \subseteq ((SA] \cdot (SA]] \subseteq A = O,$$

which shows that O is a $(0,2)$-ideal of S contained in A.

***Corollary* 1.2.** *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid with zero. Assume that A is a 0-minimal left ideal of S and O is an $\mathcal{AG}$-subgroupoid of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

***Lemma* 1.5.** *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid with zero and O be a 0-minimal $(0,2)$-ideal of S. Then $O^2 = \{0\}$ or O is a 0-minimal right (left) ideal of S.*

Proof. Let O be a 0-minimal $(0,2)$-ideal of S, then

$$(S(O^2)^2] \subseteq (SS \cdot O^2O^2] \subseteq (O^2O^2 \cdot S] = (SO^2 \cdot O^2]$$
$$\subseteq ((SO^2] \cdot O^2] \subseteq (OO^2] \subseteq O^2,$$

which shows that O^2 is a $(0,2)$-ideal of S contained in O, therefore by minimality of O, $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then

$$(OS] \subseteq (OO \cdot SS] \subseteq (SO^2] \subseteq O,$$

which shows that O is a right ideal of S. Let R be a right ideal of S contained in O, then

$$(R^2S] = (RR \cdot S] \subseteq ((RS] \cdot S] \subseteq R.$$

Thus R is a $(0,2)$-ideal of S contained in O, and again by minimality of O, $R = \{0\}$ or $R = O$.

The following Corollary follows from Lemma 1.2 and Corollary 1.2.

***Corollary* 1.3.** *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then O is a minimal $(0,2)$-ideal of S if and only if O is a minimal left ideal of S.*

***Theorem* 1.2.** *Let $(S, \cdot, \leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then A is a minimal $(2,1)$-ideal of S if and only if A is a minimal bi-ideal of S.*

Proof. Let A be a minimal $(2,1)$-ideal of S. Then

$$((((A^2S \cdot A])^2 S)((A^2S \cdot A]))] \subseteq$$
$$\subseteq (((A^2S \cdot A)^2 S)(A^2S \cdot A)] =$$
$$= ((((A^2S \cdot A)(A^2S \cdot A))S)(A^2S \cdot A)] \subseteq$$
$$\subseteq ((((AS \cdot A)(AS \cdot A))S)(AS \cdot A)] =$$
$$= ((((AS \cdot AS)(AA))S)(AS \cdot A)] \subseteq$$
$$\subseteq (((A^2S \cdot AA)S)(AS \cdot A)] \subseteq$$
$$\subseteq (((AS \cdot AS)S)(AS \cdot A)] \subseteq$$
$$\subseteq ((A^2S \cdot S)(AS \cdot A)] \subseteq$$
$$\subseteq ((AS \cdot S)(AS \cdot A)] = ((AS \cdot AS)(SA)] \subseteq$$
$$\subseteq (A^2S \cdot SA] = (AS \cdot SA^2] = ((SA^2 \cdot S)A]$$
$$\subseteq ((A^2S \cdot S)A] = ((SS \cdot AA)A] = (A^2S \cdot A],$$

and similarly we can show that $(A^2S \cdot A]^2 \subseteq \subseteq (A^2S \cdot A]$. Thus $(A^2S \cdot A]$ is a $(2,1)$-ideal of S contained in A, therefore by minimality of A, $(A^2S \cdot A] = A$. Now

$$(AS \cdot A] = ((AS)(A^2S \cdot A)] =$$
$$= (((A^2S \cdot A)S)A] = ((SA \cdot A^2S)A] =$$
$$= ((A^2(SA \cdot S))A] \subseteq (A^2S \cdot A] = A,$$

It follows that A is a bi-ideal of S. Suppose that there exists a bi-ideal B of S contained in A, then $(B^2S \cdot B] \subseteq (BS \cdot B] \subseteq B$, so B is a $(2,1)$-ideal of S contained in A, therefore $B = A$.

Conversely, assume that A is a minimal bi-ideal of S, then it is easy to see that A is a $(2,1)$-ideal of S. Let C be a $(2,1)$-ideal of S contained in A, then

$$((((C^2S\cdot C])S)((C^2S\cdot C])]\subseteq$$
$$\subseteq (((C^2S\cdot C)S)(C^2S\cdot C)]=$$
$$=((SC\cdot C^2S)(C^2S\cdot C)]=$$
$$=((SC^2\cdot CS)(C^2S\cdot C)]=$$
$$=((C(SC^2\cdot S))(C^2S\cdot C)]=$$
$$=(((C^2S\cdot C)(SC^2\cdot SS))C]\subseteq$$
$$\subseteq (((C^2S\cdot C)(S\cdot C^2S))C]\subseteq$$
$$\subseteq (((C^2S\cdot C)(C^2S))C]\subseteq$$
$$=((C^2((C^2S\cdot C)S))C]\subseteq (C^2S\cdot C].$$

This shows that $(C^2S\cdot C]$ is a bi-ideal of S, and by minimality of A, $(C^2S\cdot C]=A.$ Thus

$$A=(C^2S\cdot C]\subseteq C,$$

and therefore A is a minimal $(2,1)$-ideal of S.

***Theorem* 1.3.** *Let A be 0-minimal $(0,2)$-bi-ideal of a unitary ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ with zero. Then exactly one of the following cases occurs:*

(i) $A=(\{0,a\}]$, $a^2=0$;

(ii) *for all* $a\in A\setminus\{0\}$, $(Sa^2]=A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-ideal of S. Let $a\in A\setminus\{0\}$, then $(Sa^2]\subseteq A.$ Also $(Sa^2]$ is a $(0,2)$-bi-ideal of S, therefore $(Sa^2]=\{0\}$ or $(Sa^2]=A.$

Let $(Sa^2]=\{0\}$. Since $a^2\in A,$ we have either $a^2=a$ or $a^2=0$ or $a^2\in A\setminus\{0,a\}$. If $a^2=a$, then $a^3=a^2a=a,$ which is impossible because $a^3\in(a^2S]\subseteq(Sa^2]=\{0\}$. Let $a^2\in A\setminus\{0,a\}$, we have

$$(S\cdot(\{0,a^2\}\{0,a^2\}]]\subseteq(SS\cdot a^2a^2]=$$
$$=(Sa^2\cdot Sa^2]=\{0\}\subseteq(\{0,a^2\}],$$

and

$$(((\{0,a^2\}]S)(\{0,a^2\}]]\subseteq(\{0,a^2S\}\{0,a^2\}]=$$
$$=(a^2S\cdot a^2]\subseteq(Sa^2]=\{0\}\subseteq(\{0,a^2\}].$$

Therefore $(\{0,a^2\}]$ is a $(0,2)$-bi-ideal of S contained in A. We observe that $(\{0,a^2\}]\neq\{0\}$ and $(\{0,a^2\}]\neq A.$ This is a contradiction to the fact that A is a 0-minimal $(0,2)$-bi-ideal of S. Therefore $a^2=0$ and $A=(\{0,a\}]$. If $(Sa^2]\neq\{0\}$, then $(Sa^2]=A.$

***Corollary* 1.4.** *Let A be 0-minimal $(0,2)$-bi-ideal of a unitary ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ with zero such that $(A^2]\neq 0$. Then $A=(Sa^2]$ for every $a\in A\setminus\{0\}$.*

***Lemma* 1.6.** *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid. Then every right ideal of S is a $(0,2)$-bi-ideal of S.*

Proof. Assume that A is a right ideal of S, then

$$(SA^2]\subseteq(AA\cdot SS]\subseteq((AS]\cdot(AS]]\subseteq$$
$$\subseteq(AA]\subseteq(AS]\subseteq A,\ (AS\cdot A]\subseteq A,$$

and clearly $A^2\subseteq A,$ therefore A is a (0,2)-bi-ideal of S.

The converse of Lemma 1.2 is not true in general. Example 2.1 shows that there exists a $(0,2)$-bi-ideal $A=\{a,c,e\}$ of S which is not a right ideal of S.

***Definition* 1.3.** *An ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ with zero is said to be 0-$(0,2)$-bisimple if $(S^2]\neq\{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-ideal of S.*

***Theorem* 1.4.** *Let $(S,\cdot,\leq)$ be a unitary ordered $\mathcal{AG}$-groupoid with zero. Then $(Sa^2]=S$ for all $a\in S\setminus\{0\}$ if and only if S is 0-$(0,2)$-bisimple if and only if S is right 0-simple.*

Proof. Assume that $(Sa^2]=S$ for every $a\in S\setminus\{0\}$. Let A be a $(0,2)$-bi-ideal of S such that $A\neq\{0\}$. Let $a\in A\setminus\{0\}$, then

$$S=(Sa^2]\subseteq(SA^2]\subseteq A.$$

Therefore $S=A.$ Since $S=(Sa^2]\subseteq(S^2]$, we have $(S^2]=S\neq\{0\}$. Thus S is 0-$(0,2)$-bisimple. The converse statement follows from Corollary 1.2.

Let R be a right ideal of 0-$(0,2)$-bisimple S. Then by Lemma 1.2, R is a $(0,2)$-bi-ideal of S and so $R=\{0\}$ or $R=S$. Conversely, assume that S is right 0-simple. Let $a\in S\setminus\{0\}$, then $(Sa^2]=S$. Hence S is 0-$(0,2)$-bisimple.

***Theorem* 1.5.** *Let A be a 0-minimal $(0,2)$-bi-ideal of a unitary ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ with zero. Then either $(A^2]=\{0\}$ or A is right 0-simple.*

Proof. Assume that A is 0-minimal $(0,2)$-bi-ideal of S such that $(A^2]\neq\{0\}$. Then by using Corollary 1.2, $(Sa^2]=A$ for every $a\in A\setminus\{0\}$. Since $a^2\in A\setminus\{0\}$ for every $a\in A\setminus\{0\}$, we have $a^4=(a^2)^2\in A\setminus\{0\}$ for every $a\in A\setminus\{0\}$. Let $a\in A\setminus\{0\}$, then

$$((Aa^2]S\cdot(Aa^2]]=(a^2A\cdot S(Aa^2)]=$$
$$=(((S\cdot Aa^2)A)a^2]\subseteq(((S\cdot A)A)a^2]$$
$$\subseteq((AA\cdot SS)a^2]\subseteq((SA^2]\cdot a^2]\subseteq(Aa^2],$$

and

$$(S(Aa^2]^2]=(S((Aa^2]\cdot(Aa^2])]=$$
$$=(S((a^2A]\cdot(a^2A])]=(S(a^2(a^2A\cdot A))]=$$
$$=((aa)(S(a^2A\cdot A))]=(((a^2A\cdot A)S)a^2]\subseteq$$
$$\subseteq((AA\cdot SS)a^2]\subseteq((SA^2]\cdot a^2]\subseteq(Aa^2],$$

which shows that $(Aa^2]$ is a $(0,2)$-bi-ideal of S contained in A. Hence $(Aa^2] = \{0\}$ or $(Aa^2] = A$. Since $a^4 \in (Aa^2]$ and $a^4 \in A \setminus \{0\}$, we get $(Aa^2] = A$. Thus by using Theorem 1.2, A is right 0-simple.

2 Ideals in intra-regular ordered $\mathcal{AG}$-groupoid

Ideal theory plays a very important role in studying and exploring the structural properties of different algebraic structures. Here we study left (right) ideals which usually allow us to characterize an ordered $\mathcal{AG}$-groupoid and play the role in an ordered $\mathcal{AG}$-groupoid which is played by normal subgroups in ordered group theory and by ideals in ordered ring theory.

***Definition* 2.1.** *An element a of an ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ is called an intra-regular element of S if there exist some $x, y \in S$ such that $a \leq xa^2 \cdot y$ and S is called intra-regular if every element of S is intra-regular or equivalently, $A \subseteq (SA^2 \cdot S]$ for all $A \subseteq S$ and $a \in (Sa^2 \cdot S]$ for all $a \in S$.*

***Example* 2.1.** Let $S = \{a,b,c,d,e\}$ be an ordered $\mathcal{AG}$-groupoid with the following multiplication table and order below.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

$\leq := \{(a,a),(a,b),(c,c),(d,d),(e,e),(b,b)\}$.

By routine calculation, it is easy to verify that S is intra-regular.

***Definition* 2.2.** *An ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$ is called left (resp. right) simple if it has no proper left (resp. right) ideal and is called simple if it has no proper ideal.*

***Theorem* 2.1.** *The following conditions are equivalent for a unitary ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$:*

(i) $(aS] = S$, for some $a \in S$;

(ii) $(Sa] = S$, for some $a \in S$;

(iii) S is simple;

(iv) $(AS] = S = (SA]$, where A is any two-sided ideal of S;

(v) S is intra-regular.

Proof. $(i) \Rightarrow (ii)$: Let S be a unitary ordered $\mathcal{AG}$-groupoid and assume that $(aS] = S$ holds for some $a \in S$. Since $(aS]$ and $(Sa]$ are the left ideals of S, then $(aS] = aS$ and $(Sa] = Sa$. Therefore

$$S = (SS] = ((aS]\cdot S] = (aS \cdot S] = (SS \cdot a] = (Sa].$$

$(ii) \Rightarrow (iii)$: Let S be a unitary ordered $\mathcal{AG}$-groupoid such that $(aS] = S$ holds for some $a \in S$. Suppose that S is not left simple and let L be a proper left ideal of S, then

$$(SL] \subseteq L \subseteq S =$$
$$= (SS] \subseteq (Sa \cdot S] \subseteq ((SS \cdot ea)S] =$$
$$= ((ae \cdot SS)S] \subseteq ((ae \cdot S)(SS)] =$$
$$= ((Se \cdot a)(SS)] = ((SS)(a \cdot Se)] =$$
$$= (a(SS \cdot Se)] \subseteq (aS],$$

implies that $sl \leq at$ for some $a, s, t \in S$ and $l \in L$. Since $sl \in L$, therefore $at \in L$, but $at \in (aS]$. Thus $(aS] \subseteq L$ and therefore we have $S = (aS] \subseteq L$, which implies that $S = L$, which contradicts the given assumption. Thus S is left simple and similarly we can show that S is right simple, which shows that S is simple.

$(iii) \Rightarrow (iv)$: Let S be a simple unitary ordered $\mathcal{AG}$-groupoid and let A be any two-sided ideal of S, then $A = S$. Therefore, we have $(AS] = (SS] = (SA]$.

$(iv) \Rightarrow (v)$: Let S be a unitary ordered $\mathcal{AG}$-groupoid such that $(AS] = S = (SA]$ holds for any two-sided ideal A of S. Since $(a^2S]$ is two-sided ideal of S such that $(a^2S \cdot S] = S = (S \cdot a^2S]$. Let $a \in S$, then

$$a \in S = (a^2S \cdot S] \subseteq ((aa \cdot SS)S] =$$
$$= ((SS \cdot aa)S] \subseteq (Sa^2 \cdot S],$$

that is $a \leq (xa^2)y$ for some $x, y \in S$. Thus S is intra-regular

$(v) \Rightarrow (i)$: Let S be a unitary intra-regular ordered $\mathcal{AG}$-groupoid. Let $a \in S$, then there exist $x, y \in S$ such that $a \leq (xa^2)y$. Thus

$$a \leq (xa^2)y = (ex \cdot aa)y = (aa \cdot ex)y$$
$$= (y \cdot ex)(aa) = a((y \cdot ex)a) \in aS,$$

which shows that $S \subseteq (Sa]$ and $(Sa] \subseteq S$ is obvious. Thus $(Sa] = S$ holds for some $a \in S$.

***Corollary* 2.1.** *The following conditions are equivalent for any unitary ordered $\mathcal{AG}$-groupoid $(S,\cdot,\leq)$:*

(i) $(aS] = S$, for some $a \in S$;

(ii) $(Sa] = S$, for some $a \in S$;

(iii) S is right simple;

(iv) $(AS] = S = (SA]$, where A is any right ideal of S;

(v) S is fully regular.

***Corollary* 2.2.** *If $(S,\cdot,\leq)$ is a unitary ordered $\mathcal{AG}$-groupoid, then the following conditions are equivalent:*

(i) $(Sa] = S$, *for some* $a \in S$;

(ii) $(aS] = S$, *for some* $a \in S$.

***Corollary* 2.3.** *If* $(S,\cdot,\leq)$ *is a unitary ordered* $\mathcal{AG}$*-groupoid, then* $(eS] = S = (Se]$ *holds for* $e \in S$, *where* e *is a left identity of* S.

***Corollary* 2.4.** *The following conditions are equivalent for any unitary ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$:

(i) *S is intra-regular;*

(ii) $(Sa] = S = (aS]$ *for some* $a \in S$.

***Definition* 2.3.** *A left (resp. right) ideal* A *of an ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$ *is called semiprime if* $a \in A$ *implies* $a^2 \in A$.

***Lemma* 2.1.** *The following conditions are equivalent for a unitary ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$:

(i) *S is intra-regular*;

(ii) *Every right ideal of S is semiprime.*

Proof. $(i) \Rightarrow (ii)$: Let T be a right ideal of a unitary intra-regular ordered $\mathcal{AG}$-groupoid S. For $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2 \cdot y$. Let $a^2 \in T$, then

$$a \leq (ex \cdot a^2)y = (a^2 \cdot xe)y = (y \cdot xe)a^2 =$$
$$= a^2(xe \cdot y) \in TS \subseteq (TS] \subseteq T,$$

which implies that T is semiprime.

Now $(ii) \Rightarrow (i)$: Since $(a^2S]$ is a right ideal of a unitary ordered $\mathcal{AG}$-groupoid S containing a^2 so $a \in (a^2S]$. Thus

$$a \in (a^2S] \subseteq (a^2 \cdot SS] = (S \cdot a^2S] \subseteq (SS \cdot a^2S] =$$
$$= (Sa^2 \cdot SS] \subseteq (Sa^2 \cdot S].$$

Hence S is intra-regular.

***Corollary* 2.5.** *The following conditions are equivalent for any unitary ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$:

(i) *S is intra-regular;*

(ii) *every ideal of S is semiprime.*

***Theorem* 2.2.** *The following conditions are equivalent for a unitary ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$:

(i) *S is intra-regular*;

(ii) $L \cap R \subseteq (LR]$ *for every semiprime right ideal R and every left ideal L of S*;

(iii) $L \cap R \subseteq (LR \cdot L]$ *for every semiprime right ideal R and every left ideal L of S*.

Proof. $(i) \Rightarrow (iii)$: Let S be a unitary intra-regular ordered $\mathcal{AG}$-groupoid and L, R be any left and right ideals of S respectively such that $k \in L \cap R$. Then there exist $x, y \in S$ such that $k \leq xk^2 \cdot y$. Thus

$$k \leq (x \cdot kk)y = (k \cdot xk)y =$$
$$= (y \cdot xk)k \leq \left(y\left(x\left(xk^2 \cdot y\right)\right)\right)k =$$
$$= \left(y\left(xk^2 \cdot xy\right)\right)k = \left(xk^2 \cdot y(xy)\right)k =$$
$$= \left(x(kk) \cdot y(xy)\right)k =$$
$$= \left(k(xk) \cdot y(xy)\right)k \in ((R \cdot SL)S)L \subseteq (RL \cdot S)L =$$
$$= LS \cdot RL = LR \cdot SL \subseteq LR \cdot L,$$

which implies that $L \cap R \subseteq (LR \cdot L]$. Also by Lemma 1.3, R is semiprime.

$(iii) \Rightarrow (ii)$: Let R and L be the left and right ideals of S respectively and R be semiprime, then

$$L \cap R = R \cap L \subseteq (RL \cdot R] \subseteq$$
$$\subseteq (RL \cdot S] \subseteq (RL \cdot SS] = (SS \cdot LR]$$
$$= (L(SS \cdot R)] = (L(RS \cdot S)] \subseteq (L \cdot (RS]] \subseteq (LR].$$

$(ii) \Rightarrow (i)$: Since $a \in (Sa]$, which is a left ideal of S, and $a^2 \in (a^2S]$, which is a semiprime right ideal of S, therefore by given assumption $a \in (a^2S]$. Thus

$$a \in (Sa] \cap (a^2S] \subseteq ((Sa] \cdot (a^2S]] \subseteq (Sa \cdot a^2S] \subseteq$$
$$\subseteq (SS \cdot a^2S] = (Sa^2 \cdot SS] \subseteq (Sa^2 \cdot S].$$

Hence S is intra-regular.

***Lemma* 2.2.** *The following conditions are equivalent for a unitary ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$:

(i) *S is intra-regular*;

(ii) *every left ideal of S is idempotent.*

Proof. It is simple. We omit the proof.

***Theorem* 2.3.** *The following conditions are equivalent for a unitary ordered* $\mathcal{AG}$*-groupoid* $(S,\cdot,\leq)$:

(i) *S is intra-regular*;

(ii) $A = ((SA)^2]$, *where A is any left ideal of S.*

Proof. $(i) \Rightarrow (ii)$: Let A be a left ideal of a unitary intra-regular ordered $\mathcal{AG}$-groupoid, then $(SA] \subseteq A$ and by Lemma 1.3, $((SA)^2] = (SA] \subseteq A$. Now $A = (AA] \subseteq (SA] = ((SA)^2]$, which implies that $A = ((SA)^2]$.

$(ii) \Rightarrow (i)$: Let A be a left ideal of S, then $A = ((SA)^2] \subseteq (A^2]$, which implies that A is idempotent and by using Lemma 1.3, S is intra-regular.

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