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РАЗРЕШИМЫЕ ФОРМАЦИИ С УСЛОВИЕМ ШЕМЕТКОВА

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SOLUBLE FORMATIONS WITH THE SHEMETKOV PROPERTY

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Описаны все насыщенные разрешимые формации $\mathfrak{F}$, у которых все минимальные не $\mathfrak{F}$-группы разрешимы. Всякой локальной формации $\mathfrak{F} = LF(f)$ такой, что $f(p) = \mathfrak{S}_{\pi(f(p))}$ для всех $p \in \pi(\mathfrak{F})$ и $f(p) = \varnothing$ в противном случае, был поставлен в соответствие ориентированный граф $\Gamma(\mathfrak{F}, f)$ без петель, вершинами которого являются простые числа из $\pi(\mathfrak{F})$, и (p_i, p_j) – ребро $\Gamma(\mathfrak{F}, f)$ тогда и только тогда, когда $p_j \in \pi(f(p_i))$. С помощью графов такого типа были описаны все наследственные разрешимые формации с условием Шеметкова.

Ключевые слова: *минимальная простая группа, минимальная не $\mathfrak{F}$-группа, наследственная локальная формация, формация с условием Шеметкова, связанный с формацией граф.*

All saturated soluble formations whose all *s*-critical groups are soluble were described. With every local formation $\mathfrak{F} = LF(f)$, such that $f(p) = \mathfrak{S}_{\pi(f(p))}$ for all $p \in \pi(\mathfrak{F})$ and $f(p) = \varnothing$ otherwise, was associated directed graph $\Gamma(\mathfrak{F}, f)$ without loops whose vertices are prime numbers from $\pi(\mathfrak{F})$ and (p_i, p_j) is an edge of $\Gamma(\mathfrak{F}, f)$ if and only if $p_j \in \pi(f(p_i))$. With the help of such kind's graphs all hereditary soluble formations with the Shemetkov property were described.

Keywords: *minimal simple group, s-critical group, hereditary local formation, formation with the Shemetkov property, graph associated with formation.*

Introduction

All considered groups are finite. Recall that $\mathfrak{S}$ ($\mathfrak{S}_\pi$) is the class of all soluble groups (π-groups). Let $\mathfrak{F}$ be a class of groups. A group G is called *s*-critical for $\mathfrak{F}$ (or minimal non-$\mathfrak{F}$-group) if G is not in $\mathfrak{F}$ but all proper subgroups of G are in $\mathfrak{F}$. The set of all *s*-critical groups for $\mathfrak{F}$ is denoted by $\mathcal{M}(\mathfrak{F})$. Recall that $\pi(G)$ is the set of all prime divisors of $|G|$ for a group G and $\pi(\mathfrak{F})$ is the set of all prime divisors of orders of groups from $\mathfrak{F}$ for a class of groups $\mathfrak{F}$.

In 1924 [1] O.Yu. Shmidt described all *s*-critical groups for $\mathfrak{N}$ where $\mathfrak{N}$ is the formation of all nilpotent groups. These groups are called Shmidt groups. In 1951 [2] N. Ito showed that all *s*-critical groups for the class of *p*-nilpotent groups are also Shmidt groups. L.A. Shemetkov in the Kourovka Notebook [3] posed the following problem: "Find all local hereditary formations $\mathfrak{F}$ of finite groups such that every *s*-critical group for $\mathfrak{F}$ is either a cyclic group of prime order or a Shmidt group". Such formations are called formations with the Shemetkov property or briefly $\breve{S}$-formations.

In the soluble universe this problem was solved in 1984 [4] by V.N. Semenchuk and A.F. Vasil'ev. In particular there was shown that $\mathfrak{F}$ is a saturated hereditary $\breve{S}$-formation if and only if $\mathfrak{F} = LF(f)$ where f is a full local definition of $\mathfrak{F}$ such that $f(p) = \mathfrak{S}_{\pi(f(p))}$ for all $p \in \pi(\mathfrak{F})$ and $f(p) = \varnothing$ otherwise. Note that A.N. Skiba in 1990 [5] showed that a hereditary soluble $\breve{S}$-formation is saturated.

But in the universe of all groups formations that are $\breve{S}$-formations in the soluble universe may not be $\breve{S}$-formations. For example let $\mathfrak{F} = LF(f)$ where $f(p) = \mathfrak{S}_{\{2,3,5\}}$ for $p \in \{2,3,5\}$ and $f(p) = \varnothing$ otherwise. It is easy to see that the alternating group of degree 5 is *s*-critical for $\mathfrak{F}$.

In the general case the problem was solved independently by A. Ballester-Bolinshes and M.D. Perez-Ramos [6] and S.F. Kamornikov [8]. According to the corollary 2.4.23 [7] a hereditary local formation $\mathfrak{F}$ is $\breve{S}$-formation if and only if

1) $\mathfrak{F} = LF(f)$ where f is a full local definition of $\mathfrak{F}$ such that $f(p) = \mathfrak{G}_{\pi(f(p))}$ for all $p \in \pi(\mathfrak{F})$ and $f(p) = \varnothing$ otherwise and

2) $\mathcal{M}(\mathfrak{F})$ contains only solvable groups.

But the second condition for an arbitrary hereditary local formation is hard to verify. That is why in [7, p. 117] the following question was posed: "Describe all hereditary local formation $\mathfrak{F}$ with

$\mathcal{M}(\mathfrak{F}) \subseteq \mathfrak{S}$". Theorem A gives the answer for this question in case when $\mathfrak{F}$ is a soluble formation.

***Theorem* A.** *Let $\mathfrak{F}$ be a local soluble formation. Then $\mathcal{M}(\mathfrak{F}) \subseteq \mathfrak{S}$ if and only if $\mathfrak{F}$ does not contain the following local subformations*:

$\mathfrak{M}_1(p) = LF(f)$ *where p is a prime and*

$$f(q)=\begin{cases}\mathfrak{A}(2^p-1) & \text{if } q=2,\\ \mathfrak{A}(2) & \text{if } q\in\pi(2^{2p}-1),\\ \varnothing & \text{if } q\in\mathbb{P}\setminus\pi(2(2^{2p}-1));\end{cases}$$

$\mathfrak{M}_2(p) = LF(f)$ *where p is an odd prime and*

$$f(q)=\begin{cases}\mathfrak{A}\left(\frac{3^p-1}{2}\right) & \text{if } q=3,\\ \mathfrak{A}(3) & \text{if } q=2,\\ \mathfrak{A}(2) & \text{if } q\in\pi(3^{2p}-1)\setminus\{2\},\\ \varnothing & \text{if } q\in\mathbb{P}\setminus\pi(3(3^{2p}-1));\end{cases}$$

$\mathfrak{M}_3(p) = LF(f)$ *where p is a prime such that* $p^2+1\equiv 0 \bmod 5$ *and*

$$f(q)=\begin{cases}\mathfrak{A}\left(\frac{p-1}{2}\right) & \text{if } q=p,\\ \mathfrak{A}(2) & \text{if } q\in\pi(p^2-1)\setminus\{2\},\\ \boldsymbol{QR}_0(S_3,Z_3) & \text{if } p^2-1\equiv 0 \bmod 16 \text{ and } q=2,\\ \mathfrak{A}(3) & \text{if } p^2-1\not\equiv 0 \bmod 16 \text{ and } q=2,\\ \varnothing & \text{if } q\in\mathbb{P}\setminus\pi(p^3-p);\end{cases}$$

$\mathfrak{M}_4 = LF(f)$ *where*

$$f(q)=\begin{cases}\mathfrak{A}(3) & \text{if } q=13,\\ \boldsymbol{QR}_0(GL(2,3),SL(2,3),Z_8,Q,SD_{16}) & \text{if } q=3,\\ \boldsymbol{QR}_0(Z_3,S_3) & \text{if } q=2,\\ \varnothing & \text{if } q\notin\{2,3,13\};\end{cases}$$

$\mathfrak{M}_5(p) = LF(f)$ *where p is an odd prime and*

$$f(q)=\begin{cases}\mathfrak{A}(2^p-1) & \text{if } q=2,\\ \mathfrak{A}(4) & \text{if } q\in\pi(2^{2p}+1),\\ \mathfrak{A}(2) & \text{if } q\in\pi(2^p-1),\\ \varnothing & \text{if } q\in\mathbb{P}\setminus\pi(2(2^{2p}+1)(2^p-1)).\end{cases}$$

In this work a criterion of hereditary soluble formation to be a $\breve{S}$-formation only in terms of its local definition is given.

Let $\mathfrak{Z}$ denote the class of Sylow tower groups. T. Hawkes [9] described all groups G such that $G\notin\mathfrak{Z}$ and all proper subgroups and epimorphic images of G belong to $\mathfrak{Z}$. In his description he associated with every group G a directed graph whose vertices are elements from $\pi(G)$ and $(p_1,\ p_2)$ is the edge for $p_1, p_2\in\pi(G)$ if and only if $p_2\in\pi(G/O_{p_1',p_1}(G))$. An analogous idea is used for description of all solvable $\breve{S}$-formations.

Let $\mathfrak{F} = LF(f)$ be a local hereditary formation where f is a full local definition of $\mathfrak{F}$ such that $f(p)=\mathfrak{S}_{\pi(f(p))}$ for all $p\in\pi(\mathfrak{F})$ and $f(p)=\varnothing$ otherwise. Let associate with $\mathfrak{F}$ directed graph $\Gamma(\mathfrak{F}, f)$ without loops whose vertices are prime numbers from $\pi(\mathfrak{F})$ and (p_i, p_j) is an edge of $\Gamma(\mathfrak{F}, f)$ for $p_i\neq p_j$ if and only if $p_j\in\pi(f(p_i))$. For example $\Gamma(\mathfrak{N}, f)$ is the set of isolated vertices where $f(p)=\mathfrak{S}_p$. All considered graphs are assumed to be subgraphs of the full directed graph without loops on $\mathbb{P}$.

Let p be a prime and $\Gamma_1(p)$ be a graph whose set of vertices is $\pi(2(2^{2p}-1))$ and edges are $(2,q)$ for $q\in\pi(2^p-1)$ and $(q,2)$ for $p\in\pi(2^{2p}-1)$.

Let p be a an odd prime and $\Gamma_2(p)$ be a graph whose set of vertices is $\pi(3(3^{2p}-1))$ and edges are $(3,q)$ for $p\in\pi(\frac{3^p-1}{2})$, $(q,2)$ for $p\in\pi(3^{2p}-1)\setminus\{2\}$ and $(2,3)$.

Let p be a a prime such that $p^2+1\equiv 0 \bmod 5$ and $\Gamma_3(p)$ be a graph whose set of vertices is $\pi(p(p^2-1))$ and edges are (p,q) for $q\in\pi(\frac{p-1}{2})$, $(q,2)$ for $q\in\pi(p^2-1)\setminus\{2\}$ and $(2,3)$.

Let Γ_4 be a graph whose set of vertices is $\{2,3,13\}$ and edges are $(13,3)$, $(3,2)$ and $(2,3)$.

Let p be an odd prime and $\Gamma_5(p)$ be a graph whose set of vertices is $\pi(2(2^{2p}+1)(2^p-1))$ and edges are $(2,q)$ for $q\in\pi(2^p-1)$ and $(q,2)$ for $p\in\pi((2^{2p}+1)(2^p-1))$.

***Theorem* B.** *Let $\mathfrak{F}$ be a hereditary soluble formation. Then the following statements are equivalent*:

(1) $\mathfrak{F}$ *is formation with the Shemetkov property*;

(2) $\mathfrak{F} = LF(f)$ *where f is a full local definition of $\mathfrak{F}$ such that $f(p)=\mathfrak{S}_{\pi(f(p))}$ for all $p\in\pi(\mathfrak{F})$ and $f(p)=\varnothing$ otherwise and $\Gamma(\mathfrak{F}, f)$ does not contain graphs from $\{\Gamma_1(q),\Gamma_2(q),\Gamma_3(q),\Gamma_4,\Gamma_5(q)\}$ as subgraphs.*

1 Preliminaries

Standard notation and terminology are used. If necessary it can be found in [7], [10]. For $\pi=\varnothing$ we assume that $\mathfrak{S}_\pi=\varnothing$.

Recall that for a class of group $\mathfrak{X}$

$$\mathbf{Q}\mathfrak{X} = (G \mid \exists H\in\mathfrak{X} \text{ and epimorphism from } H \text{ onto } G)$$

$$\mathbf{R}_0\mathfrak{X} = (G \mid \exists N_i \lhd G (i=1,\dots,n) \text{ with } G/N_i\in\mathfrak{X} \text{ and } \cap_{i=1}^n N_i = 1)$$

$$\mathbf{S}\mathfrak{X} = (G \mid \exists H\in\mathfrak{X} \text{ and } G\leq H).$$

Class $\mathfrak{F} = \boldsymbol{R}_0\mathfrak{F}$ and $\mathfrak{F} = \boldsymbol{Q}\mathfrak{F}$ is called formation. Function $f: \mathbb{P} \to \{formations\}$ is called formation function. Formation $\mathfrak{F}$ is called saturated if from $G/\Phi(G) \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$. According to the well known Gashutz – Lubeseder – Schmid Theorem saturated formations are exactly local formations, i. e. formations $\mathfrak{F} = LF(f)$ defined by a formation function f:

$LF(f) = (G \mid$ if H/K is a chief factor of G and $p \in \pi(H/K)$ then $G/C_G(H/K) \in f(p))$.

Let f and g be local definitions of formation $\mathfrak{F}$. Then h is also definition of $\mathfrak{F}$ where $h(p) = f(p) \cap g(p)$ for all primes p. It means that every formation has the minimal local definition. If f is the minimal local definition of $\mathfrak{F}$ and $\mathfrak{H} = LF(h)$ then $\mathfrak{F} \subseteq \mathfrak{H}$ if and only if $f(p) \subseteq h(p)$ for all primes p.

It is known that if f is a local definition of $\mathfrak{F}$ then F is also a local definition of $\mathfrak{F}$ where $F(p) = \mathfrak{N}_p f(p)$ for all primes p. This local definition is called full.

***Lemma* 1.1.** *Let G be a simple non-abelian group, $\mathfrak{F}$ be a local formation and f be a local definition of $\mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if $G \in f(p)$ for all $p \in \pi(G)$.*

Proof. Follows from the definition of local formation. □

***Lemma* 1.2** [10, p. 272]. *Let $\mathfrak{X}$ be a class of groups. Then the smallest formation containing $\mathfrak{X}$ is $\boldsymbol{QR}_0\mathfrak{X}$.*

It is easy to see that if $G, G/N \in \mathfrak{X}$ then

$$\boldsymbol{QR}_0\mathfrak{X} = \boldsymbol{QR}_0(\mathfrak{X} \setminus \{G/N\}).$$

Note that the formation generated by cyclic group Z_m is the formation $\mathfrak{A}(m)$ of all abelian groups of the exponent dividing m.

***Lemma* 1.3** [13, p. 14]. *Let $\mathfrak{X}$ be a class of groups. The smallest local formation that contains $\mathfrak{X}$ is $LF(f)$ where*

$$f(p) = \boldsymbol{QR}_0(H/O_{p',p}(H) \mid H \in \mathfrak{X})$$

for $p \in \pi(\mathfrak{X})$ and $f(p) = \varnothing$ otherwise and f is the minimal local definition of $\mathfrak{F}$.

***Lemma* 1.4** [13, p. 14]. *Let $\mathfrak{X}$ be a class of groups. The smallest local hereditary formation that contains $\mathfrak{X}$ is $LF(f)$ where*

$$f(p) = \boldsymbol{QR}_0\boldsymbol{S}(H/O_{p',p}(H) \mid H \in \mathfrak{X})$$

for $p \in \pi(\mathfrak{X})$ and $f(p) = \varnothing$ otherwise and f is the minimal local definition of $\mathfrak{F}$.

The following theorems give descriptions of minimal simple groups:

***Theorem* 1.5.** (Thompson [11, p. 190]). *All minimal simple non-abelian groups are*:

(1) $PSL(2, 2^p)$ *where p is a prime*;

(2) $PSL(2, 3^p)$ *where p is an odd prime*;

(3) $PSL(2, p)$ *where $p > 5$ is a prime and $p^2 + 1 \equiv 0 \bmod 5$*;

(4) $Sz(2^p)$ where p is an odd prime;

(5) $PSL(3,3)$.

***Theorem* 1.6** (Dickson [11, p. 213]). *Any subgroup of $PSL(2, p^n)$ is isomorphic to one of the following groups.*

(*a*) *Elementary abelian p-groups.*

(*b*) *Cyclic groups Z_m of order m, where m is a divisor of $(p^n \pm 1)/d$ and $d = (p-1, 2)$.*

(*c*) *Dihedral groups of order $2m$, where m is defined in (b).*

(*d*) *Alternating group A_4 if $p > 2$ or $p = 2$ and $n \equiv 0 \bmod 2$.*

(*e*) *Symmetric group S_4 if $p^{2n} \equiv 1 \bmod 16$.*

(*f*) *Alternating group A_5 if $p = 5$ or $p^{2n} \equiv 1 \bmod 5$.*

(*g*) *A semi-direct product of an elementary abelian p-group of order p^m and a cyclic group of order k, where k is a divisor of $p^m - 1$ and $p^n - 1$.*

(*h*) *The group $PSL(2, p^m)$ if m is a divisor of n, or the group $PGL(2, p^m)$ if $2m$ is a divisor of n.*

Let $G \simeq PSL(2, 2^p)$ be a minimal simple group. Note that subgroups of types (*f*) and (*h*) are not proper subgroups of G. Let $H = N \rtimes C$ be a subgroup of type (g). Then it is straightforward to check that $C_H(N) = N$.

***Theorem* 1.7** (Suzuki [12]). *Any subgroup of $G \simeq Sz(q)$ is isomorphic to a subgroup of one of the following groups where $q = 2^p$.*

(*a*) *Frobenius groups of order $q^2(q-1)$, $H = QK$, $Q \lhd H$, $Q \in Syl_q(H)$ and K is cyclic of order $q - 1$.*

(*b*) *Dihedral groups of order $q(q-1)$.*

(*c*) *Cyclic groups A_i, $i = 1, 2$ of orders $q \pm r + 1$ where $r^2 = 2q$.*

(*d*) *$B_i = N_G(A_i)$ of order $4(q \pm r + 1)$.*

(*e*) *$Sz(s)$ if q is a power of s.*

2 Main Results

***2.1 Proof of the Theorem* A**

***Proposition* 2.1.** *Let $\mathfrak{F}$ be a local formation of soluble groups. Then all s-critical groups for $\mathfrak{F}$ are* *soluble if and only if all minimal simple non-abelian groups are not s-critical for* $\mathfrak{F}$.

Proof. Let $\mathfrak{F}$ be a local formation of soluble groups and G be an unsolvable s-critical group for $\mathfrak{F}$. Since $\mathfrak{F}$ is local formation, $G/\Phi(G)$ is also s-critical for $\mathfrak{F}$. So we can assume that $\Phi(G)=1$. Now there is the unique minimal normal subgroup N of G. If $N \neq G$ then groups N and G/N are soluble. Hence G is soluble a contradiction. Now $G=N$. It means that G is a minimal simple non-abelian group. □

Let G be a group, $\mathfrak{F}$ and $\mathfrak{H}$ be local formations such that $G \in \mathcal{M}(\mathfrak{F})$ and $G \in \mathcal{M}(\mathfrak{H})$. Then it is clear that $G \in \mathcal{M}(\mathfrak{H} \cap \mathfrak{F})$. It means that for every group G if $G \in \mathcal{M}(\mathfrak{F})$ for some formation $\mathfrak{F}$ then there is the smallest local formation $\mathfrak{K}$ such that $G \in \mathcal{M}(\mathfrak{K})$.

***Proposition* 2.2.** *Let G be a simple non-abelian group and*

$$f_1(p) = \boldsymbol{QR}_0(H/O_{p',p}(H) \mid H < G)$$

for $p \in \pi(G)$ *and* $f_1(p) = \varnothing$ *otherwise*;

$$f_2(p) = \boldsymbol{QR}_0\boldsymbol{S}(H/O_{p',p}(H) \mid H < G)$$

for $p \in \pi(G)$ *and* $f_2(p) = \varnothing$ *otherwise.*

Then $LF(f_1)$ *and* $LF(f_2)$ *are the smallest formation and hereditary formation among local formations* $\mathfrak{F}$ *with* $G \in \mathcal{M}(\mathfrak{F})$.

Proof. Let $f = f_2$. According to lemma 1.4 $LF(f)$ is the smallest local hereditary formation containing all proper subgroups of G.

Let us show that $G \notin LF(f)$. Assume that $G \in LF(f)$. Then $G \in f(p)$ for some $p \in \pi(G)$. It means that $G \simeq \varphi(S)$ is a homomorphic image of a subgroup S of a direct product H of groups $H_1,\ldots,H_n$ from $\mathbf{S}(H/O_{p',p}(H) \mid H < G)$. Without loose of generality one may assume that S is the minimal subgroup with this property. Let ρ_i be a projection of S on H_i. Assume that $S = ker\varphi\, ker\rho_i$ for all $i = 1,\ldots,n$. Then

$$G \simeq S/ker\varphi =$$

$$= ker\varphi\, ker\rho_i / ker\varphi \simeq ker\rho_i / ker\rho_i \cap ker\varphi$$

for all $i = 1,\ldots,n$. Since $ker\rho_i \le H$, we see $ker\rho_i = S$ for all $i=1,\ldots,n$. A contradiction. Now $ker\varphi\, ker\rho_i < S$ for some i. So $ker\rho_i \subseteq ker\varphi$. It means that G is a a homomorphic image of a subgroup of G, a contradiction. Thus $G \in \mathcal{M}(\mathfrak{F})$.

If $f = f_1$ the proof is analogues.□

Let $\mathfrak{F} = LF(f)$ be a local formation of soluble groups and G be a minimal simple non-abelian group. Let $\mathfrak{H} = LF(h)$ be a minimal local formation with $G \in \mathcal{M}(\mathfrak{H})$. Since $G \in \mathcal{M}(\mathfrak{S})$, it is clear that $\mathfrak{H}$ is soluble. So $G \in \mathcal{M}(\mathfrak{F})$ if and only if $\mathfrak{H} \subseteq \mathfrak{F}$. Proposition 2.2 gives us the minimal local definition h of $\mathfrak{H}$. So it is sufficient to verify that $h(p) \subseteq f(p)$ for all primes p.

Since all minimal simple non-abelian groups are well described it is sufficient to calculate h for all this groups. With the help of theorems 1.5, 1.6 and 1.7 we can do that.

***Corollary* 2.3.** *Let* $\mathfrak{F}$ *be a local formation of soluble groups,* p *be a prime and* $\mathfrak{M}_1(p) = LF(f)$ *where*

$$f(q) = \begin{cases} \mathfrak{A}(2^p-1) & \text{if } q = 2, \\ \mathfrak{A}(2) & \text{if } q \in \pi(2^{2p}-1), \\ \varnothing & \text{if } q \in \mathbb{P} \setminus \pi(2(2^{2p}-1)). \end{cases}$$

Then $PSL(2,2^p) \in \mathcal{M}(\mathfrak{F})$ *if and only if* $\mathfrak{M}_1(p) \subseteq \mathfrak{F}$.

Proof. Let us calculate minimal local formation $\mathfrak{F}$ with $PSL(2,2^p) \in \mathcal{M}(\mathfrak{F})$. It is possible to do this with the help of proposition 2.2. Let $G \simeq PSL(2,2^p)$. Then G has subgroups of types (*a*), (*b*), (*c*), (*g*), (*d*) and (*f*) from theorem 1.6. But the last two cases are possible only for $p = 2$. In this case subgroups of type (*g*) and (*d*) coincide and subgroup of type (*f*) is G itself.

For all $q \in \pi(G)$ and subgroups H of types (*a*) and (*b*) $H/O_{q',q}(H) \simeq 1$.

For all subgroups H of type (*c*) $H/O_{2',2}(H) \simeq 1$. Now let H be a subgroup of type (*c*) that is isomorphic to the dihedral group of order $2(2^p+1)$ $(2(2^p-1))$. Then $H/O_{q',q}(H) \simeq Z_2$ for all

$$q \in \pi(2^p+1) \ \ (q \in \pi(2^p-1)).$$

Let $H \simeq M \rtimes Z_{2^p-1}$ where M is an elementary abelian subgroup of order 2^p be a subgroup of type (*g*). Then $H/O_{2',2}(H) \simeq Z_{2^p-1}$ and $H/O_{q',q}(H) \simeq 1$ for all $q \in \pi(2^p-1)$. □

The proof of the following two corollaries is analogous.

***Corollary* 2.4.** *Let* $\mathfrak{F}$ *be a local formation of soluble groups*, p *be an odd prime and* $\mathfrak{M}_2(p) = LF(f)$ *where*

$$f(q) = \begin{cases} \mathfrak{A}\left(\dfrac{3^p-1}{2}\right) & \text{if } q = 3, \\ \mathfrak{A}(3) & \text{if } q = 2, \\ \mathfrak{A}(2) & \text{if } q \in \pi(3^{2p}-1) \setminus \{2\}, \\ \varnothing & \text{if } q \in \mathbb{P} \setminus \pi(3(3^{2p}-1)). \end{cases}$$

Then $PSL(2,3^p) \in \mathcal{M}(\mathfrak{F})$ *if and only if* $\mathfrak{M}_2(p) \subseteq \mathfrak{F}$.

***Corollary* 2.5.** *Let* $\mathfrak{F}$ *be a local formation of soluble groups*, $p > 5$ *be a prime such that* $p^2+1 \equiv 0 \bmod 5$ *and* $\mathfrak{M}_3(p) = LF(f)$ *where*

$$f(q)=\begin{cases}\mathfrak{A}\left(\frac{p-1}{2}\right) & if\ q=p,\\ \mathfrak{A}(2) & if\ q\in\pi(p^2-1)\setminus\{2\},\\ \boldsymbol{QR}_0(S_3,Z_3) & if\ p^2-1\equiv 0\ mod\ 16\ and\ q=2,\\ \mathfrak{A}(3) & if\ p^2-1\not\equiv 0\ mod\ 16\ and\ q=2,\\ \varnothing & if\ q\in\mathbb{P}\setminus\pi(p^3-p).\end{cases}$$

Then $PSL(2,p)\in\mathcal{M}(\mathfrak{F})$ *if and only if* $\mathfrak{M}_3(p)\subseteq\mathfrak{F}$.

Corollary 2.6. *Let* $\mathfrak{F}$ *be a local formation of soluble groups and* $\mathfrak{M}_4=LF(f)$ *where*

$$f(q)=\begin{cases}\mathfrak{A}(3) & if\ q=13,\\ \boldsymbol{QR}_0(GL(2,3),SL(2,3),Z_8,Q,SD_{16}) & if\ q=3,\\ \boldsymbol{QR}_0(Z_3,S_3) & if\ q=2,\\ \varnothing & if\ q\notin\{2,3,13\}.\end{cases}$$

Then $PSL(3,3)\in\mathcal{M}(\mathfrak{F})$ *if and only if* $\mathfrak{M}_4\subseteq\mathfrak{F}$.

Proof. Let us calculate minimal local formation $\mathfrak{F}$ with $PSL(3,3)\in\mathcal{M}(\mathfrak{F})$. It is possible to do it with the help of proposition 2.2. Let $G\simeq PSL(3,3)$. Then G has 3 families of isomorphic maximal subgroups $Z_{13}\leftthreetimes Z_3$, the symmetric group S_4 of degree 4 and the general affine group $GA(2,3)=E_9\leftthreetimes GL(2,3)$ where $E_9=C_{GA(2,3)}(E_9)\simeq Z_3\times Z_3$ is a minimal normal subgroup of $GA(2,3)$.

$$Z_{13}\leftthreetimes Z_3/O_{13',13}(Z_{13}\leftthreetimes Z_3)\simeq Z_3,$$
$$Z_{13}\leftthreetimes Z_3/O_{3',3}(Z_{13}\leftthreetimes Z_3)\simeq 1,$$
$$S_4/O_{2',2}(S_4)\simeq S_3,$$
$$S_4/O_{3',3}(S_4)\simeq Z_2,$$
$$GA(2,3)/O_{2',2}(GA(2,3))\simeq S_3,$$
$$GA(2,3)/O_{3',3}(GA(2,3))\simeq GL(2,3).$$

All maximal subgroups of $GA(2,3)$ are isomorphic to $GL(2,3)$ or to $E_9\leftthreetimes M_i$, $i=1,2,3$, where M_i is a maximal subgroup of $GL(2,3)$. So

$$M_1\simeq SD_{16}=\langle a,x\mid a^8=x^2=1,xax=a^3\rangle$$

is the semidihedral group of order 16; $M_2\simeq SL(2,3)$; $M_3\simeq D_{12}$ is the dihedral group of order 12.

$$GL(2,3)/O_{2',2}(GL(2,3))\simeq S_3,$$
$$GL(2,3)/O_{3',3}(GL(2,3))\simeq Z_2,$$
$$E_9\leftthreetimes M_1/O_{2',2}(E_9\leftthreetimes M_1)\simeq 1,$$
$$E_9\leftthreetimes M_1/O_{3',3}(E_9\leftthreetimes M_1)\simeq M_1\simeq SD_{16},$$
$$E_9\leftthreetimes M_2/O_{2',2}(E_9\leftthreetimes M_2)\simeq Z_3,$$
$$E_9\leftthreetimes M_2/O_{3',3}(E_9\leftthreetimes M_2)\simeq M_2\simeq SL(2,3),$$
$$E_9\leftthreetimes M_3/O_{2',2}(E_9\leftthreetimes M_3)\simeq 1,$$
$$E_9\leftthreetimes M_3/O_{3',3}(E_9\leftthreetimes M_3)\simeq Z_2\times Z_2.$$

Among maximal subgroups of $E_9\leftthreetimes SD_{16}$ there are $E_9\leftthreetimes Z_8$, $E_9\leftthreetimes D_8$ and $E_9\leftthreetimes Q$ where D_8 and Q are the dihedral group of order 8 and the quaternion group.

$$E_9\leftthreetimes Z_8/O_{3',3}(E_9\leftthreetimes Z_8)\simeq Z_8,$$
$$E_9\leftthreetimes D_8/O_{3',3}(E_9\leftthreetimes D_8)\simeq D_8,$$
$$E_9\leftthreetimes Q/O_{3',3}(E_9\leftthreetimes Q)\simeq Q.$$

Note that $SD_{16}/\langle a^2\rangle\simeq D_8$. It is not difficult to calculate that for all other subgroups H, $H/O_{p',p}(H)$ is the homomorphic image of $K/O_{p',p}(K)$ for some considered subgroup K. It means the generating sets of $f(2), f(3)$ and $f(3)$ have been calculated. □

Corollary 2.7. *Let* $\mathfrak{F}$ *be a local formation of soluble groups, p be an odd prime and* $\mathfrak{M}_5(p)=LF(f)$ *where*

$$f(q)=\begin{cases}\mathfrak{A}(2^p-1) & if\ q=2,\\ \mathfrak{A}(4) & if\ q\in\pi(2^{2p}+1),\\ \mathfrak{A}(2) & if\ q\in\pi(2^p-1),\\ \varnothing & if\ q\in\mathbb{P}\setminus\pi(2(2^{2p}+1)(2^p-1)).\end{cases}$$

Then $Sz(2^p)\in\mathcal{M}(\mathfrak{F})$ *if and only if* $\mathfrak{M}_5(p)\subseteq\mathfrak{F}$.

Proof. Let us calculate minimal local formation $\mathfrak{F}$ with $Sz(2^p)\in\mathcal{M}(\mathfrak{F})$. One can do it with the help of proposition 2.2. Let $G\simeq Sz(2^p)$. Since p is a prime it is necessary to consider only four cases from theorem 1.7.

Let H be a Frobenius group of order $2^{2p}(2^p-1)$. Then $H/O_{2',2}(H)\simeq Z_{2^p-1}$ and $H/O_{q',q}(H)\simeq 1$ for all $q\in\pi(2^p-1)$.

Let H be the dihedral group of order $2^p(2^p-1)$. Then $H/O_{2',2}(H)\simeq 1$ and $H/O_{q',q}(H)\simeq Z_2$ for all $q\in\pi(2^p-1)$.

Let H be the cyclic group of order $2^p\pm r+1$ where $r^2=2^{p+1}$. Then $H/O_{q',q}(H)\simeq 1$ for all $q\in\pi(H)$.

Let $B=N_G(H)$. Then $B/O_{2',2}(B)\simeq 1$ and $B/O_{q',q}(B)\simeq Z_4$ for all $q\in\pi(H)\setminus\{2\}$. Note that $(2^{2p}+1,2^p-1)=1$. □

Now theorem A follows from corollaries 2.3–2.7.

2.2 Proof of the theorem B

From theorem A follows

Proposition 2.8. *Let* $\mathfrak{F}=LF(f)$ *be a formation and* $f(p)=\mathfrak{S}_{\pi(f(p))}$. *Then:*

$\Gamma_1(p)$ *is subgraph of* $\Gamma(\mathfrak{F},f)$ *if and only if* $PSL(2,2^p)\in\mathcal{M}(\mathfrak{F})$;

$\Gamma_2(p)$ *is subgraph of* $\Gamma(\mathfrak{F},f)$ *if and only if* $PSL(2,3^p)\in\mathcal{M}(\mathfrak{F})$;

$\Gamma_3(p)$ *is subgraph of* $\Gamma(\mathfrak{F}, f)$ *if and only if* $PSL(2, p) \in \mathcal{M}(\mathfrak{F})$;

Γ_4 *is subgraph of* $\Gamma(\mathfrak{F}, f)$ *if and only if* $PSL(3,3) \in \mathcal{M}(\mathfrak{F})$;

$\Gamma_5(p)$ *is subgraph of* $\Gamma(\mathfrak{F}, f)$ *if and only if* $Sz(2^p) \in \mathcal{M}(\mathfrak{F})$.

Let $\mathfrak{F}$ be a hereditary soluble $\breve{S}$-formation. By [5] it is saturated. According to [4] $\mathfrak{F} = LF(f)$ where f is a full local definition of $\mathfrak{F}$ such that $f(p) = \mathfrak{S}_{\pi(f(p))}$. Assume that $\Gamma(\mathfrak{F}, f)$ contains any graph from $\{\Gamma_1(q), \Gamma_2(q), \Gamma_3(q), \Gamma_4, \Gamma_5(q)\}$ as subgraph. According to proposition 2.8 there is a minimal simple *s*-critical for $\mathfrak{F}$ group. Thus it is not a $\breve{S}$-formation. A contradiction.

Assume that $\Gamma(\mathfrak{F}, f)$ does not contain subgraphs from $\{\Gamma_1(q), \Gamma_2(q), \Gamma_3(q), \Gamma_4, \Gamma_5(q)\}$. Let G be a *s*-critical for $\mathfrak{F}$-group with $\Phi(G) = 1$. Assume that G is not solvable. Then all proper subgroups of G are soluble. Hence G is a minimal simple group. It follows from proposition 2.8 that $\Gamma(\mathfrak{F}, f)$ contains graph from $\Gamma \in \{\Gamma_1(q), \Gamma_2(q), \Gamma_3(q), \Gamma_4, \Gamma_5(q)\}$ that corresponds to *G*. A contradiction. □

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