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On the supersolubility of a group with semisubnormal factors

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Abstract. A subgroup A of a group G is called *seminormal* in G if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B. We introduce the new concept that unites subnormality and seminormality. A subgroup A of a group G is called *semisubnormal* in G if A is subnormal in G or seminormal in G. A group $G = AB$ with semisubnormal supersoluble subgroups A and B is studied. The equality $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$ is established; moreover, if the indices of subgroups A and B in G are relatively prime, then $G^{\mathfrak{U}} = G^{\mathfrak{N}^2}$. Here $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{N}^2$ are the formations of all nilpotent, supersoluble and metanilpotent groups, respectively; $H^{\mathfrak{X}}$ is the $\mathfrak{X}$-residual of H. Also we prove the supersolubility of $G = AB$ when all Sylow subgroups of A and of B are semisubnormal in G.

1 Introduction

Throughout this paper, all groups are finite, and G always denotes a finite group. We use the standard notation and terminology of [13].

In 1953, Huppert [12] gave an example of a non-supersoluble group with supersoluble non-conjugate subgroups A and B of index 2. Since A and B are normal in G, it follows that G is soluble and $G = AB$; see [13, Theorem II.3.9]. Baer [3] obtained the supersolubility of a group $G = AB$ such that A and B are normal supersoluble subgroups and the derived subgroup G' is nilpotent. Baer's result was extended by weakening normality to subnormality in [20, Theorem 3]. Besides, $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$ for a group $G = AB$ with supersoluble subnormal subgroups A and B; see [20, Theorem 2]. Here $\mathfrak{N}$ and $\mathfrak{U}$ are the formations of all nilpotent and supersoluble groups, respectively; $G^{\mathfrak{U}}$ and $(G')^{\mathfrak{N}}$ are the corresponding residuals of G.

It is well known that every normal subgroup permutes with any subgroup of a group. Hence if A and B are normal subgroups of $G = AB$, then A permutes with every subgroup of B and B permutes with every subgroup of A. In this case, a group $G = AB$ is called a *mutually permutable product* of A and B; see [4, p. 149]. If any subgroups of A and of B are permutable, then a group $G = AB$ is called a *totally permutable product* of A and B; see [4, p. 149].

Asaad and Shaalan in [2, Theorem 3.8] extended Baer's theorem by considering mutually permutable subgroups. They also proved in [2, Theorem 3.1] that if G is a totally permutable product of supersoluble subgroups A and B, then G is supersoluble.

Asaad and Shaalan's results were developed by other authors; see, for instance, the references in [4]. The result associated with total permutability was generalized in works [1, 10, 16].

A subgroup A of a group G is called *seminormal* in G if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B; see [23]. It is obvious that any subgroup of prime index is seminormal. If the subgroups A and B of $G = AB$ are mutually permutable, then A and B are seminormal in G.

Example 1.1. Let Z_n be a cyclic group of order n. A group

$$G = Z_7 \rtimes \operatorname{Aut} Z_7 = Z_7 \rtimes (Z_2 \times Z_3)$$

is the product of subgroups $A \simeq Z_2 \times Z_3$ and $B \simeq Z_7 \rtimes Z_2$ that are seminormal in G. But A and B are not mutually permutable since A does not permute with some order 2 subgroups of B.

Groups with some seminormal subgroups have been investigated by many authors; see, for example, [7, 11, 15, 17, 23]. In particular, the supersolubility of a group with seminormal Sylow subgroups was obtained in [11, 17].

We introduce the following concept that unites subnormality and seminormality.

Definition. A subgroup A of a group G is called *semisubnormal* in G if A is subnormal in G or seminormal in G.

Let A and B be semisubnormal subgroups of $G = AB$. In the present paper, we prove that G is supersoluble in the following cases:

- A is nilpotent and B is supersoluble; see Theorem A;
- A and B are supersoluble and G' is nilpotent; see Theorem B;
- all Sylow subgroups of A and of B are semisubnormal in G; see Theorem D.

In Theorem C, for a group $G = AB$ with supersoluble semisubnormal subgroups A and B, we obtain the equality $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$. Besides, if the indices of the subgroups A and B in G are relatively prime, then $G^{\mathfrak{U}} = G^{\mathfrak{N}^2}$. Here $\mathfrak{N}^2$ is the formation of all metanilpotent groups.

From these theorems, we deduce some corollaries that present independent interest. The above-mentioned results of works [2,3] are covered by Theorems A–D.

In Section 5, as an application, we localize previous results to the p-supersoluble case.

2 Preliminary results

In this section, we give some definitions and basic results which are essential in the sequel.

A group whose chief factors have prime order is called *supersoluble*. Recall that a *p-closed* group is a group with a normal Sylow p-subgroup and a *p-nilpotent* group is a group with a normal Hall p'-subgroup.

We say a group G has a *Sylow tower* if there is a normal series such that each quotient is isomorphic to a Sylow subgroup. Let G have order $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_1 > p_2 > \dots > p_k$. We say G has an *ordered Sylow tower of supersoluble type* if there exists a series

$$1 = G_0 < G_1 < G_2 < \cdots < G_{k-1} < G_k = G$$

of normal subgroups of G such that G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for each $i = 1, 2, \dots, k$.

Denote by G', $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, center, Fitting and Frattini subgroups of G, respectively; $O_p(G)$ and $O_{p'}(G)$ the greatest normal p- and p'-subgroups of G, respectively. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. Denote by $\pi(G)$ the set of all prime divisors of the order of G. A group G is called *primary* if $|\pi(G)| = 1$.

Let $\mathfrak{X}$ be non-empty formation. Then $G^{\mathfrak{X}}$ denotes the $\mathfrak{X}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{X}$.

Lemma 2.1 ([6, Theorems 1.2, 1.4, 1.6–1.8, Corollary 3.2]). *The following statements hold.*

(1) *The class $\mathfrak{U}$ is a hereditary saturated formation.*

(2) *Every minimal normal subgroup of a supersoluble group has prime order.*

(3) *Let N be a normal subgroup of G, and assume that G/N is supersoluble. If N is either cyclic, or $N \leq Z(G)$, or $N \leq \Phi(G)$, then G is supersoluble.*

(4) *Each supersoluble group has an ordered Sylow tower of supersoluble type.*

(5) *The derived subgroup of a supersoluble group is nilpotent.*

(6) *A group G is supersoluble if and only if every maximal subgroup of G has prime index.*

If H is a subgroup of G, then $H_G = \bigcap_{x \in G} H^x$ is called *the core* of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*.

Lemma 2.2 ([20, Lemma 6]). *Let G be a soluble group. Assume that $G \notin \mathfrak{U}$, but $G/K \in \mathfrak{U}$ for every non-trivial normal subgroup K of G. Then*

(1) *G contains a unique minimal normal subgroup N,*

$$N = F(G) = \mathrm{O}_p(G) = C_G(N) \quad \text{for some } p \in \pi(G),$$

(2) *$Z(G) = \mathrm{O}_{p'}(G) = \Phi(G) = 1$,*

(3) *G is primitive, $G = N \rtimes M$, where M is maximal in G with trivial core,*

(4) *N is an elementary abelian subgroup of order p^n, $n > 1$,*

(5) *if V is a subgroup G and $G = VN$, then $V = M^x$ for some $x \in G$.*

Lemma 2.3 ([5, Propositions 2.2.8, 2.2.11]). *Let $\mathfrak{F}$ and $\mathfrak{H}$ be formations, and let K be normal in G. Then*

(1) *$(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$,*

(2) *$G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$,*

(3) *if $\mathfrak{H} \subseteq \mathfrak{F}$, then $G^{\mathfrak{F}} \leq G^{\mathfrak{H}}$.*

Recall that a group G is said to be *siding* if every subgroup of the derived subgroup G' is normal in G; see [22, Definition 2.1]. Metacyclic groups, t-groups (groups in which every subnormal subgroup is normal) are siding. The group $G = (Z_6 \times Z_2) \rtimes Z_2$ ([24], IdGroup(G) = [24,8]) is siding, but it is not metacyclic and is not a t-group.

Lemma 2.4. *Let G be siding. Then the following statements hold.*

(1) *If N is normal in G, then G/N is siding.*

(2) *If H is a subgroup of G, then H is siding.*

(3) *G is supersoluble.*

Proof. (1) By [18, Lemma 4.6], $(G/N)' = G'N/N$. Let A/N be an arbitrary subgroup of $(G/N)'$. Then

$$A \leq G'N, \quad A = A \cap G'N = (A \cap G')N.$$

Since $A \cap G' \le G'$, we have $A \cap G'$ is normal in G. Hence $(A \cap G')N/N$ is normal in G/N.

(2) Since $H \le G$, it follows that $H' \le G'$. Let A be an arbitrary subgroup of H'. Then $A \le G'$, and A is normal in G. Therefore, A is normal in H.

(3) We proceed by induction on the order of G. Let $N \le G'$ and $|N| = p$, where p is prime. By the hypothesis, N is normal in G. By induction, G/N is supersoluble and G is supersoluble by Lemma 2.1 (3). □

Remark 2.5. By Lemma 2.4, the class of all siding groups is a hereditary homomorph. The supersoluble group $G = S_3 \times S_3$ ([24], IdGroup(G) = [36,10])) is not siding. Really, the derived subgroup $G' = \langle a \rangle \times \langle b \rangle$ is an elementary abelian group of order 9, but the subgroup $\langle ab \rangle$ of G' is not normal in G. Moreover, all primitive quotients of G are isomorphic to either a cyclic group of order 2, or S_3, hence are siding. Thus the class of all siding groups is not a Schunck class and formation.

3 Properties of semisubnormal subgroups

Lemma 3.1. *The following statements hold.*

(1) *If H is a semisubnormal subgroup of G and $H \le X \le G$, then H is semisubnormal in X.*

(2) *If H is a semisubnormal subgroup of G and N is normal in G, then HN/N is semisubnormal in G/N.*

(3) *If H is a semisubnormal subgroup of G and Y is a non-empty set of elements from G, then $H^Y = \langle H^y \mid y \in Y \rangle$ is semisubnormal in G. In particular, H^g is semisubnormal in G for any $g \in G$.*

Proof. If H is subnormal in G, then statements (1)–(3) are true; see [14, Chapter 2]. If H is seminormal, then these statements were proved in [15, Lemmas 2 and 5]. Thus statements (1)–(3) are true. □

Lemma 3.2. *The following statements hold.*

(1) *Let p be the greatest prime in $\pi(G)$, and let P be a Sylow p-subgroup of G. If P is semisubnormal in G, then P is normal in G.*

(2) *If any Sylow subgroup of G is semisubnormal in G, then G is supersoluble.*

(3) *Let H be a maximal subgroup of G. If H is semisubnormal in G, then the index of H in G is prime.*

(4) *If every maximal subgroup of G is semisubnormal in G, then G is supersoluble.*

(5) *If the index of H in G is prime, then H is semisubnormal in G.*

Proof. (1) It is clear that if P is subnormal in G, then P is normal in G. If P is seminormal in G and p is maximal in $\pi(G)$, then by [17, Lemma 4], P is normal in G.

(2) Suppose that G has at least one subnormal Sylow subgroup P. Then P is normal in G and therefore seminormal in G. Hence any Sylow subgroup of G is seminormal in G. By [17, Corollary 6], G is supersoluble.

(3) If H is subnormal in G, then H is normal in G and $|G : H|$ is prime. Let H be seminormal in G, and let K be a subgroup of G such that $HK = G$ and HK_1 is a subgroup of G for every subgroup K_1 of K. Let r be a prime which divides the index $|G : H|$, and let R be a Sylow r-subgroup of K. Then $HR = G$ and $G = H\langle x\rangle$ for $x \in R \setminus H$. We choose an element x of minimal order. Then $H\langle x^r\rangle = \langle x^r\rangle H = H$ and $|G : H| = r$.

(4) Let M be a maximal subgroup of G. By (3), the index of M in G is prime. By Lemma 2.1 (6), G is supersoluble.

(5) Let $|G : H| = r$, and let R be a Sylow r-subgroup of G. Then R is not contained in H and there exists an element $x \in R \setminus H$. Let

$$|x| = r^a \quad \text{and} \quad |\langle x\rangle \cap H| = r^{a_1}.$$

It is obvious that $a > a_1$; hence

$$|\langle x\rangle H| = \frac{|\langle x\rangle||H|}{|\langle x\rangle \cap H|} = \frac{r^a \frac{|G|}{r}}{r^{a_1}} \geq |G|, \quad \langle x\rangle H = G.$$

Now x^r belongs to H and H is seminormal in G, and therefore is semisubnormal in G. □

Example 3.3. A group with seminormal 2-maximal subgroups is supersoluble; see [23]. A group with subnormal 2-maximal subgroups can be non-supersolvable. Any non-supersoluble Schmidt group (a non-nilpotent group whose proper subgroups are all nilpotent) confirms this fact. Such a group $G = P \rtimes Q$, where $|P| = p^m$, $|Q| = q$ and $m > 1$, is the order of p modulo q, for example, A_4. Since P is a minimal normal subgroup of G, it follows that P and Q^y, $y \in P$, are all maximal subgroups of G. It is clear that all 2-maximal subgroups are subnormal in G. Hence a group with semisubnormal 2-maximal subgroups can be non-supersolvable.

Recall that $A^G = \langle A^g \mid g \in G \rangle$ is the smallest normal subgroup of G containing A.

Lemma 3.4. *The following statements hold.*

(1) *If A is a semisubnormal 2-nilpotent subgroup of G, then A^G is soluble.*

(2) *Let p be the smallest prime divisor of order of G. If A is semisubnormal in G and p does not divide the order of A, then p does not divide the order of A^G.*

Proof. (1) If A is subnormal in G, then by [18, Theorem 5.31], A^G is soluble. If A is seminormal in G, then A^G is soluble by [15, Lemma 10].

(2) If A is a subnormal p'-subgroup of G, then by [18, Theorem 5.31], A^G is a p'-subgroup. If A is a seminormal p'-subgroup of G, then A^G is a p'-subgroup by [15, Lemma 11]. □

Lemma 3.5. *Let G be soluble. If G has a subgroup H of prime index, then G/H_G is supersoluble.*

Proof. Suppose that H is not normal in G. Then $H_G \neq H$ and G/H_G is primitive with primitivator H/H_G. By [18, Theorem 4.42],

$$G/H_G = (P/H_G) \rtimes (H/H_G), \quad P/H_G = C_{G/H_G}(P/H_G).$$

Let $|G : H| = p$, where p is prime. Then

$$|G/H_G : H/H_G| = |G : H| = p, \quad |P/H_G| = p.$$

The subgroup H/H_G is cyclic, as the automorphism group of P/H_G of prime order. Hence G/H_G is supersoluble. If H is normal in G, then $H = H_G$ and G/H_G is supersoluble. □

Let X be seminormal in G. Then there exists a subgroup Y such that $G = XY$ and XY_1 is a proper subgroup in G for every proper subgroup Y_1 of Y. Here a subgroup Y is called a *supersupplement* to X in G.

Lemma 3.6. *Let A be a seminormal subgroup of a soluble group G, and let r be the greatest prime in $\pi(G)$. If A is r-closed, then A_r is subnormal in G.*

Proof. Let Y be a supersupplement to A in G, and let X be a maximal subgroup of Y of prime index. By hypothesis, A permutes with X. Then by induction, A_r is subnormal in AX. Since AX is a subgroup of G of prime index, it follows that, by Lemma 3.5, $G/(AX)_G$ is supersoluble. Let $|G : AX| = t$. If $t = r$, then $AX/(AX)_G$ is a r'-group. Hence $A_r \leq (AX)_G$. Since A_r is subnormal in AX, we have A_r is subnormal in $(AX)_G$ and therefore is subnormal in G. If $t \neq r$, then $t < r$ and $G/(AX)_G$ is a r'-group. Thus $A_r \leq (AX)_G$ and A_r is subnormal in G. □

Lemma 3.7. *Let A and B be semisubnormal in G and $G = AB$. If A and B have an ordered Sylow tower of supersoluble type, then G has an ordered Sylow tower of supersoluble type.*

Proof. We proceed by induction on $|G|$. Since A is 2-nilpotent, it follows, by Lemma 3.4 (1), that A^G is soluble and $G = A^G B$ is soluble. Let $r \in \pi(G)$ with r maximal. It is clear that a Sylow r-subgroup A_r is normal in A. Let A be seminormal in G. Then by Lemma 3.6, A_r is subnormal in G. If A is subnormal in G, then A_r is subnormal in G. Similarly, a Sylow r-subgroup B_r of B is subnormal in G. Since $R = A_r B_r$ is a Sylow subgroup of G, we have that G is r-closed. The subgroups $AR/R \simeq A/A \cap R$ and $BR/R \simeq B/B \cap R$ are semisubnormal in $G/R = (AR/R)(BR/R)$ and have an ordered Sylow tower of supersoluble type. By induction, G/R has an ordered Sylow tower of supersoluble type; hence G has an ordered Sylow tower of supersoluble type. □

4 On the supersolubility of a factorized group with semisubnormal factors

Theorem A. *Let A and B be semisubnormal subgroups of G and $G = AB$. If A is nilpotent and B is supersoluble, then G is supersoluble.*

Proof. If A is subnormal in G, then A^G is the nilpotent normal subgroup of G. Hence, in the factorization $G = AB$, we can replace the subgroup A by the nilpotent normal subgroup A^G. Further, we assume that A is seminormal in G. Let Y be a supersupplement to A in G.

Assume that the claim is false, and let G be a minimal counterexample. If N is a non-trivial normal subgroup of G, then the subgroups AN/N and BN/N are semisubnormal in G/N by Lemma 3.1 (2), $AN/N \simeq A/A \cap N$ is nilpotent and $BN/N \simeq B/B \cap N$ is supersoluble. Then by induction,

$$G/N = (AN/N)(BN/N)$$

is supersoluble. By Lemmas 2.1 (4) and 3.7, G has an ordered Sylow tower of supersoluble type, and therefore we apply Lemma 2.2. We keep for G the notation of this lemma; in particular, $N = G_p$ is the Sylow p-subgroup for the greatest $p \in \pi(G)$. Since $G = AB$, it follows that $N = A_p B_p$, where A_p and B_p are Sylow p-subgroups of A and B, respectively; see [13, Theorem VI.4.6].

Suppose that $A_p = 1$. Then $N = B_p \leq B$. We choose a minimal normal subgroup N_1 of B such that $N_1 \leq N$. Since B is supersoluble, we have $|N_1| = p$ by Lemma 2.1 (2). Since $N_1 \leq N \leq Y$, it follows that there exists a subgroup $AN_1 = [N_1]A$ and N_1 is normal in G; this contradicts Lemma 2.2 (4). Thus the assumption $A_p = 1$ is false and $A_p \neq 1$.

Assume that $B_p = 1$. Hence $N = A_p \leq A$ and $N = A$ by Lemma 2.2 (1). Then $B \cap N = 1$ and B is maximal in G. Since B is semisubnormal in G, we have, by Lemma 3.2 (3), the index of B in G is prime; this contradicts Lemma 2.2 (4).

Let Y_1 be a Hall p'-subgroup of Y. Then AY_1 is a subgroup of G and A_p is normal in AY_1. Since N is abelian, a Sylow p-subgroup Y_p of Y centralizes A_p and A_p is normal in G; hence $A_p = N$. Because A is nilpotent and by Lemma 2.2 (1), it follows that $A = N$. Since B is supersoluble, we have B_p is normal in B. In this case, B_p is normal in $N = A$ and therefore is normal in G. Thus $B_p = N$ and $G = AB = NB = B$ is supersoluble, a contradiction. The theorem is proved. □

Corollary 4.1. *Let A and B be subgroups of G and $G = AB$. Suppose that A is nilpotent and B is supersoluble. Then G is supersoluble in each of the following cases.*

(1) *A and B are mutually permutable; see* [2, Theorem 3.2].

(2) *A and B are subnormal in G; see* [20, Lemma 10].

(3) *A and B are seminormal in G.*

(4) *One of the subgroups A or B is seminormal in G; the other is subnormal in G.*

(5) *The indices of A and B in G are prime; see* [21, Theorem A].

(6) *One of the subgroups A or B is semisubnormal in G; the index of the other in G is prime.*

Recall that if every subnormal subgroup of a group G is normal in G, then G is called a *t-group*. In 1957, Gaschütz [9] proved that the soluble t-groups can be represented as a semidirect product of a normal abelian Hall subgroup of odd order and a Dedekind subgroup. In [8, Theorem 2], Cossey obtained that if $G = AB$, where A and B are the normal soluble t-subgroups of G, then G is supersoluble. If A and B are subnormal in G, then G can be non-supersoluble; see [8].

Corollary 4.2. *If $G = AB$, A is a supersoluble semisubnormal subgroup of G and B is a normal siding subgroup of G, then G is supersoluble.*

Proof. We use induction on the order of G. If N is a non-trivial normal subgroup of G, then AN/N is semisubnormal in G/N by Lemma 3.1 (2) and is supersoluble, BN/N is a normal siding subgroup of G/N by Lemma 2.4 (1). By induction, $G/N = (AN/N)(BN/N)$ is supersoluble.

Let A be seminormal in G, and let U be a supersupplement to A in G. Since G is soluble, then U has a subgroup U_1 of prime index, and hence $M = AU_1$ is a subgroup of prime index in G. If A is subnormal in G, then the segment of a subnormal series between the subgroups A and G can be compacted to a composition series. Because G is soluble, it follows that there exists a maximal subgroup M of G of prime index such that $A \leq M$. By Dedekind's identity, $M = A(M \cap B)$. Since A is semisubnormal in M by Lemma 3.1 (1) and $M \cap B$ is a normal siding subgroup of M, we have that M is supersoluble by induction. If B is nilpotent, then by Theorem A, G is supersoluble. Hence B is non-nilpotent and $B' \neq 1$.

Let N be a minimal normal subgroup of G such that $N \leq B'$. If N is not contained in M, then $G = N \rtimes M$ and $|N|$ is prime. By Lemma 2.1 (3), G is supersoluble. Suppose that N is contained in M and N_1 is a subgroup of N of prime order such that N_1 is normal in M. Then N_1 is normal in B and therefore is normal in G. By Lemma 2.1 (3), G is supersoluble. □

Example 4.3. The non-supersoluble group

$$G = Z_3 \times ((S_3 \times S_3) \rtimes Z_2)$$

([24], IdGroup = [216,157]) is the product of a normal supersoluble subgroup $A \simeq S_3 \times S_3$ and a subnormal siding subgroup

$$B \simeq Z_3 \times Z_3 \times S_3, \quad B' \simeq Z_3.$$

A subgroup that is isomorphic to Z_4 is a supplement to B. Therefore, in Corollary 4.2, the condition of normality of the siding factor cannot be weakened to subnormality and seminormality.

Corollary 4.4. *Let A and B be subgroups of G, and let $G = AB$. Suppose that A is supersoluble and that B is normal and siding. Then G is supersoluble in each of the following cases.*

(1) *A is subnormal in G; see* [22, Corollary 3.3].

(2) *A is normal in G.*

(3) *A is normal in G, and B is a soluble t-group; see* [19, Theorem 3].

(4) *A is seminormal in G.*

(5) *The indices of A and B in G are prime; see* [21, Theorem B].

Theorem B. *Let A and B be semisubnormal supersoluble subgroups of G, and let $G = AB$. If the derived subgroup G' is nilpotent, then G is supersoluble.*

Proof. Assume that the claim is false, and let G be a minimal counterexample. If N is a non-trivial normal subgroup of G, then the subgroups AN/N and BN/N are semisubnormal in G/N by Lemma 3.1 (2) and are supersoluble. Since

$$(G/N)' = G'N/N \simeq G'/G' \cap N,$$

it follows that the derived subgroup $(G/N)'$ is nilpotent. Consequently, G/N satisfies the hypothesis of the theorem, and by induction, G/N is supersoluble. By Lemmas 2.1 (4) and 3.7, G has an ordered Sylow tower of supersoluble type, and therefore we apply Lemma 2.2. We keep for G the notation of this lemma; in particular, $N = G_p$ is the Sylow p-subgroup for the greatest $p \in \pi(G)$. By hypothesis, G' is nilpotent; hence $N = G'$ and G/N is abelian.

Suppose that $AN = G$. Then $A \cap N = 1$, and A is a maximal subgroup of G. Since A is semisubnormal in G, then by Lemma 3.2 (3), the index of A in G is prime; this contradicts Lemma 2.2 (4). Thus the assumption is false, and $AN < G$. By Lemma 3.1 (1), A is semisubnormal in AN, and AN is supersoluble by induction. Similarly, we have $BN < G$ and BN is supersoluble. Now $G = (AN)(BN)$ is the product of normal supersoluble subgroups AN and BN, and G' is nilpotent. By Baer's theorem [3], G is supersoluble. The theorem is proved. □

Corollary 4.5. *Let A and B be supersoluble subgroups of G, and let $G = AB$. If G' is nilpotent, then G is supersoluble in each of the following cases.*

(1) *A and B are mutually permutable; see* [2, Theorem 3.8].

(2) *A and B are subnormal in G; see* [20, Theorem 3].

(3) *A and B are seminormal in G.*

(4) *One of the subgroups A or B is seminormal in G; the other is subnormal in G.*

(5) *The indices of A and B in G are prime; see* [21, Corollary 3.6].

(6) *One of the subgroups A or B is semisubnormal in G; the index of the other in G is prime.*

Corollary 4.6. *Let A and B be semisubnormal supersoluble subgroups of G with relatively prime indices in G. If G is metanilpotent, then G is supersoluble.*

Proof. Since $(|G : A|, |G : B|) = 1$, we have $G = AB$. We use induction on the order of G. From the proof of Theorem B, we obtain that $G = N \rtimes M$ is a primitive group, where M is a maximal subgroup of G and $N = F(G) = G_p$ is a unique minimal normal subgroup of G, where p is the greatest in $\pi(G)$. Besides, AN and BN are proper subgroups of G. The subgroups A and N are semisubnormal in AN and supersoluble; moreover, $(|AN : A|, |AN : N|) = 1$. Since AN is metanilpotent, by induction, AN is supersoluble. Similarly, BN is supersoluble. Since G/N is nilpotent, AN and BN are subnormal in G. Now $G = (AN)(BN)$ is the product of subnormal supersoluble subgroups AN and BN such that its indices in G are relatively prime. By [20, Corollary 3.1], G is supersoluble. □

Corollary 4.7. *Let A and B be semisubnormal supersoluble subgroups of G with relatively prime indices in G. If $|\pi(G)| \le 2$, then G is supersoluble.*

Proof. By Lemma 3.7, G has an ordered Sylow tower of supersoluble type. By hypothesis, $|\pi(G)| \le 2$; hence G is metanilpotent. By Corollary 4.6, G is supersoluble. □

Example 4.8. Huppert [12] and Baer [3] gave the first examples of non-supersoluble groups that are products of two normal supersoluble subgroups. The groups in these examples are metanilpotent and have orders $2^3 \cdot 5^2$ and $2^3 \cdot p^2$. Hence we cannot omit the condition $(|G : A|, |G : B|) = 1$ in Corollaries 4.6 and 4.7.

Let p, q be primes, and let $\mathfrak{S}_{\{p,q\}}$ be the formation of all $\{p,q\}$-groups. For a group G, we introduce the following notation:

$$\mathfrak{B}(G) = \bigcap_{\forall\{p,q\}\subseteq\pi(G)} G^{\mathfrak{S}_{\{p,q\}}}.$$

It is clear that $\mathfrak{B}(G)$ is normal in G, and if G is $\{p,q\}$-nilpotent, then

$$\pi(\mathfrak{B}(G)) \cap \{p,q\} = \emptyset.$$

In particular, if G is non-primary and has a Sylow tower, then

$$|\mathfrak{B}(G)| \le |\pi(G)| - 2.$$

Theorem C. *Let A and B be semisubnormal supersoluble subgroups of G and $G = AB$. Then*

(1) $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$,

(2) *if* $(|G : A|, |G : B|) = 1$, *then* $G^{\mathfrak{U}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G) = (G')^{\mathfrak{N}}$.

Proof. (1) If G is supersoluble, then $G^{\mathfrak{U}} = 1$ and G' is nilpotent by Lemma 2.1 (5). Consequently, $(G')^{\mathfrak{N}} = 1 = G^{\mathfrak{U}}$, and the statement is true. Further, we assume that G is non-supersoluble. By Lemmas 2.1 (4) and 3.7, G has an ordered Sylow tower of supersoluble type. Since $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, we have

$$G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} \leq G^{\mathfrak{U}}$$

by Lemma 2.3 (2) and (3). Next we check the reverse inclusion. For this, we prove that $G/(G')^{\mathfrak{N}}$ is supersoluble. The derived subgroup

$$(G/(G')^{\mathfrak{N}})' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}}$$

is nilpotent. The quotients

$$G/(G')^{\mathfrak{N}} = (A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}})(B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}),$$

$$A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \simeq A/A \cap (G')^{\mathfrak{N}}, \quad B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \simeq B/B \cap (G')^{\mathfrak{N}},$$

hence the subgroups $A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ and $B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ are supersoluble, and by Lemma 3.1 (2), these subgroups are semisubnormal in $G/(G')^{\mathfrak{N}}$. By Theorem B, $G/(G')^{\mathfrak{N}}$ is supersoluble. Thus $G^{\mathfrak{U}} \leq (G')^{\mathfrak{N}}$, and (1) is proved.

(2) First we prove that $G^{\mathfrak{U}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$. It is obvious that all quotients of G satisfy the hypothesis of the theorem. Since $G/G^{\mathfrak{N}^2} \in \mathfrak{N}^2$, it follows that $G/G^{\mathfrak{N}^2}$ is supersoluble by Corollary 4.6. Hence $G^{\mathfrak{U}} \leq G^{\mathfrak{N}^2}$. Since $G/G^{\mathfrak{S}_{\{p,q\}}} \in \mathfrak{S}_{\{p,q\}}$, we have $G/G^{\mathfrak{S}_{\{p,q\}}}$ is supersoluble by Corollary 4.7. By Remak's lemma [18, Lemma 2.33], $G/\mathfrak{B}(G)$ is isomorphic to a subgroup of direct product

$$\prod_{\forall \{p,q\} \subseteq \pi(G)} G/G^{\mathfrak{S}_{\{p,q\}}}.$$

Thus $G/\mathfrak{B}(G)$ is supersoluble, and $G^{\mathfrak{U}} \leq G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$.

Next we prove the reverse inclusion. By Lemma 2.1 (5), every supersoluble group is metanilpotent. Hence $\mathfrak{U} \subseteq \mathfrak{N}^2$ and $G^{\mathfrak{N}^2} \leq G^{\mathfrak{U}}$ by Lemma 2.3 (3). Therefore,

$$G^{\mathfrak{N}^2} \cap \mathfrak{B}(G) \leq G^{\mathfrak{N}^2} \leq G^{\mathfrak{U}}.$$

So the equality $G^{\mathfrak{U}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$ is proved. By (1), we have

$$G^{\mathfrak{U}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G) = (G')^{\mathfrak{N}}.$$

The theorem is proved. □

Theorem D. *Let A and B be subgroups of G and $G = AB$. If all Sylow subgroup of A and of B are semisubnormal in G, then G is supersoluble.*

Proof. We use induction on the order of G. Let t be the smallest in $\pi(G)$. We consider $r \neq t$. By Lemma 3.4 (2), t does not divide the order of A_r^G. Similarly, t does not divide the order of B_r^G, and the subgroup

$$H = \prod_{r \neq t} A_r^G B_r^G$$

is a normal t'-subgroup of G, i.e. $H = G_{t'}$. In particular, G is soluble. By [13, Theorem VI.4.6], $G_{t'} = A_{t'} B_{t'}$. By induction, $G_{t'}$ is supersoluble; hence G has an ordered Sylow tower of supersoluble type.

Let N be a non-trivial proper normal subgroup of G. Then

$$G/N = (AN/N)(BN/N).$$

Let S/N be a Sylow s-subgroup of AN/N, and let T be a Sylow s-subgroup of $S \cap A$. Then $TN/N = S/N$, and T is a Sylow s-subgroup of A.

Since T is semisubnormal in G, by Lemma 3.1 (2), $TN/N = S/N$ is semisubnormal in G/N. Similarly, if K/N is a Sylow subgroup of BN/N, then K/N is semisubnormal in G/N. By induction, G/N is supersoluble. By Lemma 2.2, G is primitive, and we use for G the notation of this lemma. In particular, $N = G_p$, and p is the greatest in $\pi(G)$.

Let $N_1 \leq N$ and $|N_1| = p$. Since M is a Hall p'-subgroup of G, we have $M = A_{p'} B_{p'}$.

Let $\pi(A_{p'}) = \pi_1 \cup \pi_2$. If $r \in \pi_1$, then A_r is seminormal in G, and if $r \in \pi_2$, then A_r is subnormal in G. It is obvious that π_1 and π_2 can be chosen so that $\pi_1 \cap \pi_2 = \emptyset$. Suppose that $r \in \pi_1$. Then A_r is seminormal in G, and there is a subgroup U such that $G = A_r U$ and A_r permutes with every subgroups of U. Because $N \leq U$, it follows that A_r permutes with N_1. Since it is true for any $r \in \pi_1$, we have A_{π_1} permutes with N_1. Let $r \in \pi_2$. Then A_r is subnormal in G, and $(A_r)^G$ is a r-group. Hence $N \leq (A_r)^G$. It is impossible because $p \neq r$. Thus $\pi_2 = \emptyset$ and $\pi(A_{p'}) = \pi_1$. Therefore, $A_{p'}$ permutes with N_1.

Similarly, $B_{p'}$ permutes with N_1. Hence M permutes with N_1. Now MN_1 is a subgroup of G, and N_1 is normal in MN_1. Since N is abelian, then N_1 is normal in $NM = G$; this contradicts $|N| > p$. The theorem is proved. □

Let G be a product of two subgroups. The following example shows that if all maximal subgroups of these subgroups are semisubnormal in G, then G can be non-supersoluble.

Example 4.9. The alternating group $G = A_4$ of degree 4 is a product of subgroups $A = Z_3$ and $B = Z_2 \times Z_2$. It is clear that all maximal subgroups of A and of B are semisubnormal in G. But G is non-supersoluble.

5 Applications to p-supersoluble groups

A group is said to be *p-soluble* if the order of each of its chief factors is either a p-power or coprime to p. A group is said to be *p-supersoluble* if the order of each of its factors is either equal to p or coprime to p. We write $p\mathfrak{S}$ for the class of all p-soluble groups and $p\mathfrak{U}$ for the class of all p-supersoluble groups. The classes of all p-closed and p-nilpotent groups are equal to the products $\mathfrak{N}_p\mathfrak{E}_{p'}$ and $\mathfrak{E}_{p'}\mathfrak{N}_p$, respectively, where $\mathfrak{N}_p$ is the class of all p-groups and $\mathfrak{E}_{p'}$ is the class of all p'-groups. The classes $p\mathfrak{S}$, $\mathfrak{N}_p\mathfrak{E}_{p'}$ and $\mathfrak{E}_{p'}\mathfrak{N}_p$ are radical hereditary saturated formations and $\mathfrak{N}_p\mathfrak{E}_{p'} \cup \mathfrak{E}_{p'}\mathfrak{N}_p \subseteq p\mathfrak{S}$.

Lemma 5.1 ([13, Theorem VI.9.1]). *The following statements hold.*

(1) *The class $p\mathfrak{U}$ is a hereditary saturated formation.*

(2) *Each minimal normal subgroup of a p-supersoluble group is either a p'-subgroup or a group of order p. In particular, the p-rank of a p-supersoluble group is equal to* 1.

(3) *Let N be a normal subgroup of G and $G/N \in p\mathfrak{U}$. If N is cyclic or*

$$N \in \{Z(G), \mathrm{O}_{p'}(G), \Phi(G)\},$$

then $G \in p\mathfrak{U}$.

(4) *The derived subgroup of a p-supersoluble group is p-nilpotent.*

Lemma 5.2 ([19, Lemma 4]). *Let G be a p-supersoluble group, let P be a Sylow p-subgroup of G, and let H be a Hall p'-subgroup of G. If $\mathrm{O}_{p'}(G) = 1$, then the following statements hold.*

(1) *P is normal in G and $F(G) = P$.*

(2) *If $\Phi(G) = 1$, then $P = P_1 \times P_2 \times \ldots \times P_t$, where P_i is a normal subgroup of G of prime order for any i. In particular, P is elementary abelian.*

(3) *H is abelian of exponent dividing $p - 1$.*

(4) *G is supersoluble.*

Lemma 5.3 ([19, Lemma 5]). *Suppose that a p-soluble group G does not belong to $p\mathfrak{U}$, but $G/K \in p\mathfrak{U}$ for every non-trivial normal subgroup K of G. Then the following hold.*

(1) *$Z(G) = \mathrm{O}_{p'}(G) = \Phi(G) = 1$.*

(2) *G has a unique minimal normal subgroup N, $N = F(G) = \mathrm{O}_p(G) = C_G(N)$.*

(3) *G is a primitive and $G = N \rtimes M$, where M is a maximal subgroup of G with trivial core.*

(4) *N is an elementary abelian group of order p^n, $n > 1$.*

(5) *If M is abelian, then M is cyclic of order dividing $p^n - 1$, and n is the smallest positive integer such that $p^n \equiv 1 \pmod{|M|}$.*

Lemma 5.4 ([19, Corollary 1.1]). *Let A and B be normal p-supersoluble subgroups of G, and let $G = AB$. If G' is p-nilpotent, then G is p-supersoluble.*

Theorem E. *Let A and B be semisubnormal subgroups of a p-soluble group G and $G = AB$. If A is p-nilpotent and B is p-supersoluble, then G is p-supersoluble.*

Proof. We use induction on the order of G. Let N be a non-trivial normal subgroup of G. The quotients

$$G/N = (AN/N)(BN/N),$$
$$AN/N \simeq A/A \cap N, \quad BN/N \simeq B/B \cap N,$$

hence AN/N is p-nilpotent and BN/N is p-supersoluble. By Lemma 3.1 (2), these subgroups are semisubnormal in G/N. Consequently, G/N satisfies the hypothesis of the theorem, and by induction, G/N is p-supersoluble. By Lemma 5.3, G has a unique minimal normal subgroup N, $N = F(G) = \mathrm{O}_p(G) = C_G(N)$ and N is an elementary abelian group of order p^n, $n > 1$.

Suppose that $AN = G$. Then $A \cap N = 1$, and A is a maximal subgroup of G. By hypothesis, A is semisubnormal in G, and by Lemma 3.2 (3), the index of M in G is prime; this contradicts $|N| > p$. Thus the assumption is false, and $AN < G$. Similarly, $BN < G$.

The subgroups A and N are semisubnormal in AN, A is p-nilpotent, and N is p-supersoluble. By induction, AN is p-supersoluble. Similarly, BN is p-supersoluble since N and B are semisubnormal in BN, N is p-nilpotent and B is p-supersoluble.

Since $\mathrm{O}_{p'}(G) = 1$ and $N = C_G(N) = \mathrm{O}_p(G)$, we have

$$\mathrm{O}_{p'}(AN) = 1 = \mathrm{O}_{p'}(BN).$$

By Lemma 5.2, AN and BN are supersoluble and p-closed; besides, its Hall p'-subgroups are abelian. Hence A is nilpotent, and B is supersoluble. By Theorem A, G is supersoluble. □

Theorem F. *Let A and B be semisubnormal p-supersoluble subgroups of a p-soluble group G and $G = AB$. If G' is p-nilpotent, then G is p-supersoluble.*

Proof. By induction and by Lemma 5.3, we obtain that $G = N \rtimes M$ is a primitive group, where M is a maximal subgroup of G and $N = F(G) = O_p(G)$ is a unique minimal normal subgroup of G for some prime $p \in \pi(G)$. By hypothesis, the derived subgroup of G is p-nilpotent. Since $\mathrm{O}_{p'}(G) = 1$, it follows that G' is a p-group. Thus $N = G'$ and G/N is abelian.

Besides, AN and BN are proper subgroups of G. The subgroups A and N are semisubnormal in AN and p-supersoluble. Since $(AN)'$ is p-nilpotent, by induction, AN is p-supersoluble. Similarly, BN is p-supersoluble. Since G/N is abelian, we have AN and BN are normal in G. Now $G = (AN)(BN)$ is the product of normal p-supersoluble subgroups AN and BN. By Lemma 5.4, G is p-supersoluble, a contradiction. □

Theorem G. *Let $G = AB$ be a p-soluble group, and let A and B be semisubnormal p-supersoluble subgroups of G. Then $G^{p\mathfrak{U}} = (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$.*

Proof. If G is p-supersoluble, then we have $G^{p\mathfrak{U}} = 1$ and G' is p-nilpotent by Lemma 5.1 (4). Consequently, $G^{p\mathfrak{U}} = 1 = (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$, and the statement is true. Further, we assume that G is non-p-supersoluble.

Since $p\mathfrak{U} \subseteq \mathfrak{E}_{p'}\mathfrak{N}_p\mathfrak{A}$, we have

$$G^{(\mathfrak{E}_{p'}\mathfrak{N}_p\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{E}_{p'}\mathfrak{N}_p} = (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} \leq G^{p\mathfrak{U}}$$

by Lemma 2.3 (2) and (3). Next we check the reverse inclusion. For this, we prove that $G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$ is p-supersoluble. The derived subgroup

$$(G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p})' = G'(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} = G'/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$$

is p-nilpotent. The quotients

$$G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} = (A(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p})(B(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}),$$
$$A(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} \simeq A/A \cap (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p},$$
$$B(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} \simeq B/B \cap (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p},$$

hence the subgroups $A(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$ and $B(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$ are p-supersoluble, and by Lemma 3.1 (2), these subgroups are semisubnormal in $G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$. By Theorem F, $G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$ is p-supersoluble. The theorem is proved. □

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