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Finite soluble groups with all n-maximal subgroups $\mathfrak{F}$-subnormal

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Abstract. We describe finite soluble groups in which every n-maximal subgroup is $\mathfrak{F}$-subnormal.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use $\mathfrak{U}$, $\mathfrak{N}$ and $\mathfrak{N}^r$ to denote the class of all supersoluble groups, the class of all nilpotent groups and the class of all soluble groups of nilpotent length at most r ($r \geq 1$). The symbol $\mathbb{P}$ denotes the set of all primes, $\pi(G)$ denotes the set of prime divisors of the order of G. If p is a prime, then we use $\mathfrak{G}_p$ to denote the class of all p-groups.

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \emptyset$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ for any group G, *hereditary* if $H \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$ and H is a subgroup of G. A group G is called $\mathfrak{F}$-*critical* provided G does not belong to $\mathfrak{F}$ but every proper subgroup of G belongs to $\mathfrak{F}$. The *Gaschütz product* $\mathfrak{M} \circ \mathfrak{H}$ of the formations $\mathfrak{M}$ and $\mathfrak{H}$ is the class of all groups G such that $G^{\mathfrak{H}} \in \mathfrak{M}$.

For any formation function $f : \mathbb{P} \to \{\text{group formation}\}$, the symbol $\mathrm{LF}(f)$ denotes the collection of all groups G such that one has either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G and every $p \in \pi(H/K)$. It is well known that for any non-empty saturated formation $\mathfrak{F}$, there is a unique formation function F such that $\mathfrak{F} = \mathrm{LF}(F)$ and $F(p) = \mathfrak{G}_p \circ F(p) \subseteq \mathfrak{F}$ for all primes p (see [2, Chapter IV, Proposition 3.8]). The formation function F is called the *canonical local satellite* of $\mathfrak{F}$. A chief factor H/K of G is called $\mathfrak{F}$-*central* in G provided $G/C_G(H/K) \in F(p)$ for all primes p dividing $|H/K|$, otherwise it is called $\mathfrak{F}$-*eccentric*.

Fix some ordering ϕ of $\mathbb{P}$. The record $p\phi q$ means that p precedes q in ϕ and $p \neq q$. Recall that a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is called *ϕ-dispersive* whenever $p_1 \phi p_2 \phi \cdots \phi p_n$ and for every i there is a normal subgroup of G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$. Furthermore, if ϕ is such that $p\phi q$ always implies $p > q$, then every ϕ-dispersive group is called *Ore dispersive*.

By definition, every formation is 0-*multiply saturated* and for $n \geq 1$ a formation $\mathfrak{F}$ is called *n-multiply saturated* if $\mathfrak{F} = \mathrm{LF}(f)$, where every non-empty value of the function f is an $(n-1)$-multiply saturated formation (see [20, 21]). In fact, almost saturated formations met in mathematical practice are n-multiply saturated for every natural n. For example, the formations of all soluble groups, all nilpotent groups, all p-soluble groups, all p-nilpotent groups, all p-closed groups, all p-decomposable groups, all Ore dispersive groups, all metanilpotent groups are n-multiply saturated for all $n \geq 1$. Nevertheless, the formations of all supersoluble groups and all p-supersoluble groups are saturated, but they are not 2-multiply saturated formations.

Recall that a subgroup H of G is called a 2-*maximal* (*second maximal*) subgroup of G whenever H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define 3-*maximal subgroups*, and so on.

The interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its n-maximal subgroups. One of the earliest publication in this direction is the article of Huppert [9] who established the supersolubility of a group G whose all second maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of G are normal in G, then the commutator subgroup G' of G is nilpotent and the chief rank of G is at most 2. These two results were developed by many authors. Among the recent results on n-maximal subgroups we can mention [8], where the solubility of groups is established in which all 2-maximal subgroups enjoy the cover-avoidance property, and [5, 6, 14], where new characterizations of supersoluble groups in terms of 2-maximal subgroups were obtained. The classification of nonnilpotent groups whose all 2-maximal subgroups are TI-subgroups appeared in [13]. Description was obtained in [3] of groups whose every 3-maximal subgroup permutes with all maximal subgroups. The nonnilpotent groups are described in [4] in which every two 3-maximal subgroups are permutable. The groups are described in [15] whose all 3-maximal subgroups are S-quasinormal, that is, permute with all Sylow subgroups. Subsequently this result was strengthened in [16] to provide a description of the groups whose all 3-maximal subgroups are subnormal.

Despite of all these and many other known results about n-maximal subgroups, the fundamental work of Mann [17] still retains its value. It studied the structure of groups whose n-maximal subgroups are subnormal. Mann proved that if all

n-maximal subgroups of a soluble group G are subnormal and $|\pi(G)| \geq n+1$, then G is nilpotent; but if $|\pi(G)| \geq n-1$, then G is ϕ-dispersive for some ordering ϕ of $\mathbb{P}$. Finally, in the case $|\pi(G)| = n$ Mann described G completely.

Let $\mathfrak{F}$ be a non-empty formation. Recall that a subgroup H of a group G is said to be $\mathfrak{F}$-*subnormal in* G if either $H = G$ or there exists a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that H_{i-1} is a maximal subgroup of H_i and $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$, for $i = 1, \ldots, n$.

The main goal of this article is to prove the following formation analogs of Mann's theorems.

Theorem A. *Let $\mathfrak{F}$ be an r-multiply saturated formation such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{N}^{r+1}$ for some $r \geq 0$. If every n-maximal subgroup of a soluble group G is $\mathfrak{F}$-subnormal in G and $|\pi(G)| \geq n+r+1$, then $G \in \mathfrak{F}$.*

Theorem B. *Let $\mathfrak{F} = \mathrm{LF}(F)$ be a saturated formation such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{U}$, where F is the canonical local satellite of $\mathfrak{F}$. Let G be a soluble group with $|\pi(G)| \geq n+1$. Then all n-maximal subgroups of G are $\mathfrak{F}$-subnormal in G if and only if G is a group of one of the following types:*

(I) *$G \in \mathfrak{F}$.*

(II) *$G = A \rtimes B$, where $A = G^{\mathfrak{F}}$ and B are Hall subgroups of G, while G is Ore dispersive and satisfies the following:*

   (1) *A is either of the form $N_1 \times \cdots \times N_t$, where each N_i is a minimal normal subgroup of G, which is a Sylow subgroup of G, for $i = 1, \ldots, t$, or a Sylow p-subgroup of G of exponent p for some prime p and the commutator subgroup, the Frattini subgroup, and the center of A coincide, while $A/\Phi(A)$ is an $\mathfrak{F}$-eccentric chief factor of G,*

   (2) *every n-maximal subgroup of G belongs to $\mathfrak{F}$ and induces on the Sylow p-subgroup of A an automorphism group which is contained in $F(p)$ for every prime divisor p of $|A|$.*

In the proof of Theorem B we often use Theorem A and the following useful

Theorem C. *Let $\mathfrak{F}$ be a hereditary saturated formation such that every $\mathfrak{F}$-critical group is soluble and it has a normal Sylow p-subgroup $G_p \neq 1$ for some prime p. Then every 2-maximal subgroup of G is $\mathfrak{F}$-subnormal in G if and only if either $G \in \mathfrak{F}$ or G is an $\mathfrak{F}$-critical group and $G^{\mathfrak{F}}$ is a minimal normal subgroup of G.*

Theorem D. *Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{U}$. If every n-maximal subgroup of a soluble group G is $\mathfrak{F}$-subnormal in G and $|\pi(G)| \geq n$, then G is ϕ-dispersive for some ordering ϕ of $\mathbb{P}$.*

All unexplained notation and terminology are standard. The reader is referred to [2] or [1] if necessary.

2 Preliminary results

Let $\mathfrak{F}$ be a non-empty formation. Recall that a maximal subgroup H of G is said to be $\mathfrak{F}$-*normal* in G if $G/H_G \in \mathfrak{F}$, otherwise it is said to be $\mathfrak{F}$-*abnormal* in G.

We use the following results.

Lemma 2.1. *Let $\mathfrak{F}$ be a formation and H an $\mathfrak{F}$-subnormal subgroup of G.*

(1) *If $\mathfrak{F}$ is hereditary and $K \leq G$, then $H \cap K$ is an $\mathfrak{F}$-subnormal subgroup in K, cf.* [1, Lemma 6.1.7 (2)].

(2) *If N is a normal subgroup in G, then HN/N is an $\mathfrak{F}$-subnormal subgroup in G/N, cf.* [1, Lemma 6.1.6 (3)].

(3) *If K is a subgroup of G such that K is $\mathfrak{F}$-subnormal in H, then K is $\mathfrak{F}$-subnormal in G, cf.* [1, Lemma 6.1.6 (1)].

(4) *If $\mathfrak{F}$ is hereditary and K is a subgroup of G such that $G^{\mathfrak{F}} \leq K$, then K is $\mathfrak{F}$-subnormal in G, cf.* [1, Lemma 6.1.7 (1)].

The following lemma is evident.

Lemma 2.2. *Let $\mathfrak{F}$ be a hereditary formation. If $G \in \mathfrak{F}$, then every subgroup of G is $\mathfrak{F}$-subnormal in G.*

Lemma 2.3. *Let $\mathfrak{F}$ be a hereditary saturated formation. If every n-maximal subgroup of G is $\mathfrak{F}$-subnormal in G, then every $(n-1)$-maximal subgroup of G belongs to $\mathfrak{F}$ and every $(n+1)$-maximal subgroup of G is $\mathfrak{F}$-subnormal in G.*

Proof. We first show that every $(n-1)$-maximal subgroup of G belongs to $\mathfrak{F}$. Let H be an $(n-1)$-maximal subgroup of G and K a maximal subgroup of H. Then K is an n-maximal subgroup of G and so, by hypothesis, K is $\mathfrak{F}$-subnormal in G. Hence K is $\mathfrak{F}$-subnormal in H by Lemma 2.1 (1). Thus all maximal subgroups of H are $\mathfrak{F}$-normal in H. Therefore $H \in \mathfrak{F}$ since $\mathfrak{F}$ is saturated.

Now, let E be an $(n+1)$-maximal subgroup of G, and let E_1 and E_2 be an n-maximal and an $(n-1)$-maximal subgroup of G, respectively, such that $E \leq E_1 \leq E_2$. Then, by the above, $E_2 \in \mathfrak{F}$, so $E_1 \in \mathfrak{F}$. Hence E is $\mathfrak{F}$-subnormal in E_1 by Lemma 2.2. By hypothesis, E_1 is $\mathfrak{F}$-subnormal in G. Therefore E is $\mathfrak{F}$-subnormal in G by Lemma 2.1(3). The lemma is proved. □

Lemma 2.4 (see [19, Chapter VI, Theorem 24.2]). *Let $\mathfrak{F}$ be a saturated formation and G a soluble group. If $G^{\mathfrak{F}} \neq 1$ and every $\mathfrak{F}$-abnormal maximal subgroup of G belongs to $\mathfrak{F}$, then the following hold:*

(1) *$G^{\mathfrak{F}}$ is a p-group for some prime p,*

(2) *$G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is an $\mathfrak{F}$-eccentric chief factor of G,*

(3) *if $G^{\mathfrak{F}}$ is a non-abelian group, then the commutator subgroup, the Frattini subgroup, and the center of G coincide and are of exponent p,*

(4) *if $G^{\mathfrak{F}}$ is abelian, then $G^{\mathfrak{F}}$ is elementary,*

(5) *if $p > 2$, then $G^{\mathfrak{F}}$ is of exponent p, for $p = 2$ the exponent of $G^{\mathfrak{F}}$ is at most 4,*

(6) *every pair of $\mathfrak{F}$-abnormal maximal subgroups of G are conjugate in G.*

Lemma 2.5 (see [19, Chapter VI, Theorem 24.5]). *Let $\mathfrak{F}$ be a saturated formation. Let G be an $\mathfrak{F}$-critical group and G has a normal Sylow p-subgroup $G_p \neq 1$ for some prime p. Then:*

(1) *$G_p = G^{\mathfrak{F}}$,*

(2) *$F(G) = G_p\Phi(G)$,*

(3) *$G_{p'} \cap C_G(G_p/\Phi(G_p)) = \Phi(G) \cap G_{p'}$, where $G_{p'}$ is some complement of G_p in G.*

Lemma 2.6 (see [19, Chapter VI, Theorems 26.3 and 26.5]). *Let G be a $\mathfrak{U}$-critical group. Then:*

(1) *G is soluble and $|\pi(G)| \leq 3$,*

(2) *if G is not a Schmidt group, then G is Ore dispersive,*

(3) *$G^{\mathfrak{U}}$ is the unique normal Sylow subgroup of G,*

(4) *if S is a complement of $G^{\mathfrak{U}}$ in G, then $S/S \cap \Phi(G)$ is either a primary cyclic group or a Miller–Moreno group.*

Let $\mathfrak{F}$ be a class of groups. Recall that the product of all normal subgroups of a group G whose G-chief factors are $\mathfrak{F}$-central in G is called *$\mathfrak{F}$-hypercentre of G* and denoted by $Z_{\mathfrak{F}}(G)$, see [2, p. 389].

Lemma 2.7 (see [7, Lemma 2.14]). *Let $\mathfrak{F}$ be a saturated formation and F be the canonical local satellite of $\mathfrak{F}$. Let E be a normal p-subgroup of a group G. Then $E \leq Z_{\mathfrak{F}}(G)$ if and only if $G/C_G(E) \in F(p)$.*

Lemma 2.8 (see [20, Corollary 7.14]). *The Gaschütz product of any two n-multiply saturated formations is an n-multiply saturated formation for every $n \geq 0$.*

We shall also need the following evident lemma.

Lemma 2.9. *If $G = AB$, then $G = AB^x$ for all $x \in G$.*

Let $\mathfrak{F}$ be a class of groups and t be a natural number with $t \geq 2$. Recall that $\mathfrak{F}$ is called Σ_t*-closed* if $\mathfrak{F}$ contains every group G such that G has subgroups $H_1, \dots, H_t$ whose indices are pairwise coprime and $H_i \in \mathfrak{F}$, for $i = 1, \dots, t$.

Lemma 2.10 (see [19, Chapter I, Lemma 4.11]). *Every formation of nilpotent groups is Σ_3-closed.*

If $\mathfrak{F} = \mathrm{LF}(f)$ and $f(p) \subseteq \mathfrak{F}$ for all primes p, then f is called an *integrated local satellite* of $\mathfrak{F}$. Let $\mathfrak{X}$ be a set of groups. The symbol l_n form $\mathfrak{X}$ denotes the intersection of all n-multiply saturated formations $\mathfrak{F}$ such that $\mathfrak{X} \subseteq \mathfrak{F}$. In view of [1, Remark 3.1.7], l_n form $\mathfrak{X}$ is an n-multiply saturated formation.

Lemma 2.11 (see [20, Theorem 8.3]). *Let $\mathfrak{F}$ be an n-multiply saturated formation for some $n \geq 1$. Then $\mathfrak{F}$ has an integrated local satellite f such that, for all primes p, $f(p) = l_{n-1}$* form$(G/O_{p',p}(G) | G \in \mathfrak{F})$.

Lemma 2.12 (see [22, Section 1.4]). *Every r-multiply saturated formation contained in $\mathfrak{N}^{r+1}$ is hereditary for any $r \geq 0$.*

Lemma 2.13 (see [19, p. 35]). *For any ordering ϕ of $\mathbb{P}$ the class of all ϕ-dispersive groups is a saturated formation.*

Lemma 2.14 (see [7, Corollary 1.6]). *Let $\mathfrak{F}$ be a saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap \Phi(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$.*

Lemma 2.15 (see [19, Theorem 15.10]). *Let $\mathfrak{F}$ be a saturated formation and G be a group such that $G^{\mathfrak{F}}$ is nilpotent. Let H and M be subgroups of G, $H \in \mathfrak{F}$, $H \leq M$ and $HF(G) = G$. If H is $\mathfrak{F}$-subnormal in M, then $M \in \mathfrak{F}$.*

3 Proof of Theorem A

First we give two proposition which may be independently interesting since they generalize some known results.

Proposition 3.1. *Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are soluble subgroups of G. If the three indices $|G : N_G(A_1')|$, $|G : N_G(A_2')|$ and $|G : N_G(A_3')|$ are pairwise coprime, then G is soluble.*

Proof. Suppose that this proposition is false and let G be a counterexample with $|G|$ minimal.

(1) *If N is a minimal normal subgroup of G, then G/N is soluble. Hence N is the unique minimal normal subgroup of G and $C_G(N) = 1$.*

It is clear that $A_i'N/N = (A_iN)'N/N = (A_iN/N)'$, for $i = 1, 2, 3$. Hence $N_G(A_i')N/N \leq N_{G/N}(A_iN/N)'$. On the other hand, $A_iN/N \simeq A_i/A_i \cap N$ is soluble for all $i = 1, 2, 3$. Therefore the hypothesis holds for G/N, so G/N is soluble by the choice of G. Hence N is non-abelian and N is the unique minimal normal subgroup of G. Therefore, since $C_G(N)$ is normal in G, $C_G(N) = 1$.

(2) *At least two of the subgroups A_1, A_2, A_3 are non-abelian.* (This directly follows from Kegel–Wielandt's theorem on solubility of products of nilpotent groups [10, Chapter VI, Hauptsatz 4.3] and the choice of G.)

(3) *There are two pairs $i_1 \neq j_1$ and $i_2 \neq j_2$, where $j_1 = i_2$, such that the subgroup A_{i_a} has a minimal normal subgroup L_a such that ${L_a}^G \leq N_G(A'_{j_a})$.*

Let L_1 be a minimal normal subgroup of A_1. Then L_1 is a p-group for some prime p since A_1 is soluble by hypothesis. On the other hand, again by hypothesis, $|G : N_G(A_2')|$ and $|G : N_G(A_3')|$ are coprime, so at least one of the subgroups $N_G(A_2')$ or $N_G(A_3')$ contains a Sylow p-subgroup P of G. Without loss of generality we may assume that $P \leq N_G(A_2')$. Moreover, in view of Lemma 2.9 and Sylow's theorem we may assume that $L_1 \leq P$. Hence

$$L_1^G = L_1^{A_1A_2} = L_1^{A_1N_G(A_2')} \leq N_G(A_2').$$

Now let L_2 be a minimal normal subgroup of A_2. Then, as above, we have either ${L_2}^G \leq N_G(A_1')$ or ${L_2}^G \leq N_G(A_3')$.

Final contradiction. In view of (3) we may assume that A_1 has a minimal normal subgroup L_1 such that ${L_1}^G \leq N_G(A_2')$, and A_2 has a minimal normal subgroup L_2 such that ${L_2}^G \leq N_G(A_i')$ for some i. In view of (2), at least one of the subgroups A_2 or A_i is not abelian. Without loss of generality we may suppose that $A_2' \neq 1$. In view of (1), we have $N \leq N_G(A_2')$. Let $H = A_2' \cap N$. Suppose that $H \neq 1$. Then H is a normal non-identity subgroup of N. Hence N is soluble, contrary to (1). Finally, consider the case when $H = 1$. In this case we have $A_2' \leq C_G(N) \leq N$, so $A_2' = 1$ by (1). This contradiction completes the proof of the result. □

Corollary 3.2. *Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are soluble subgroups of G. If the three indices $|G : N_G(A_1)|$, $|G : N_G(A_2)|$ and $|G : N_G(A_3)|$ are pairwise coprime, then G is soluble.*

Corollary 3.3 (H. Wielandt). *If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself soluble.*

Proposition 3.4. *Let $\mathfrak{M}$ be an r-multiply saturated formation and let, for some $r \geq 0$, $\mathfrak{N} \subseteq \mathfrak{M} \subseteq \mathfrak{N}^{r+1}$. Then, for any prime p, both formations $\mathfrak{M}$ and $\mathfrak{G}_p \circ \mathfrak{M}$ are Σ_{r+3}-closed.*

Proof. Let M be the canonical local satellite of the formation $\mathfrak{M}$. Let $\mathfrak{F}$ be one of the formations $\mathfrak{M}$ or $\mathfrak{G}_p \circ \mathfrak{M}$. Let G be any group such that for some subgroups $H_1, \dots, H_{r+3}$ of G whose indices $|G : H_1|, \dots, |G : H_{r+3}|$ are pairwise coprime we have $H_1, \dots, H_{r+3} \in \mathfrak{F}$. We shall prove that $G \in \mathfrak{F}$. Suppose that this is false and let G be a counterexample with $r + |G|$ minimal. Let N be a minimal normal subgroup of G.

(1) *$N = G^{\mathfrak{F}}$ is the only minimal normal subgroup of G and $N \leq O_q(G)$ for some prime q. Hence if $\mathfrak{F} = \mathfrak{G}_p \circ \mathfrak{M}$, then $q \neq p$.*

It is clear that the hypothesis holds for G/N, so $G/N \in \mathfrak{F}$ by the choice of G. Hence $N = G^{\mathfrak{F}}$ since $G \notin \mathfrak{F}$. Moreover, N is a q-group for some prime q since G is soluble by Proposition 3.1. Finally, if $\mathfrak{F} = \mathfrak{G}_p \circ \mathfrak{M}$ and $p = q$, then

$$G \in \mathfrak{G}_p \circ (\mathfrak{G}_p \circ \mathfrak{M}) = \mathfrak{G}_p \circ \mathfrak{F} = \mathfrak{F},$$

a contradiction. Hence we have (1).

Since the indices $|G : H_1|, \dots, |G : H_{r+3}|$ are pairwise coprime, in view of (1) we may assume without loss of generality that $N \leq H_i$ for all $i = 2, \dots, r+3$.

(2) $C_G(N) = N$.

First we show that $N \not\leq \Phi(G)$. Suppose that $N \leq \Phi(G)$. If $r > 0$, then $\mathfrak{F}$ is saturated by Lemma 2.8, so $G \in \mathfrak{F}$. This contradiction shows that $r = 0$ and so $\mathfrak{F} = \mathfrak{G}_p \circ \mathfrak{M}$ by Lemma 2.10 and the choice of G. Hence $q \neq p$ by (1). Let $O/N = O_p(G/N)$ and P be a Sylow p-subgroup of O. Then

$$G = ON_G(P) = NPN_G(P) = NN_G(P) = N_G(P)$$

by the Frattini Argument since $N \leq \Phi(G)$. Hence in view of (1), $O_p(G/N) = 1$ and so $G/N \in \mathfrak{M}$ since $G/N \in \mathfrak{F} = \mathfrak{G}_p \circ \mathfrak{M}$. But then G is a p'-group. Hence $H_1, H_2, H_3 \in \mathfrak{M}$. Thus $G \in \mathfrak{M} \subseteq \mathfrak{F}$ by Lemma 2.10. This contradiction shows that $N \not\leq \Phi(G)$. But then $C_G(N) = N$ by (1) and [2, Chapter A, Theorem 15.2].

(3) $r > 0$.

Suppose that $r = 0$. Then $\mathfrak{F} = \mathfrak{G}_p \circ \mathfrak{M}$, where $\mathfrak{M}$ is a formation of nilpotent groups. Since $N \leq H_2 \in \mathfrak{F}$ and, by (2), $C_G(N) = N$, $O_p(H_2) = 1$. Hence H_2 is a p'-group. Similarly, H_3 is a p'-group. Hence $G = H_1 H_2$ is a p'-group. But then $H_1 \in \mathfrak{M}$, so $G \in \mathfrak{F}$ by Lemma 2.10. This contradiction shows that we have (3).

(4) $H_i/N \in M(q)$ *for all* $i = 2, \ldots, r+3$.

Let $i \in \{2, \ldots, r+3\}$. Then $H_i \in \mathfrak{M}$. Indeed, if $\mathfrak{F} = \mathfrak{G}_p \circ \mathfrak{M}$, then $q \neq p$ by (1). On the other hand, in view of (2), we get $C_G(N) = N$. Hence $O_p(H_i) = 1$, which implies that $H_i \in \mathfrak{M}$. But then, by claim (2) and Lemma 2.7, we conclude that $H_i/N = H_i/C_{H_i}(N) \in M(q)$.

(5) $G/N \in M(q)$.

By Lemma 2.11 and [2, Chapter IV, Proposition 3.8], $M(q) = \mathfrak{G}_q \circ \mathfrak{M}_0$, where $\mathfrak{M}_0 = l_{r-1}\text{form }(G/O_{q',q}(G) | G \in \mathfrak{M})$. Since $\mathfrak{M} \subseteq \mathfrak{N}^{r+1}$, $G/O_{q',q}(G) \in \mathfrak{N}^r$, so $\mathfrak{M}_0 \subseteq \mathfrak{N}^r$ since $\mathfrak{M}_0$ is an $(r-1)$-multiply saturated formation. Therefore the minimality of $r + |G|$ and claim (4) imply that $G/N \in M(q)$.

Final contradiction. Since N is a q-group by (1), from claim (5) it follows that $G \in \mathfrak{G}_q \circ M(q) = M(q) \subseteq \mathfrak{M} \subseteq \mathfrak{G}_p \circ \mathfrak{M}$. This contradiction completes the proof of the proposition. □

Corollary 3.5 (see [12, Satz 1.3]). *Every saturated formation contained in* $\mathfrak{N}^2$ *is* Σ_4*-closed.*

Corollary 3.6. *The class of all soluble groups of nilpotent length at most* r $(r \geq 2)$ *is* Σ_{r+2}*-closed.*

Proof. It is clear that $\mathfrak{N}^r$ is a hereditary formation. Moreover, in view of Lemma 2.8, $\mathfrak{N}^r$ is an $(r-1)$-multiply saturated formation. So $\mathfrak{N}^r$ is Σ_{r+2}-closed by Proposition 3.4. □

Proof of Theorem A. Suppose that the theorem is false and consider some counterexample G of minimal order. Take a maximal subgroup M of G. By hypothesis, all $(n-1)$-maximal subgroups of M are $\mathfrak{F}$-subnormal in G, and so they are $\mathfrak{F}$-subnormal in M by Lemmas 2.1 (1) and 2.12. The solubility of G implies that either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$, so $M \in \mathfrak{F}$ by the choice of G. Hence G is an $\mathfrak{F}$-critical group.

Since G is soluble, G has a maximal subgroup T with $|G : T| = p^a$ for any prime p dividing $|G|$. On the other hand, $\mathfrak{F}$ is Σ_{r+3}-closed by Proposition3.4. Hence $|\pi(G)| \leq r + 2$. Moreover, by hypothesis, $|\pi(G)| \geq n + r + 1$. Therefore $n = 1$. Thus all maximal subgroups of G are $\mathfrak{F}$-normal, so $G/\Phi(G) \in \mathfrak{F}$. But $\mathfrak{F}$ is a saturated formation and hence $G \in \mathfrak{F}$. This contradiction completes the proof of the result. □

Corollary 3.7 (see [17, Theorem 6]). *If each* n*-maximal subgroup of a soluble group* G *is subnormal and* $|\pi(G)| \geq n + 1$, *then* G *is nilpotent.*

Corollary 3.8 (see [11, Theorem A]). *If every n-maximal subgroup of a soluble group G is $\mathfrak{U}$-subnormal in G and $|\pi(G)| \geq n+2$, then G is supersoluble.*

Corollary 3.9. *Let $\mathfrak{F}$ be the class of all groups G with $G' \leq F(G)$. If every n-maximal subgroup of a soluble group G is $\mathfrak{F}$-subnormal in G and $|\pi(G)| \geq n+2$, then $G \in \mathfrak{F}$.*

Corollary 3.10. *If every n-maximal subgroup of a soluble group G is $\mathfrak{N}^r$-subnormal in G $(r \geq 1)$ and $|\pi(G)| \geq n+r$, then $G \in \mathfrak{N}^r$.*

4 Proofs of Theorems B, C, and D

Proof of Theorem C. First suppose that every 2-maximal subgroup of G is $\mathfrak{F}$-subnormal in G. Assume that $G \notin \mathfrak{F}$. We shall show that G is an $\mathfrak{F}$-critical group and $G^{\mathfrak{F}}$ is a minimal normal subgroup of G. Let M be a maximal subgroup of G and T be a maximal subgroup of M. By hypothesis, T is $\mathfrak{F}$-subnormal in G. Therefore T is $\mathfrak{F}$-normal in M by Lemma 2.1 (1), so $M/T_M \in \mathfrak{F}$. Since T is arbitrary and $\mathfrak{F}$ is saturated, $M \in \mathfrak{F}$. Consequently, all maximal subgroups of G belong to $\mathfrak{F}$. Hence G is an $\mathfrak{F}$-critical group. Then, by hypothesis, G is soluble and it has a normal Sylow p-subgroup $G_p \neq 1$ for some prime p. Thus $G_p = G^{\mathfrak{F}}$ by Lemma 2.5. On the other hand, by Lemma 2.4, $G_p/\Phi(G_p)$ is a chief factor of G.

Let M be an $\mathfrak{F}$-abnormal maximal subgroup of G. Then $G_p \nleq M$ by Lemma 2.1 (4), so $G = G_pM$ and $M = (G_p \cap M)G_{p'} = \Phi(G_p)G_{p'}$, where $G_{p'}$ is a Hall p'-subgroup of G. Assume that $\Phi(G_p) \neq 1$. It is clear that $\Phi(G_p) \nleq \Phi(M)$. Let T be a maximal subgroup of M such that $\Phi(G_p) \nleq T$. Then $M = \Phi(G_p)T$. Since T is $\mathfrak{F}$-subnormal in G, there is a maximal subgroup L of G such that $T \leq L$ and $G/L_G \in \mathfrak{F}$. Then $G_p \leq L_G$, so

$$G = G_pM = G_p\Phi(G_p)T = G_pT \leq L,$$

a contradiction. Hence $\Phi(G_p) = 1$. Therefore $G_p = G^{\mathfrak{F}}$ is a minimal normal subgroup of G.

Now suppose that G is an $\mathfrak{F}$-critical group and $G^{\mathfrak{F}}$ is a minimal normal subgroup of G. Let T be a 2-maximal subgroup of G and M be a maximal subgroup of G such that T is a maximal subgroup of M. Since $M \in \mathfrak{F}$, T is $\mathfrak{F}$-subnormal in M by Lemma 2.2. Therefore, if M is $\mathfrak{F}$-normal in G, then T is $\mathfrak{F}$-subnormal in G by Lemma 2.1 (3). Assume that M is $\mathfrak{F}$-abnormal in G. Then $G^{\mathfrak{F}} \nleq M$ by Lemma 2.1 (4). Therefore, since $G^{\mathfrak{F}}$ is a minimal normal subgroup of G by hypothesis, $G = G^{\mathfrak{F}} \rtimes M$ and $G^{\mathfrak{F}}T$ is a maximal $\mathfrak{F}$-normal subgroup of G. Moreover, since G is an $\mathfrak{F}$-critical group, $G^{\mathfrak{F}}T \in \mathfrak{F}$ and hence T is $\mathfrak{F}$-subnormal in $G^{\mathfrak{F}}T$ by Lemma 2.2. Hence T is $\mathfrak{F}$-subnormal in G. The theorem is proved. □

From Theorem C and Lemma 2.6 we get

Corollary 4.1 (see [11, Theorem 3.1]). *Every 2-maximal subgroup of G is $\mathfrak{U}$-subnormal in G if and only if G is a $\mathfrak{U}$-critical group and $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

Proof of Theorem B. First suppose that all n-maximal subgroups of G are $\mathfrak{F}$-subnormal in G. We shall show, in this case, that either $G \in \mathfrak{F}$ or G is a group of type II. Assume that this is false and consider a counterexample G for which $|G| + n$ is minimal. Therefore $A = G^{\mathfrak{F}} \neq 1$. Then:

(a) *The hypothesis holds for every maximal subgroup of G.*

Let M be a maximal subgroup of G. By hypothesis, all $(n-1)$-maximal subgroups of M are $\mathfrak{F}$-subnormal in G, and so they are $\mathfrak{F}$-subnormal in M by Lemmas 2.1 (1) and 2.12. Moreover, the solubility of G implies that one has either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$.

(b) *If M is a maximal subgroup of G and $|\pi(M)| = |\pi(G)|$, then $M \in \mathfrak{F}$.*

In view of hypothesis and Lemmas 2.1 (1) and 2.12, all $(n-1)$-maximal subgroups of M are $\mathfrak{F}$-subnormal in M. Since $|\pi(M)| = |\pi(G)| \geq n+1 = n-1+2$, $M \in \mathfrak{F}$ by Theorem A.

(c) *If W is a Hall q'-subgroup of G for some $q \in \pi(G)$, then either $W \in \mathfrak{F}$ or W is a group of type II.*

If W is not a maximal subgroup of G, then there is a maximal subgroup V of G such that $W \leq V$ and $|\pi(V)| = |\pi(G)|$. By (b), $V \in \mathfrak{F}$. Hence $W \in \mathfrak{F}$ by Lemma 2.12. Suppose that W is a maximal subgroup of G. Then, by (a), the hypothesis holds for W, so either $W \in \mathfrak{F}$ or W is a group of the type II by the choice of G.

(d) *The hypothesis holds for G/N, where N is a minimal normal subgroup of G.*

Arguing similarly as in the proof of necessity in [11, Theorem B], we see that in the case when N is not a Sylow subgroup of G the hypothesis holds for G/N. Consider the case when N is a Sylow p-subgroup of G. Let E be a Hall p'-subgroup of G. It is clear that $|\pi(E)| = |\pi(G)| - 1$ and E is a maximal subgroup of G. Let H/N be an $(n-1)$-maximal subgroup of G/N. Then H is an $(n-1)$-maximal subgroup of G and $H = H \cap NE = N(H \cap E)$. There is a chain of subgroups $H = H_0 < H_1 < \cdots < H_{n-1} = G$ of G, where H_{i-1} is a maximal subgroup of H_i $(i = 1, \ldots, n-1)$. Then $H_{i-1} \cap E$ is a maximal subgroup of $H_i \cap E$, for $i = 1, \ldots, n-1$. Indeed, suppose that for some i there is a subgroup K of $H_i \cap E$ such that $H_{i-1} \cap E \leq K \leq H_i \cap E$. Then $(H_{i-1} \cap E)N \leq KN \leq (H_i \cap E)N$, so $H_{i-1} = H_{i-1} \cap EN \leq KN \leq H_i \cap EN = H_i$. Whence either $KN = H_{i-1}$

or $KN = H_i$. If $KN = H_{i-1}$, then $H_{i-1} \cap E = KN \cap E = K(N \cap E) = K$. In the second case we have $H_i \cap E = KN \cap E = K(N \cap E) = K$. Therefore $H_{i-1} \cap E$ is a maximal subgroup of $H_i \cap E$, so $H \cap E$ is an $(n-1)$-maximal subgroup of E. Since E is a maximal subgroup of G, $H \cap E$ is an n-maximal subgroup of G. Hence $H \cap E$ is $\mathfrak{F}$-subnormal in G by hypothesis. Therefore $H/N = (H \cap E)N/N$ is $\mathfrak{F}$-subnormal in G/N by Lemma 2.1(2).

(e) $|\pi(G)| > 2$.

If $|\pi(G)| = 2$, then $n = 1$ and so all maximal subgroups of G are $\mathfrak{F}$-normal in G by hypothesis. Hence $G \in \mathfrak{F}$ since $\mathfrak{F}$ is a saturated formation, a contradiction.

(f) *G is an Ore dispersive group* (see Claim (a) in the proof of [11, Theorem B]).

(g) *A is a nilpotent group.*

Suppose that this is false. Let N be a minimal normal subgroup of G. Then by claim (d), $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N \simeq G^{\mathfrak{F}}/G^{\mathfrak{F}} \cap N$ is a nilpotent group. It is known that the class of all nilpotent groups is a saturated formation. Hence in the case when G has a minimal normal subgroup $R \neq N$ we have that the group $G^{\mathfrak{F}}/(G^{\mathfrak{F}} \cap N) \cap (G^{\mathfrak{F}} \cap R) \simeq G^{\mathfrak{F}}$ is nilpotent. Thus N is the unique minimal normal subgroup of G and $N \leq G^{\mathfrak{F}}$. If $N \leq \Phi(G)$, then $G^{\mathfrak{F}}/G^{\mathfrak{F}} \cap \Phi(G) \simeq (G^{\mathfrak{F}}/N)/((G^{\mathfrak{F}} \cap \Phi(G))/N)$ is nilpotent, so $G^{\mathfrak{F}}$ is nilpotent by Lemma 2.14. Therefore $N \nleq \Phi(G)$. Hence $\Phi(G) = 1$ and there is a maximal subgroup L of G such that $G = N \rtimes L$ and $L_G = 1$. Thus $C_G(N) = N$ by [2, A, Theorem 15.2] and $N \neq A$.

Case 1: $|\pi(G)| = 3$. By hypothesis, either all maximal subgroups of G or all its 2-maximal subgroups are $\mathfrak{F}$-subnormal in G. In the first case we infer that $G \in \mathfrak{F}$, which contradicts the choice of G. Hence all 2-maximal subgroups of G are $\mathfrak{F}$-subnormal. Since $\mathfrak{F} \subseteq \mathfrak{U}$, in view of Lemma 2.6, every $\mathfrak{F}$-critical group has a normal Sylow subgroup. Whence Theorem C and Lemma 2.12 imply that G is an $\mathfrak{F}$-critical group and $A = G^{\mathfrak{F}}$ is a minimal normal subgroup of G. Therefore $A = N$, a contradiction.

Case 2: $|\pi(G)| \geq 4$. Assume that N is a p-group, and take a Sylow subgroup P of G such that $N \leq P$. Observe that if $N \neq P$, then $L \in \mathfrak{F}$ by (b), and so $A = N$, a contradiction. Hence $N = P$.

Case 2.1: $|\pi(G)| = 4$. (1) *All 3-maximal subgroups of G are $\mathfrak{F}$-subnormal in G and L is an $\mathfrak{F}$-critical group.*

Since $G \notin \mathfrak{F}$ and $|\pi(G)| = 4$, either all 2-maximal subgroups of G or all its 3-maximal subgroups are $\mathfrak{F}$-subnormal in G. In the first case G is an $\mathfrak{F}$-critical group and $A = G^{\mathfrak{F}}$ is a minimal normal subgroup of G by Theorem C and Lemma 2.12. Hence we get $A = N$, a contradiction. Therefore all 3-maximal subgroups of G

are $\mathfrak{F}$-subnormal in G. Thus all second maximal subgroups of G belong to $\mathfrak{F}$ by Lemmas 2.3 and 2.12. Consequently, either $L \in \mathfrak{F}$ or L is an $\mathfrak{F}$-critical group. But in the first case $N = A$, a contradiction. Therefore L is an $\mathfrak{F}$-critical group.

(2) *$L = Q \rtimes (R \rtimes T)$, where Q, R, T are Sylow subgroups of G, $Q = L^{\mathfrak{F}}$ is a minimal normal subgroup of L, and $G^{\mathfrak{F}} = PQ$.*

Since $N = P$ is a Sylow p-subgroup of G and $|\pi(G)| = 4$, $|\pi(L)| = 3$. Hence in view of (f), $L = Q \rtimes (R \rtimes T)$, where Q, R, T are Sylow subgroups of G. Moreover, $Q = L^{\mathfrak{F}}$ by Lemma 2.5 and Q is a minimal normal subgroup of L by Theorem C and Lemma 2.12 since every 2-maximal subgroup of L is $\mathfrak{F}$-subnormal in L by (1) and Lemmas 2.1 (1) and 2.12. Finally, since $G/N \notin \mathfrak{F}$ and $G/PQ \simeq L/Q \in \mathfrak{F}$, we have $G^{\mathfrak{F}} = PQ$.

(3) *$V = PQR$ is not supersoluble. Hence $V \notin \mathfrak{F}$.*

Assume that V is a supersoluble group. Since $F(V)$ is a characteristic subgroup of V and V is a normal subgroup of G, $F(V)$ is normal in G. Hence every Sylow subgroup of $F(V)$ is normal in G. But N is the unique minimal normal subgroup of G. Therefore $F(V) = N = P$. Thus $V/P \simeq QR$ is an abelian group. Hence R is normal in L and so $R \leq F(L)$. In view of Lemma 2.5, $F(L) = Q\Phi(L)$. Whence $R \leq \Phi(L)$. This contradiction shows that V is not supersoluble. Thus $V \notin \mathfrak{F}$ since $\mathfrak{F} \subseteq \mathfrak{U}$ by hypothesis.

(4) *V is a maximal subgroup of G. Hence $|T| = t$ is a prime.*

If V is not a maximal subgroup of G, then there is a maximal subgroup U of G such that $V \leq U$ and $|\pi(U)| = |\pi(G)|$. Hence $U \in \mathfrak{F}$ by (b), so $V \in \mathfrak{F}$ by Lemma 2.12, a contradiction. Therefore V is a normal maximal subgroup of G. Whence $|T|$ is a prime.

(5) *$|Q| = q$ is a prime and $R = \langle x \rangle$ is a cyclic group.*

Since V is a maximal subgroup of G by (4), all 2-maximal subgroups of V are $\mathfrak{F}$-subnormal in V by (1) and Lemmas 2.1 (1) and 2.12. Hence, in view of (3), V is an $\mathfrak{F}$-critical group by Theorem C and Lemma 2.12. Therefore, in fact, V is a $\mathfrak{U}$-critical group by (3) since $\mathfrak{F} \subseteq \mathfrak{U}$. Hence QR is supersoluble. Since V is normal in G and $\Phi(G) = 1$, $\Phi(V) = 1$. Therefore QR is a Schmidt group by Lemma 2.6. Hence R is cyclic and Q is a minimal normal subgroup of QR by Lemma 2.4. Whence $|Q|$ is a prime.

(6) *$|R| = r$ is a prime and $C_L(Q) = Q$.*

By (5), L is a supersoluble group. Suppose that $|R| = r^b$ is not a prime and let M be a maximal subgroup of L such that $|L : M| = r$. Let $W = PM$. Then $\pi(W) = \pi(G)$, so $W \in \mathfrak{F}$ by (b) and hence W is supersoluble. As $C_G(N) = N$, it follows that $F(W) = P$. Hence $W/P \simeq M$ is abelian. It is clear that $Q \leq M$, so $M \leq C_L(Q)$. Hence $T \leq F(L)$. On the other hand, we have $F(L) = Q\Phi(L)$

by Lemma 2.5. Therefore $T \not\leq F(L)$. This contradiction shows that $|R| = r$ and so $C_L(Q) = Q$ by Lemma 2.5.

(7) $1 \neq C_G(x) \cap PQ = P_1 \leq P$.

Suppose that $C_G(x) \cap PQ = 1$. Then, by Thompson's theorem [18, Theorem 10.5.4], PQ is a nilpotent group, so $Q \leq C_G(P) = P$, a contradiction. Thus $C_G(x) \cap PQ \neq 1$. Suppose that q divides $|C_G(x) \cap PQ|$. Then, by (5), for some $a \in P$ we have $Q^a \leq C_G(x) \cap PQ$, so $\langle Q^a, RT\rangle \leq N_G(R)$. Hence if E is a Hall p'-subgroup of $N_G(R)$, then $E \simeq L$. Therefore L has a normal r-subgroup, so $C_L(Q) \neq Q$, a contradiction. Thus $C_G(x) \cap PQ = P_1 \leq P$.

Final contradiction for Case 2.1. Let $D = \langle P_1, RT\rangle$. Then $D \leq N_G(R)$. If q divides $|D|$, then, as above, we have $C_L(Q) \neq Q$. Thus $D \cap Q^a = 1$ for all $a \in P$. Moreover, if $P \leq D$, then $PR = P \times R$ and $R \leq C_G(P) = P$. Therefore $P \not\leq D$ and D is not a maximal subgroup of G. Hence D is a k-maximal subgroup of G for some $k \geq 2$. Then there is a 3-maximal subgroup S of G such that $RT \leq S \leq D$. By hypothesis, S is $\mathfrak{F}$-subnormal in G. Hence at least one of the maximal subgroups L or PRT is $\mathfrak{F}$-normal in G, contrary to (2).

Case 2.2: $|\pi(G)| > 4$. If $\pi(L) = \{p_1, \dots, p_t\}$, then $t > 3$. Let E_i be a Hall p_i'-subgroup of L and $X_i = PE_i$. We shall show that $E_i \in \mathfrak{F}$ for all $i = 1, \dots, t$. By (c), either $X_i \in \mathfrak{F}$ or X_i is a group of type II, for $i = 1, \dots, t$. In the former case we have $E_i \simeq X_i/P \in \mathfrak{F}$. Assume that X_i is a group of type II. Then $X_i^{\mathfrak{F}}$ is nilpotent, so $X_i^{\mathfrak{F}} \leq F(X_i)$. But since P is normal in X_i and $C_G(P) = P$, $F(X_i) = P$. Hence $X_i^{\mathfrak{F}} = P$, so $E_i \in \mathfrak{F}$. Since $t > 3$, Proposition 3.4 implies that $L \in \mathfrak{F}$. Therefore $A = N$, a contradiction. Hence we have (g).

(h) *A is a Hall subgroup of G.*

Suppose that this is false. Since G is Ore dispersive by (f), for the greatest prime divisor p of $|G|$ the Sylow p-subgroup P is normal in G. Assume that P is not a minimal normal subgroup of G. Then there is a maximal subgroup M of G such that $G = PM$ and $P \cap M \neq 1$. Since $|\pi(M)| = |\pi(G)|$, $M \in \mathfrak{F}$ by (b). Hence $G/P \simeq M/M \cap P \in \mathfrak{F}$, so $A = G^{\mathfrak{F}} \leq P$. Suppose that $\Phi(P) \neq 1$. Let N be a minimal normal subgroup of G such that $N \leq \Phi(P)$. By (d), the hypothesis holds for G/N, so either $G/N \in \mathfrak{F}$ or G/N is a group of type II by the choice of G. If $G/N \in \mathfrak{F}$, then $A = N \leq \Phi(P)$. Since P is normal in G, $\Phi(P) \leq \Phi(G)$. Thus $A \leq \Phi(G)$ and so $G \in \mathfrak{F}$, a contradiction. Hence G/N is a group of type II. Therefore $AN/N = G^{\mathfrak{F}}N/N = (G/N)^{\mathfrak{F}}$ is a Hall subgroup of G/N. Consequently, $AN = P$. Hence $A\Phi(P) = P$, so $A = P$, a contradiction. Thus $\Phi(P) = 1$. By Maschke's theorem, $P = N_1 \times \cdots \times N_k$ is the direct product of some minimal normal subgroups of G. If $N_1 \neq P$, then $G/N_1 \in \mathfrak{F}$

and $G/N_2 \in \mathfrak{F}$ by Theorem A. Consequently, so is G. This contradiction shows that P is a minimal normal subgroup of G.

By (d), the hypothesis holds for G/P, so either $G/P \in \mathfrak{F}$ or G/P is a group of type II by the choice of G. If $G/P \in \mathfrak{F}$, then $A = P$, a contradiction. Hence G/P is a group of type II. Therefore $AP/P = G^{\mathfrak{F}}P/P = (G/P)^{\mathfrak{F}}$ is a Hall subgroup of G/P. If $P \leq A$, then $A = P \rtimes A_{p'}$, where $A_{p'}$ is a Hall p'-subgroup of A. But since $A_{p'} \simeq A/P$ and A/P is a Hall subgroup of G/P, A is a Hall subgroup of G. Therefore $P \cap A = 1$, so A is a Hall subgroup of G since $AP/P \simeq A/A \cap P \simeq A$.

(i) *A is either of the form $N_1 \times \cdots \times N_t$, where each N_i is a minimal normal subgroup of G, which is a Sylow subgroup of G, for $i = 1, \ldots, t$, or a Sylow p-subgroup of G of exponent p for some prime p and the commutator subgroup, the Frattini subgroup, and the center of A coincide, while $A/\Phi(A)$ is an $\mathfrak{F}$-eccentric chief factor of G.*

If A is not a minimal normal subgroup of G, then arguing similarly as in the proof of claim (c) in [11, Theorem B], we have (i).

(j) *Every n-maximal subgroup of G belongs to $\mathfrak{F}$ and induces on the Sylow p-subgroup of A the automorphism group which is contained in $F(p)$ for every prime divisor p of $|A|$.*

Let H be any n-maximal subgroup of G. Suppose that H is a maximal subgroup of V, where V is an $(n-1)$-maximal subgroup of G. Since $V \in \mathfrak{F}$ by Lemmas 2.3 and 2.12, $H \in \mathfrak{F}$.

Let $E = AH$. Since A is normal in E and A is nilpotent by (g), $A \leq F(E)$. Whence $E = F(E)H$. Since H is $\mathfrak{F}$-subnormal in G, H is $\mathfrak{F}$-subnormal in E by Lemmas 2.1 (1) and 2.12. Therefore $E \in \mathfrak{F}$ by Lemma 2.15. Let P be a Sylow p-subgroup of A and K/L a chief factor of E such that $1 \leq L < K \leq P$. Since $E \in \mathfrak{F}$, $E/C_E(K/L) \in F(p)$. Hence $P \leq Z_{\mathfrak{F}}(E)$, so $E/C_E(P) \in F(p)$ by Lemma 2.7. Then

$$H/C_H(P) = H/C_E(P) \cap H \simeq HC_E(P)/C_E(P) \in F(p).$$

Now suppose that either $G \in \mathfrak{F}$ or G is a group of type II. If $G \in \mathfrak{F}$, then every subgroup of G is $\mathfrak{F}$-subnormal in G by Lemmas 2.2 and 2.12. Let G be a group of type II. Take an n-maximal subgroup H of G. Put $E = G^{\mathfrak{F}}H$. Let P be a Sylow p-subgroup of $G^{\mathfrak{F}}$ and K/L a chief factor of E such that $1 \leq L < K \leq P$. By hypothesis, $H/C_H(P) \in F(p)$, so

$$H/C_H(K/L) \simeq (H/C_H(P))/(C_H(K/L)/C_H(P)) \in F(p).$$

Since $G^{\mathfrak{F}}$ is normal in E and $G^{\mathfrak{F}}$ is nilpotent,

$$G^{\mathfrak{F}} \leq F(E) \leq C_E(K/L).$$

Hence

$$\begin{aligned} E/C_E(K/L) &= E/C_E(K/L) \cap E \\ &= E/C_E(K/L) \cap G^{\mathfrak{F}}H \\ &= E/G^{\mathfrak{F}}(C_E(K/L) \cap H) \\ &= G^{\mathfrak{F}}H/G^{\mathfrak{F}}C_H(K/L) \\ &\simeq H/G^{\mathfrak{F}}C_H(K/L) \cap H \\ &= H/C_H(K/L)(G^{\mathfrak{F}} \cap H) \\ &= H/C_H(K/L), \end{aligned}$$

so $E/C_E(K/L) \in F(p)$ since $F(p)$ is hereditary by Lemma 2.12 and [1, Proposition 3.1.40]. Then $P \leq Z_{\mathfrak{F}}(E)$, whence $G^{\mathfrak{F}} \leq Z_{\mathfrak{F}}(E)$. Thus $E/Z_{\mathfrak{F}}(G) \in \mathfrak{F}$. Hence $E \in \mathfrak{F}$, so H is an $\mathfrak{F}$-subnormal subgroup of $G^{\mathfrak{F}}H = E$ by Lemmas 2.2 and 2.12. Since $G^{\mathfrak{F}} \leq G^{\mathfrak{F}}H$, $G^{\mathfrak{F}}H$ is $\mathfrak{F}$-subnormal in G by Lemmas 2.1 (4) and 2.12. Consequently, in view of Lemma 2.1 (3), H is $\mathfrak{F}$-subnormal in G. The theorem is proved. □

Corollary 4.2 (see [11, Theorem B]). *Given a soluble group G with the property that $|\pi(G)| \geq n + 1$, all n-maximal subgroups of G are $\mathfrak{U}$-subnormal in G if and only if G is a group of one of the following types:*

(I) *G is supersoluble.*

(II) *$G = A \rtimes B$, where $A = G^{\mathfrak{U}}$ and B are Hall subgroups of G, while G is Ore dispersive and satisfies the following:*

  (1) *A is either of the form $N_1 \times \dots \times N_t$, where each N_i is a minimal normal subgroup of G, which is a Sylow subgroup of G, for $i = 1, \dots, t$, or a Sylow p-subgroup of G of exponent p for some prime p and the commutator subgroup, the Frattini subgroup, and the center of A coincide, every chief factor of G below $\Phi(G)$ is cyclic, while $A/\Phi(A)$ is a noncyclic chief factor of G,*

  (2) *for every prime divisor p of the order of A every n-maximal subgroup H of G is supersoluble and induces on the Sylow p-subgroup of A an automorphism group which is an extension of some p-group by abelian group of exponent dividing $p - 1$.*

For a proof of Theorem D see the proof of [11, Theorem C].

Finally, note that there are examples which show that the restrictions on $|\pi(G)|$ in Theorems A, B, and D cannot be weakened.

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