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FINITE GROUPS WITH GIVEN SYSTEMS OF K-$\mathfrak{U}$-SUBNORMAL SUBGROUPS

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A subgroup H of a finite group G is called $\mathfrak{U}$-subnormal in Kegel's sense or K-$\mathfrak{U}$-subnormal in G if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \ldots \leq H_t = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is supersoluble for any $i = 1, \ldots, t$. We describe finite groups for which every 2-maximal or every 3-maximal subgroup is K-$\mathfrak{U}$-subnormal.

1. Introduction

All groups considered in the present paper are finite, the symbol G denotes a finite group. By $\mathfrak{U}$ we denote the class of all supersolvable groups, the symbol $G^{\mathfrak{U}}$ denotes the intersection of all normal subgroups N of G for which $G/N \in \mathfrak{U}$, and the symbol $\pi(G)$ denotes the set of prime divisors of the order G.

Let ϕ be an ordered set of prime numbers. The notation $p\phi q$ means that p precedes q in the ordering ϕ, $p \neq q$. Recall that a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ is called ϕ-dispersive if $p_1 \phi p_2 \phi \ldots \phi p_n$ and, for any i, the group G has a normal subgroup of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$. Moreover, if the ordering ϕ is such that $p\phi q$ always implies that $p > q$, then a ϕ-dispersive group is called Ore dispersive.

A subgroup H of G is called a 2-maximal *(second maximal)* subgroup of G if H is a maximal subgroup of some maximal subgroup of G. Similarly, we can define 3-*maximal subgroups,* etc.

The works devoted to the study of n-maximal subgroups $(n > 1)$ form an extensively developed branch of the theory of finite groups enriched with a great number of fundamental theorems and informative examples. The first results in this direction were obtained by Rédei [1] who described unsolvable groups with Abelian second maximal subgroups and by Huppert [2] who established the supersolvability of a group for which all second maximal subgroups are normal. In addition, Huppert proved that if all 3-maximal subgroups of G are normal in G, then the commutant G' is a nilpotent group and the principal rank of G does not exceed 2. Later, the Rédei and Huppert results were generalized by numerous researchers (Yanko, Suzuki, Gagen, Deskins, Mann, Spenser, Schmidt, Vedernikov, Pal'chik, Kontotovich, Berkovich, Agrawal, Asaad, Flavell, et al.).

In recent years, the number of mathematicians studying the n-maximal groups considerably increased (Ballester-Bolinches, Ezquerro, W. Guo, X. Guo, Shum, B. Li, Sh. Li, Belonogov, Vasil'ev, Vasil'eva, Monakhov, Semenchuk, Skiba, Tyutyanov, Knyagina, Murashko, Andreeva, Lutsenko, Legchekova, et al.), which reveals the undoubted urgency of this direction. Thus, in [3], X. Guo and Shum proved that G is solvable if all its 2-maximal subgroups have the cover-avoidance property. In [4, 5, 6], W. Guo, Shum, Skiba, and B. Li obtained new characterizations of supersolvable groups in terms of 2-maximal subgroups. In [7], Sh. Li proposed a classification of nonnilpotent groups for which all 2-maximal subgroups are TI-subgroups. In [8], Belonogov gave a description of π-nondecomposable groups in which all 2-maximal subgroups are π-decomposable. In [9], W. Guo, Lutsenko, and Skiba described nonnilpotent groups in which any two 3-maximal subgroups are permutable. The description of the groups all 2-maximal or all 3-maximal subgroups of which are subnormal can be found in [10]. In [11], Ballester-Bolinches, Ezquerro, and Skiba proposed a new classification of groups in which the second maximal subgroups of the Sylow subgroups cover or isolate the principal factors of some basic series. In [12], Knyagina and Monakhov studied the groups for which every n-maximal subgroup is permutable with any Schmidt subgroup.

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In particular, they established the fact that if $n = 1, 2, 3,$ then a group is metanilpotent; if $n \geq 4$ and the group is solvable, then the nilpotent length of the group does not exceed $n - 1$. In [13], Monakhov and Knyagina, investigated the groups in which all 2-maximal subgroups are $\mathbb{P}$-subnormal.

Recall that a subgroup H of G is called $\mathfrak{U}$-*subnormal* in G if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$ for all $i = 1, \ldots, n$. A subgroup H is called $\mathfrak{U}$-*subnormal in Kegel's sense* [14] or K-$\mathfrak{U}$-*subnormal* (see [15, p. 236]) in G if one can find a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_t = G$$

such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$ for all $i = 1, \ldots, t$. It is clear that each $\mathfrak{U}$-subnormal subgroup is K-$\mathfrak{U}$-subnormal. The converse assertion is also true for a solvable group G. In [16, 17], the authors obtained the characterizations of solvable groups in which all n-maximal subgroups are $\mathfrak{U}$-subnormal and, hence, K-$\mathfrak{U}$-subnormal.

Note that every subnormal subgroup is K-$\mathfrak{U}$-subnormal. The converse statement is, generally speaking, not true. Thus, in a symmetric group of degree 3, a subgroup of order 2 is K-$\mathfrak{U}$-subnormal but, at the same time, it is not subnormal. This elementary example and the results presented in [10, 16, 17] lead to the following natural questions:

Question 1.1. *What is the structure of a group G under the condition that every 2-maximal subgroup of G is K-$\mathfrak{U}$-subnormal?*

Question 1.2. *What is the structure of a group G under the condition that every 3-maximal subgroup of G is K-$\mathfrak{U}$-subnormal?*

An important role in the investigation of Questions 1.1 and 1.2 is played by the minimal onsupersolvable groups. Recall that G is called a *minimal nonsupersolvable* group if G is not supersolvable but each proper subgroup of G is supersolvable. The minimal nonsupersolvable groups were described by Huppert [2] and Doerk [18]. We say that G is a special Doerk–Huppert group or an SDH-*group* if G is a minimal nonsupersolvable group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.

The problem of finding the answer to Question 1.1 goes back to [16, 17]. The following theorem is a corollary of Theorem 3.1 in [16] (or Theorem C in [17]) and Lemma 2.2 (see Sec. 2):

Theorem A. *All 2-maximal subgroups of G are K-$\mathfrak{U}$-subnormal in G if and only if G is either supersolvable or an SDH-group.*

In the present paper, we analyze Question 1.2 on the basis of Theorem A. Since every subgroup of a supersolvable group is K-$\mathfrak{U}$-subnormal, it is, in fact, necessary to consider only the case of a nonsupersolvable group G. In this case, by virtue of Theorem A in [16] or Theorem A in [17], we conclude that

$$|\pi(G)| \leq 4.$$

For $|\pi(G)| = 2$, the answer to Question 1.2 is given in [19] (Theorem 1.2). In the present paper, we give the complete solution of this problem for $|\pi(G)| = 3$ and $|\pi(G)| = 4$.

The following theorems are proved:

Theorem B. *Let G be a nonsupersolvable group with $|\pi(G)| = 3$, let p, q, and r be different prime divisors of $|G|$, let P, Q, and R be a Sylow p-subgroup, a q-subgroup, and an r-subgroup of G, respectively. Every*

3-maximal subgroup of G is K-$\mathfrak{U}$-subnormal in G if and only if G is ϕ-dispersive, e.g., $G = P \rtimes (Q \rtimes R)$, and the following conditions are satisfied:

(i) Every 2-maximal subgroup of QR induces in P an Abelian group of automorphisms with an exponent dividing $p-1$. Every maximal subgroup of QR induces in P a group of automorphisms which is either irreducible or Abelian with exponent dividing $p-1$.

(ii) If P is a minimal normal subgroup of G and $P \neq G^{\mathfrak{U}}$, then either $G^{\mathfrak{U}} = Q$ or $G^{\mathfrak{U}} = PQ$, each proper subgroup of G containing PQ is supersolvable, and R induces on Q an irreducible group of automorphisms. Moreover, $G^{\mathfrak{U}} = Q$ if and only if PR is supersolvable.

(iii) If $\Phi(P) \neq 1$, then $G^{\mathfrak{U}} = P$, $P/\Phi(P)$ is a noncyclic principal factor in G and Q and R are cyclic groups; moreover, r divides $q-1$ and qr divides $p-1$. Furthermore, if G is a minimal nonsupersolvable group, then $|\Phi(P)| = p$. If G is not a minimal nonsupersolvable group, then $\Phi(P)QR$ is an SDH-group and, hence, $\Phi(P)$ is a minimal normal subgroup of G.

(iv) If P is not a minimal normal subgroup of G and $\Phi(P) = 1$, then $P = P_1 \times P_2$, where P_1 and P_2 are minimal normal subgroups of G and at least one of these subgroups is not cyclic. Moreover, in this case, Q and R are cyclic groups, r divides $q-1$, and qr divides $p-1$.

It follows from Theorem 1.2 in [19] that if $|\pi(G)| = 2$, then the nonsupersolvable group G for which all 3-maximal subgroups are K-$\mathfrak{U}$-subnormal may have no normal Sylow subgroups. It follows from Theorem B that, for $|\pi(G)| = 3$, every group G of this kind is ϕ-dispersive for a certain ordering ϕ of the set $\pi(G)$. The following theorem shows that, for $|\pi(G)| = 4$, G is an Ore-dispersive group:

Theorem C. *Let G be a nonsupersolvable group with $|\pi(G)| = 4$, let $p > q > r > t$ be different prime divisors of $|G|$, and let P, Q, R, and T be a Sylow p-subgroup, a q-subgroup, an r-subgroup, and a t-subgroup of G, respectively. Every 3-maximal subgroup of G is K-$\mathfrak{U}$-subnormal in G if and only if the following conditions are satisfied:*

(i) G is a dispersive Ore group.

(ii) P is a minimal normal subgroup of G.

(iii) Every 2-maximal subgroup of QRT induces on P an Abelian group of automorphisms with exponent dividing $p-1$. Every maximal subgroup of QRT induces on P a group of automorphisms, which is either irreducible or Abelian with exponent dividing $p-1$.

(iv) If $P \neq G^{\mathfrak{U}}$, then either $G^{\mathfrak{U}} = Q$ or $G^{\mathfrak{U}} = PQ$, Q is a minimal normal subgroup of G and each proper subgroup of G containing PQ is supersolvable.

(v) R and T are cyclic groups. Moreover, if QRT is supersolvable, then Q is a cyclic group.

In the present paper, we use the standard terminology. For the notation, if necessary, see [15, 20, 21].

2. Preliminary Results

In what follows, we need the following lemmas:

Lemma 2.1. *Let H and K be subgroups of G and let H be K-$\mathfrak{U}$-subnormal in G.*

(i) $H \cap K$ is a K-$\mathfrak{U}$-subnormal subgroup of K [15] [6.1.7(2))].

(ii) If N is a normal subgroup of G, then HN/N is a K-$\mathfrak{U}$-subnormal subgroup of G/N [15] [6.1.6(3)].

(iii) *If K is a K-$\mathfrak{U}$-subnormal subgroup of H, then K is a K-$\mathfrak{U}$-subnormal subgroup of G [15] [6.1.6(1)].*

(iv) *If $G^{\mathfrak{U}} \leq K$, then K is a K-$\mathfrak{U}$-subnormal subgroup of G [15] [6.1.7(1)].*

(v) *If $K \leq H$ and H supersolvable, then K is a K-$\mathfrak{U}$-subnormal subgroup of G.*

Lemma 2.2. *If each n-maximal subgroup of G is K-$\mathfrak{U}$-subnormal in G, then each $(n-1)$-maximal subgroup of G is supersolvable and each $(n+1)$-maximal subgroup of G is K-$\mathfrak{U}$-subnormal in G.*

Proof. First, we show that each $(n-1)$-maximal subgroup of G is supersolvable. Let H be an $(n-1)$-maximal subgroup of G and let K be an arbitrary maximal subgroup of H. Then K is an n-maximal subgroup of G. By virtue of the condition of the lemma, K is K-$\mathfrak{U}$-subnormal in G. Therefore, by Lemma 2.1 (i), K is K-$\mathfrak{U}$-subnormal in H. Hence, either K is normal in H or $H/K_H \in \mathfrak{U}$. If K is normal in H, then $|H : K|$ is a prime number. Let $H/K_H \in \mathfrak{U}$. Thus, we can also conclude that

$$|H : K| = |H/K_H : K/K_H|$$

is a prime number. Since the subgroup K is arbitrary, all maximal subgroups of H have prime numbers in H. Therefore, the subgroup H is supersolvable.

Now let E be a certain $(n+1)$-maximal subgroup of G and let E_1 and E_2 be an n-maximal subgroup and an $(n-1)$-maximal subgroup of G, respectively, such that $E \leq E_1 \leq E_2$. As shown above, E_2 is supersolvable. Hence, E_1 is also supersolvable. Therefore, by Lemma 2.1 (v), E is K-$\mathfrak{U}$-supersolvable in E_1. Since, by the condition of the lemma, the subgroup E_1 is K-$\mathfrak{U}$-subnormal in G, by Lemma 2.1 (iii), E is K-$\mathfrak{U}$-subnormal in G.

The lemma is proved.

In [16, 17], one can find the characterizations of solvable groups in which all n-maximal subgroups are $\mathfrak{U}$-subnormal and, hence, K-$\mathfrak{U}$-subnormal. In particular, the following lemma is true:

Lemma 2.3 {see [16] (Theorems B and C) or [17] (Theorems B and D)}**.** *Let G be a solvable group all n-maximal subgroups of which are K-$\mathfrak{U}$-subnormal in G.*

(i) *If $|\pi(G)| \geq n$, then G is ϕ-dispersive for a certain ordering ϕ of the set $\pi(G)$.*

(ii) *If $|\pi(G)| \geq n+1$, then G is Ore dispersive. Moreover, if G is nonsupersolvable, then $G = A \rtimes B$, where $A = G^{\mathfrak{U}}$ and B are Hall subgroups of G and A either has the form $N_1 \times \ldots \times N_t$, where N_i, $i = 1, \ldots, t$, is a minimal normal subgroup of G, which is a Sylow subgroup of G, or is a Sylow p-subgroup of G with exponent p for a certain prime number p.*

Recall that G is called a *Schmidt group* if G is not nilpotent but each proper subgroup of G is nilpotent.

Lemma 2.4. *Let G be a minimal nonsupersolvable group. The following assertions are true:*

(i) *G is solvable and $|\pi(G)| \leq 3$ [2].*

(ii) *If G is not a Schmidt group, then G is Ore dispersive [2].*

(iii) *$G^{\mathfrak{U}}$ is a unique normal Sylow subgroup of G [2, 18].*

(iv) *$G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})$ is a noncyclic principal factor of the group $G/\Phi(G)$ [18].*

(v) *If S is a complement to $G^{\mathfrak{U}}$ in G, then $S/S \cap \Phi(G)$ is either a primary cyclic group or a Miller–Moreno group [18].*

(vi) If $|\pi(G)| = 3$, $p > q > r$ are different prime divisors of $|G|$, and Q and R are a Sylow q-subgroup and an r-subgroup of G, respectively, then Q and R are cyclic groups. Moreover, r divides $q-1$ and qr divides $p-1$ [22] (Theorem 10).

3. Proofs of Theorems B and C

Recall that a maximal subgroup M of G is called $\mathfrak{U}$-*normal* in G if $G/M_G \in \mathfrak{U}$; otherwise, M is called $\mathfrak{U}$-*abnormal* in G. Note that, in the case where G is solvable, a maximal subgroup M is $\mathfrak{U}$-normal in G if and only if $|G\colon M|$ is a prime number.

Proof of Theorem B. *Necessity.* Let W be a maximal subgroup of the group G. By virtue of the condition of the theorem and Lemma 2.1 (i), each 2-maximal subgroup of W is K-$\mathfrak{U}$-subnormal in W. Hence, by Theorem A, the group W is either supersolvable or an SDH-group. In particular, all 2-maximal subgroups of G are supersolvable.

First, we show that the group G is solvable. Since each maximal subgroup of G is either supersolvable or an SDH-group, by virtue of Lemma 2.4 (i), each proper subgroup of G is solvable. If an identity subgroup is a unique 3-maximal subgroup of G, then all 2-maximal subgroups of G have prime orders and, hence, each maximal subgroup of G is supersolvable. Therefore, the group G is either supersolvable or a minimal nonsupersolvable group.

Therefore, by virtue of Lemma 2.4 (i), G is solvable. Now let T be a nonidentity 3-maximal subgroup of G. Since T is a K-$\mathfrak{U}$-subnormal subgroup of G, there exists a proper subgroup H of G such that $T \le H$ and either $G/H_G \in \mathfrak{U}$ or H is normal in G. If $G/H_G \in \mathfrak{U}$, then G is solvable by virtue of solvability of the subgroup H_G. Assume that H is normal in G. Let E/H be an arbitrary 3-maximal subgroup of G/H. Then E is a 3-maximal subgroup of G and, hence, E is K-$\mathfrak{U}$-subnormal in G. Therefore, by Lemma 2.1 (ii), E/H is K-$\mathfrak{U}$-subnormal in G/H. Then the condition of the theorem is true for G/H. By induction, we conclude that G/H is solvable and, hence, the group G is also solvable.

Since G is solvable, by virtue of Lemma 2.3 (i), the group G is ϕ-dispersive for a certain ordering ϕ of the set $\pi(G)$. Let $G = P \rtimes (Q \rtimes R)$.

(i) Let $V < E < QR$, where E is a maximal subgroup of QR and let V be a maximal subgroup of E. Then PE is a maximal subgroup of G and PV is a maximal subgroup of PE. Therefore, PV is supersolvable.

Assume that P is not a minimal normal subgroup of PE. Then PE is not an SDH-group. Hence, PE is supersolvable. Therefore, $PE/O_{p',p}(PE)$ is an Abelian group of exponent dividing $p-1$ [23] (Sections 1, 1.4) and [23] (Appendix 3.2). Moreover, $O_{p',p}(PE) = PC_E(P)$ and, hence,

$$PE/O_{p',p}(PE) \simeq E/C_E(P).$$

Therefore, E induces on P a group of automorphisms of exponent dividing $p-1$.

(ii) Assume that P is a minimal normal subgroup of G and $P \ne G^{\mathfrak{U}}$. Then $M = QR$ is a maximal subgroup of G and M is not a supersolvable group. Hence, M is an SDH-group. Therefore, $Q = M^{\mathfrak{U}}$ is a minimal normal subgroup of M.

It is easy to see that $G^{\mathfrak{U}} \le PQ$. If $P \le G^{\mathfrak{U}}$, then $G^{\mathfrak{U}} = PQ$ because $Q = M^{\mathfrak{U}} \le G^{\mathfrak{U}}$. Assume that $P \not\le G^{\mathfrak{U}}$. Then $G^{\mathfrak{U}} \cap P = 1$ because P is a minimal normal subgroup of G. Hence, $G^{\mathfrak{U}} = Q$.

It is obvious that, in the case where $G^{\mathfrak{U}} = Q$, the group PR is supersolvable. Assume that PR is supersolvable. We show that, in this case, $G^{\mathfrak{U}} = Q$. Assume that $G^{\mathfrak{U}} = PQ$. Then QR is a $\mathfrak{U}$-abnormal subgroup of G

and, hence,

$$|P| = |G\colon QR| \geq p^2$$

in view of solvability of the group G. Since PR is supersolvable, R is a K-maximal subgroup of PR for some $k \geq 2$. However, PR is a maximal subgroup of G because Q is a minimal normal subgroup of $M = QR$. Therefore, R is a $(k+1)$-maximal subgroup of G and, hence, by virtue of the condition of the theorem and Lemma 2.2, R is a K-$\mathfrak{U}$-subnormal subgroup of G. Therefore, there exists a proper subgroup H of G such that $R \leq H$ and either H is normal in G or $G/H_G \in \mathfrak{U}$. Assume that H is normal in G. Then $M \cap H$ is normal in M. Since R is a maximal subgroup of M, $R \leq M \cap H$ and R is not normal in M, we have $M \cap H = M$. Hence, $M = H$ is normal in G, which contradicts the accepted assumption. Therefore, $G^{\mathfrak{U}} \leq H$. Thus,

$$M = QR = M^{\mathfrak{U}}R \leq G^{\mathfrak{U}}R \leq H,$$

which means that the subgroup $M = H$ is $\mathfrak{U}$-normal in G. This contradiction shows that $G^{\mathfrak{U}} = Q$.

Finally, we show that each proper subgroup of the group G containing PQ is supersolvable. Assume that this is not true and let V be a proper subgroup of G such that $PQ \leq V$ and V is not supersolvable. Since each 2-maximal subgroup of G is supersolvable, V is a maximal subgroup of G and, hence, V is an SDH-group. By virtue of Lemma 2.4 (i), $|\pi(V)| \leq 3$. If $|\pi(V)| = 3$, then, by virtue of Lemma 2.4 (vi), Q is a cyclic group and, hence, QR is supersolvable, which contradicts the considered case. Therefore, $|\pi(V)| = 2$ and, hence, $V = PQ$. By virtue of Lemma 2.4 (v), $Q/Q \cap \Phi(V)$ is either a primary cyclic group or a Miller–Moreno group. Since V is normal in G and $\Phi(V)$ is characteristic in V, $\Phi(V)$ is normal in G. However, Q is a minimal normal subgroup of $M = QR$. Hence, $Q \cap \Phi(V) = 1$ and Q is an Abelian group. This implies that the group Q is cyclic. The obtained contradiction shows that V is supersolvable.

(iii) Assume that $\Phi(P) \neq 1$. Since $\Phi(P)$ is a characteristic subgroup of P, this subgroup is normal in G. Therefore, in the analyzed case, every maximal subgroup of G containing P is supersolvable.

We show that $P/\Phi(P)$ is a noncyclic principal factor of the group G. If all maximal subgroups of G are supersolvable, then this statement follows from Lemma 2.4 (iv). Otherwise, consider a nonsupersolvable maximal subgroup V of G. Then $P \nleq V$ and V is an SDH-group. Let V_p be a Sylow p-subgroup of V. Then

$$1 \neq \Phi(P) \leq V_p = P \cap V$$

is normal in V and, hence, $V_p = V^{\mathfrak{U}} = \Phi(P)$ is a minimal normal subgroup of V. Hence, $P/\Phi(P)$ is a noncyclic principal factor in G. Therefore, $P = G^{\mathfrak{U}}$.

Assume that G is a minimal nonsupersolvable group. Then, by Lemma 2.4 (vi), Q and R are cyclic groups, r divides $q-1$, and qr divides $p-1$. Assume that $|\Phi(P)| \geq p^2$. Let M be a maximal subgroup of the group G such that $P \nleq M$. Then $G = PM$ and

$$M = (P \cap M)QR = \Phi(P)QR$$

because $P/\Phi(P)$ is the principal factor of the group G. Since the group M is supersolvable, there exists a 2-maximal subgroup E of M such that $|M\colon E| = p^2$. Therefore, $M = \Phi(P)E$ and, hence, $G = PE$. Since E is a K-$\mathfrak{U}$-subnormal subgroup of G, there exists a proper subgroup H of G such that $E \leq H$ and either H is normal in G or $G/H_G \in \mathfrak{U}$. If H is normal in G, then it is obvious that G/H is supersolvable. Therefore, $P \leq H$ and, hence, $G = PE \leq H$. We arrive at a contradiction. In the case where $G/H_G \in \mathfrak{U}$, we arrive at a contradiction in a similar way. Thus, $|\Phi(P)| = p$.

Finally, we assume that G is not a minimal nonsupersolvable group. Since each maximal subgroup of G containing P is supersolvable, there exists a nonsupersolvable maximal subgroup M such that $PM = G$. Without loss of generality of the proof, we can assume that $M = \Phi(P)QR$. Since M is not supersolvable, M is an SDH-group. Therefore, $\Phi(P) = M^{\mathfrak{U}}$ is a minimal normal subgroup of M and, hence, $\Phi(P)$ is a minimal normal subgroup of G. Moreover, by virtue of Lemma 2.4 (vi), Q and R are cyclic groups, furthermore, r divides $q-1$ and qr divides $p-1$. This yields (iii).

(iv) Assume that P is not a minimal normal subgroup of G and $\Phi(P) = 1$.

By virtue of the Maschke theorem, $P = P_1 \times P_2$, where P_1 is a minimal normal subgroup of G and P_2 is a normal subgroup of G. Then $L = P_2QR$ is a maximal subgroup of G. We show that P_2 is also a minimal normal subgroup of G. If L is an SDH-group, then $P_2 = L^{\mathfrak{U}}$ is a minimal normal subgroup of L. Hence, P_2 is also a minimal normal subgroup of G. Assume that the group L is supersolvable. Then $G/P_1 \simeq L$ is a supersolvable group. If P_1QR is supersolvable, then

$$G/P_2 \simeq P_1QR$$

is also supersolvable. Hence, the group G is also supersolvable, which is a contradiction. Therefore, P_1QR is not a supersolvable group. However, each 2-maximal subgroup of G is supersolvable. Thus, P_1QR is a maximal subgroup of G and, hence, P_2 is a minimal normal subgroup of G.

Since the group G is not supersolvable, at least one subgroup $L = P_2QR$ or $T = P_1QR$ is nonsupersolvable. Let T be an SDH-group. Then $T^{\mathfrak{U}} = P_1$ and, hence, P_1 is not cyclic. Moreover, by virtue of Lemma 2.4 (vi), the groups Q and R are cyclic. Moreover, r divides $q-1$ and qr divides $p-1$.

Sufficiency. Let E be an arbitrary nonidentity 3-maximal subgroup of the group G and let M be a maximal subgroup of G such that E is a 2-maximal subgroup of M. To prove that the subgroup E is K-$\mathfrak{U}$-subnormal in G, in view of solvability of G, Lemma 2.1 (iii), and Theorem A, it suffices to determine a $\mathfrak{U}$-normal maximal subgroup L of G such that $E \leq L$ and L is either supersolvable or an SDH-group.

We first assume that $G^{\mathfrak{U}} \leq P$.

If $P \leq M$, then $M = P \rtimes V$, where V is a maximal subgroup of QR. Hence, V induces on P a group of automorphisms, which is either irreducible or an Abelian group of exponent dividing $p-1$ by virtue of assertion (i) of the theorem. If $V/C_V(P)$ is an Abelian group of exponent dividing $p-1$, then M is supersolvable [23] (Sections 1, 1.4)). Hence, E is K-$\mathfrak{U}$-subnormal in G because M is a $\mathfrak{U}$-normal subgroup of G by Lemma 2.1 (iv). If $V/C_V(P)$ is an irreducible group of automorphisms of the subgroup P, then V is a maximal subgroup of PV. Hence, by virtue of assertion (i) of the theorem, PV is an SDH-group. Reasoning as above, we can show that E is K-$\mathfrak{U}$-subnormal in G.

Assume that $P \not\leq M$. Without loss of generality of the proof, we can assume that $QR \leq M$. If $\Phi(P) \neq 1$, then, by virtue of assertion (iii) of the theorem, the group $M = \Phi(P)QR$ is an SDH-group. Therefore, $|M : E|$ is divided by at least one of the numbers q or r and, hence, for some maximal subgroup D of QR, we have $E \leq PD$. By virtue of assertion (i) of the theorem, the group PD is supersolvable. Therefore, E is a K-$\mathfrak{U}$-subnormal subgroup of G. Finally, we consider the case where $\Phi(P) = 1$. If P is a minimal normal subgroup of G, then $M = QR$ is a supersolvable group. Thus, E is K-$\mathfrak{U}$-subnormal in G by virtue of assertion (i) of the theorem. Assume that $P = P_1 \times P_2$, where P_1 and P_2 are minimal normal subgroups of the group G and at least one of these subgroups is not cyclic. Without loss of generality of the proof, we can assume that $M = P_1QR$. It is easy to see that P_1 is a minimal normal subgroup of M. Hence, QR is a maximal subgroup of M. Since $G^{\mathfrak{U}} \leq P$, QR is supersolvable. Therefore, $|M : E|$ is divided by at least one of the numbers q or r. Reasoning as above, we conclude that E is a K-$\mathfrak{U}$-subnormal subgroup of G.

We now assume that $G^{\mathfrak{U}} \not\leq P$. Then, by virtue of the assertions (ii)–(iv) of the theorem, P is a minimal normal subgroup of G for which every maximal subgroup of G containing PQ is supersolvable and either

$G^{\mathfrak{U}} = Q$ or $G^{\mathfrak{U}} = PQ$. If $PQ \le M$, then the subgroup M is supersolvable and $\mathfrak{U}$-normal in G. Hence, E is K-$\mathfrak{U}$-subnormal in G. Assume that $PQ \not\le M$. Then, by virtue of assertion (ii), M is conjugate to one of the subgroups QR or PR. If $M = QR$, then r divides $|M\colon E|$. Therefore, for some maximal subgroup D of this type in QR such that $|QR\colon D| = r$, we get $E \le PD$. Since $PQ \le PD$, the subgroup PD is supersolvable and $\mathfrak{U}$-normal in G. Thus, E is K-$\mathfrak{U}$-subnormal in G. We now consider the case where $M = PR$. If M is supersolvable, then $G^{\mathfrak{U}} = Q$ by assertion (ii). Hence, QR is a $\mathfrak{U}$-normal subgroup of G. Therefore,

$$|P| = |G\colon QR| = p.$$

Thus, r divides $|M\colon E|$ and, hence, there exists a maximal subgroup W of G such that $|G\colon W| = r$. Therefore, E is K-$\mathfrak{U}$-subnormal in G. Finally, if the group M is not supersolvable, then M is an SDH-group by virtue of assertion (i). Reasoning as above, we establish that the subgroup E is K-$\mathfrak{U}$-subnormal in G.

The theorem is proved.

Proof of Theorem C. *Necessity.* As in the proof of necessity in Theorem B, we can show that every maximal subgroup of G is either supersolvable or an SDH-group. In particular, all 2-maximal subgroups of G are supersolvable.

(i) Reasoning as in the proof of necessity of Theorem B, we can show that the group G is solvable. By virtue of Lemma 2.3 (ii), the group G is Ore dispersive, i.e.,

$$G = P \rtimes (Q \rtimes (R \rtimes T)).$$

(ii) We now show that P is a minimal normal subgroup of G.

Assume that this is not true. First, we note that, in view of the fact that $|\pi(G)| = 4$, G is not a minimal nonsupersolvable group by virtue of Lemma 2.4(iii). Let M be a maximal subgroup of G such that $P \not\le M$. Then $G = PM$ and $M \cap P \ne 1$. Therefore, $|\pi(M)| = 4$ and, hence, M is supersolvable by virtue of Lemma 2.4 (i). Now let L be an arbitrary maximal subgroup of G containing P. If L is an SDH-group, then $P = L^{\mathfrak{U}}$ is a minimal normal subgroup of L. Therefore, P is a minimal normal subgroup of G, which is a contradiction. Hence, L is supersolvable. Thus, all maximal subgroups of the group G are supersolvable. Therefore, G is a minimal nonsupersolvable group. The obtained contradiction proves that P is a minimal normal subgroup of G.

(iii) Let $V < E < QRT$, where E is a maximal subgroup of QRT and V is a maximal subgroup of E. Then PE is a maximal subgroup of G and PV is a maximal subgroup of PE. Hence, PV is supersolvable.

Assume that P is not a minimal normal subgroup of PE. Then PE is not an SDH-group. Hence, PE is supersolvable. Therefore, $PE/O_{p',p}(PE)$ is an Abelian group with exponent dividing $p-1$ [23] (Sections 1 and 1.4 and Appendix 3.2). Moreover, $O_{p',p}(PE) = PC_E(P)$ and, hence,

$$PE/O_{p',p}(PE) \simeq E/C_E(P).$$

Therefore, E induces on P a group of automorphisms with exponent dividing $p-1$.

(iv) Assume that $P \ne G^{\mathfrak{U}}$. Then $W = QRT$ is not supersolvable. Since, by (ii), W is a maximal subgroup of G, by virtue of the result presented above, W is an SDH-group. Hence, $Q = W^{\mathfrak{U}}$ is a minimal normal subgroup of W.

It is clear that $G^{\mathfrak{U}} \leq PQ$. Moreover, $Q = W^{\mathfrak{U}} \leq G^{\mathfrak{U}}$. Hence, by virtue of Lemma 2.3(ii), either $G^{\mathfrak{U}} = Q$ or $G^{\mathfrak{U}} = PQ$. If $G^{\mathfrak{U}} = Q$, then Q is a minimal normal subgroup of G because Q is a minimal normal subgroup of W. In the second case, by Lemma 2.3(ii), Q is a minimal normal subgroup of G.

Finally, let M be a maximal subgroup of G such that $PQ \leq M$. Since, as shown above, Q is normal in G, M is not an SDH-group. Hence, M is supersolvable.

(v) Since, by (ii), P is a minimal normal subgroup of G, QRT is a maximal subgroup of G. Hence, the group QRT is either supersolvable or an SDH-group. If QRT is an SDH-group, then R and T are cyclic by Lemma 2.4 (vi).

Assume that QRT is supersolvable. In this case, $G^{\mathfrak{U}} = P$. Since G is not a minimal nonsupersolvable group, there exists a maximal subgroup M of G such that $P \leq M$ and M is an SDH-group. Since $G^{\mathfrak{U}} = P \leq M$, M is $\mathfrak{U}$-normal in G by Lemma 2.1(iv). Hence, $|G\colon M|$ is a prime number due to the solvability of the group G. Moreover, by virtue of Lemma 2.4(i), $|\pi(M)| = 3$. If $|G\colon M| = t$, then $|T| = t$. In addition, the subgroups Q and R are cyclic by Lemma 2.4(vi). Reasoning as above, we conclude that, in the cases where $|G\colon M| = q$ and $|G\colon M| = r$, the subgroups Q, R, and T are cyclic.

Sufficiency. Let E be an arbitrary nonidentity 3-maximal subgroup of the group G and let M be a maximal subgroup of G such that E is a 2-maximal subgroup of M. To prove that the subgroup E is K-$\mathfrak{U}$-subnormal in G, by virtue of Lemma 2.1(iii), Theorem A, and the solvability of the group G, it suffices to determine a $\mathfrak{U}$-normal maximal subgroup L of G such that $E \leq L$ and L is either supersolvable or an SDH-group.

First, we assume that $P = G^{\mathfrak{U}}$. If $P \leq M$, then, by Lemma 2.1(iv), M is a $\mathfrak{U}$-normal subgroup of G. Moreover, $M = P \rtimes V$, where V is a maximal subgroup of QRT. Hence, V induces a group of automorphisms on P, which is either irreducible or an Abelian group with exponent dividing $p-1$ by virtue of the assertion (iii) of the theorem. If $V/C_V(P)$ is an Abelian group with exponent dividing $p-1$, then M is supersolvable [23] (Sections 1 and 1.4). Hence, E is K-$\mathfrak{U}$-subnormal in G because, by Lemma 2.1(iv), M is a $\mathfrak{U}$-normal subgroup of G. If $V/C_V(P)$ is an irreducible group of automorphisms of the subgroup P, then V is a maximal subgroup of PV. Hence, by virtue of assertion (iii) of the theorem, PV is an SDH-group. Thus, E is a K-$\mathfrak{U}$-subnormal subgroup of G.

Assume that $P \nleq M$. Without loss of generality, it is possible to assume that $M = QRT$. Since E is a 2-maximal subgroup of M, by virtue of assertion (iii), E induces on P an Abelian group of automorphisms with exponent dividing $p-1$. As above, we conclude that the group PE is supersolvable. Hence, E is K-$\mathfrak{U}$-subnormal in G because PE is K-$\mathfrak{U}$-subnormal in G by Lemma 2.1 (iv).

We now assume that $P \neq G^{\mathfrak{U}}$. In this case, by virtue of the assertion (iv) of the theorem, either $G^{\mathfrak{U}} = Q$ or $G^{\mathfrak{U}} = PQ$, Q is a minimal normal subgroup of G, and each maximal subgroup of G containing PQ is supersolvable. If $PQ \leq M$, then M is supersolvable and, hence, a $\mathfrak{U}$-normal subgroup of G.

Thus, E is K-$\mathfrak{U}$-subnormal in G. Assume that $PQ \nleq M$. Therefore, by virtue of assertions (iv) and (v), M is conjugate to one of the subgroups PRT or QRT. Let $M = QRT$. It is easy to see that Q is a minimal normal subgroup of M. Hence, $|M\colon E|$ is divided by at least one of the numbers r or t. Therefore, there exists a maximal subgroup V of G such that $E \leq V$ and $|G\colon V| \in \{r, t\}$. Since $PQ \leq V$, V is supersolvable. Hence, as above, we conclude that E is K-$\mathfrak{U}$-subnormal in G. Finally, we consider the case where $M = PRT$. Since P is a minimal normal subgroup of M and RT is supersolvable, $|M\colon E|$ is divided by at least one of the numbers r or t. Hence, as above, we conclude that E is K-$\mathfrak{U}$-subnormal in G.

The theorem is proved.

In conclusion, we note that one can easily construct examples illustrating that there exist groups satisfying the conditions of Theorems B and C. Moreover, in Theorems B and C and in Theorem 1.2 from [19], all second maximal subgroups are supersolvable. Groups with supersolvable second maximal subgroups were partially described by Semenchuk in [24].

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