ISSN 0001-4346, Mathematical Notes, 2013, Vol. 94, No. 3, pp. 117–130. © Pleiades Publishing, Ltd., 2013.
Original Russian Text © Yufeng Liu, Wenbin Guo, V. A. Kovaleva, A. N. Skiba, 2013, published in Matematicheskie Zametki, 2013, Vol. 94, No. 3, pp. 455–472.

{FirstPag

Criteria for p-Solvability and p-Supersolvability of Finite Groups

Yufeng Liu[1*], Wenbin Guo[2], V. A. Kovaleva[3***], and A. N. Skiba[3****]**

[1]Shandong Institute of Business and Technology, Shandong, China
[2]University of Science and Technology of China, China
[3]Gomel State University, Gomel, Belarus

Received February 22, 2010; in final form, January 14, 2013

Abstract—Let A, K, and H be subgroups of a group G and $K \le H$. Then we say that A covers the pair (K,H) if $AH = AK$ and isolates the pair (K,H) if $A \cap H = A \cap K$. A pair (K,H) in G is said to be maximal if K is a maximal subgroup of H. In the present paper, we study finite groups in which some subgroups cover or isolate distinguished systems of maximal pairs of these groups. In particular, generalizations of a series of known results on (partial) *CAP*-subgroups are obtained.

DOI: 10.1134/S0001434613090125

Keywords: *finite group, (partial) CAP-subgroup, supersolvable group, quasinormal subgroup, subgroup covering or isolating a pair of subgroups.*

1. INTRODUCTION

{ssec1:u4

All groups under consideration are finite. Denote by the symbol $\mathcal{U}$ the class of all supersolvable groups. Recall that by the $\mathcal{U}$*-coradical* of a group G we mean the intersection of all normal subgroups N of G for which $G/N \in \mathcal{U}$; the $\mathcal{U}$-coradical of a group G is denoted by the symbol $G^{\mathcal{U}}$. We use the terminology adopted in [1] and [2].

Let A, K, and H be subgroups of a group G and $K \le H$. Then we say that A *covers* the pair (K,H) if $AH = AK$ and *isolates* the pair (K,H) if $A \cap H = A \cap K$. Note that the relation $AH = AK$ is equivalent to $H \le K(A \cap H)$, and $A \cap H = A \cap K$ is equivalent to $A \cap H \le K$. A subgroup A of a group G is said to be *quasinormal* [3] or *permutable* ([2], [4]) in G if $AE = EA$ for all subgroups E of G. Quasinormal subgroups have many interesting properties. In particular, if A is a quasinormal subgroup of G, then, for any maximal pair (K,H) in G (i.e., a pair of the form (K,H), where K is a maximal subgroup of H), the subgroup A either covers or isolates (K,H).

The following example shows that, even if some subgroup of a group G covers or isolates every maximal pair (K,H) in G, this subgroup can be not quasinormal.

Example. Let p and q be primes, where q divides $p-1$. Let $A = \langle a \rangle$ be a cyclic group of order p^2 and B be a group of order q. Let $G = A \wr B = [K]B$, where $K = A_1 \times A_2 \times \cdots \times A_q$ is the base of the regular wreath product G. Let $L = \langle a^p \rangle^{\natural}$. Then $G/L \simeq \langle a^p \rangle \wr B$ and $L \le \Phi(G)$. Hence the group G is supersolvable. Let R be a subgroup of order p of the group A_1. Suppose that R is quasinormal in G. Since R is a Sylow p-subgroup of RB, it follows that $B \le N_G(R)$, and therefore R is normal in G; a contradiction. Hence R is not quasinormal in G. On the other hand, since the group G is supersolvable and R is subnormal in G, it follows that R covers or isolates every maximal pair in G (see Corollary 4.3 below).

[*]E-mail: yfliu65@163.com
[**]E-mail: wbguo@ustc.edu.cn
[***]E-mail: vika.kovalyova@rambler.ru
[****]E-mail: alexander.skiba49@gmail.com

It should be noted that the theory of covering and isolating maximal pairs has an immediate relationship to the theory of *CAP*-subgroups. Recall that a subgroup A of a group G is said to be a *CAP-subgroup* of G [2, A, Definition 10.8] if A either covers or isolates every pair (K, H), where H/K is a principal factor in G. A subgroup A is said to be a *partial CAP-subgroup* of G ([5], [6]) if A either covers or isolates every pair (K, H), where H/K is a factor of some chosen principal series in G. Obviously, every *CAP*-subgroup of a group G either covers or isolates every maximal pair (K, H) in G such that $L \leq K < H \leq T$, where T/L is a principal factor in G. On the other hand, every partial *CAP*-subgroup of G either covers or isolates every maximal pair (K, H) in G such that $G_{i-1} \leq K < H \leq G_i$ for some i, where $1 = G_0 < G_1 < \cdots < G_n = G$ is a chosen principal series in G.

In the present paper, we study groups in which some subgroups cover or isolate distinguished systems of maximal pairs of these groups. In particular, generalizations of a series of known results concerning (partial) *CAP*-subgroups are obtained.

2. PRELIMINARY RESULTS

{ssec2:u4

The following results are used in the present paper.

{lem2.1:u

Lemma 2.1. *Let N be a normal subgroup of a group G and (K, H) a maximal pair in G. If N isolates (K, H), then (KN, HN) is a maximal pair in G and $|HN : KN| = |H : K|$.*

Proof. Let R be a subgroup of G such that $KN \leq R \leq HN$. Then

$$R = N(R \cap H) \qquad \text{and} \qquad K \leq R \cap H \leq H.$$

Hence either $R \cap H = K$ or $R \cap H = H$. If $R \cap H = K$, then

$$R = R \cap HN = N(R \cap H) = KN.$$

If $R \cap H = H$, then

$$R = R \cap HN = N(R \cap H) = HN.$$

Therefore, (KN, HN) is a maximal pair in G. Since N isolates (K, H), it follows that $H \cap N = K \cap N$, and therefore $|HN : KN| = |H : K|$. □

{lem2.2:u

Lemma 2.2. *Let M be a subgroup of a group G and (K, H) a maximal pair in G. If $H \leq V \leq G$ and M either covers or isolates (K, H), then $M \cap V$ either covers or isolates (K, H).*

Proof. Since $H \leq V$, it follows that $M \cap H \cap V = M \cap H$. If M covers the pair (K, H), then

$$H = K(M \cap H) = K(M \cap V \cap H),$$

i.e., $M \cap V$ covers (K, H). If M isolates (K, H), then

$$M \cap H \leq K, \qquad (M \cap V) \cap H \leq K,$$

i.e., $M \cap V$ isolates (K, H). □

The next lemma is well known.

{lem2.3:u

Lemma 2.3. *Let A and B be proper subgroups of a group G such that $G = AB$. Then $G = AB^x$ and $G \neq AA^x$ for any $x \in G$.*

{lem2.4:u

Lemma 2.4. *Let G be a group and p a prime divisor of the order of G. Let a subgroup E of G either covers or isolates every maximal pair (K, H) in G such that H is not p-solvable. Then G contains a chain of subgroups $E = E_0 \leq E_1 \leq \cdots \leq E_{n-1} \leq E_n = G$ such that either E_{i-1} is normal in E_i or $E_i/(E_{i-1})_{E_i}$ is p-solvable for $i = 1, \ldots, n$.*

Proof. Let M be a maximal subgroup of G such that $E \leq M$. Suppose that G is not p-solvable. Since M^x is maximal in G for any $x \in G$, it follows that E either covers or isolates the pair (M^x, G). If E covers (M^x, G) for some x, then $EM^x = G$, and therefore $MM^x = G$, which contradicts Lemma 2.3. Hence E isolates (M^x, G) for any $x \in G$, and therefore $E \leq M_G$. We see by induction that there is a chain of subgroups

$$E = E_0 \leq E_1 \leq \cdots \leq E_{t-1} \leq E_t = M_G$$

such that either E_{i-1} is normal in E_i or $E_i/(E_{i-1})_{E_i}$ is p-solvable for $i = 1, \ldots, t$. Since M_G is normal in G, this completes the proof of the lemma. □

A subgroup H of a group G is said to be *primitive* [7] or $\cap$-*indecomposable* [8] in G if H differs from the intersection of all subgroups of G in which H is contained properly.

{lem2.5:u

Lemma 2.5 ([8, c. 133]). *If K is a subgroup of a group G and E is a $\cap$-indecomposable subgroup of K, then G admits a $\cap$-indecomposable subgroup X such that $E = K \cap X$.*

{lem2.6:u

Lemma 2.6. *Let $G = MN$, where N is a minimal normal subgroup of a group G. If $E \leq N \cap M$ and E is subnormal in G, then $E \leq M_G$.*

Proof. Since E is subnormal in G, then $N \leq N_G(E)$ by [2, A, Theorem 14.5], and therefore

$$E^G = E^{NM} = E^M \leq M.$$

Thus, $E \leq M_G$. □

{lem2.7:u

Lemma 2.7 ([9, Lemma 2.8]). *Let G be a p-supersolvable group. If $O_{p'}(G) = 1$, then G is supersolvable.*

{lem2.8:u

Lemma 2.8 ([10, Lemma 2.8]). *Let $G = [N]M$, where N is a minimal normal subgroup of a group G and M is a solvable maximal subgroup of G. Then N is an Abelian group.*

{lem2.9:u

Lemma 2.9 ([11, Lemma 1]). *If N is a normal subgroup of a group G and V is a CAP-subgroup in G, then NV is a CAP-subgroup of G.*

{lem2.10:

Lemma 2.10. *Let E be a solvable normal subgroup of G. Suppose that every maximal subgroup of every Sylow subgroup of E is a CAP-subgroup of G. If M is a maximal subgroup of G such that $EM = G$ and V is a maximal subgroup of some Sylow subgroup in E, then there is an element $x \in G$ such that V covers or isolates the pair (M^x, G).*

Proof. Let $|G : M| = q^a$, and let V be a maximal subgroup of a Sylow p-subgroup P of G. Suppose that $V \nleq M^x$ for any $x \in G$. Then $q = p$. We claim that $VM = G$. By Lemma 2.9, without any loss of generality of the proof, we may assume that $M_G = 1$, and therefore $G = [N]M$ for some minimal normal subgroup N in G contained in E. Suppose that $N \nleq V$. Then $V \cap N = 1$, and, since V is a maximal subgroup of P, we obtain $V = 1$. Thus, $N \leq V$, and therefore $G = VM$. □

3. CRITERIA FOR THE p-SOLVABILITY AND SOLVABILITY OF A GROUP {ssec3:u4

In this section, using the theory of covering and isolating maximal pairs, we give new criteria for the p-solvability and solvability of a group.

Let p be a prime. We say that a subgroup A of a group G is a *weak CAP$_p$-subgroup* of G if G admits a composition series

$$1 = G_0 < G_1 < \cdots < G_n = G$$

such that A either covers or isolates every maximal pair (K, H) in G such that $G_{i-1} \leq K < H \leq G_i$ for some i, where p divides $|G_i/G_{i-1}|$ and H is not a p-solvable group.

Recall that by the *nilpotent coradical* of a group G one means the intersection of all normal subgroups N in G such that G/N is nilpotent.

{th3.1:u4

Theorem 3.1. *Let G be a group and p a prime. The following assertions are equivalent*:

(1) *G is p-solvable*;

(2) *every subgroup in G is a weak CAP_p-subgroup of G*;

(3) *every maximal subgroup of G is a weak CAP_p-subgroup of G*;

(4) *every 2-maximal subgroup of G is a weak CAP_p-subgroup of G*;

(5) *every Sylow p-subgroup of G is a weak CAP_p-subgroup of G*;

(6) *either G is a primary group, i.e., a group whose order is a power of some prime, or G contains two p-solvable maximal subgroups M_1 and M_2 such that*

$$(|G : M_1|, |G : M_2|) = r^a q^b$$

for some prime numbers r and q and some $a, b \in \{0\} \cup \mathbb{N}$ and M_1 and M_2 are weak CAP_p-subgroups of G;

(7) *every nonsupersolvable Schmidt subgroup in G is a weak CAP_p-subgroup of G.*

Proof. $(1) \Rightarrow (2)$ Since the group G is p-solvable, it follows that every subgroup of G is also p-solvable, and therefore every subgroup of G is a weak CAP_p-subgroup by definition.

The implications $(2) \Rightarrow (3)$–(5) and $(2) \Rightarrow (7)$ are obvious.

(3),(4) $\Rightarrow$ (1) Suppose that every 2-maximal (every maximal) subgroup M of G is a weak CAP_p-subgroup of G. We claim that G is p-solvable. Suppose that this is not the case, and let G be a counterexample of minimal order. Let us show first that G/N is p-solvable, where N is an arbitrary minimal normal subgroup of G. If N is a maximal or 2-maximal subgroup of G, then this is clear. Let N be a subgroup of G that is not maximal (in Case (3)) or not 2-maximal (in Case (4)). Let us prove that the condition of the theorem holds for G/N. Let M/N be a maximal (2-maximal) subgroup of G/N. Then M is a maximal (2-maximal) subgroup of G. Therefore, by the condition of the theorem, M is a weak CAP_p-subgroup of G, and thus there is a composition series $1 = G_0 < G_1 < \cdots < G_n = G$ in G such that M either covers or isolates every maximal pair (Q, R) in G such that $G_{i-1} \le Q < R \le G_i$ for some i, where p divides $|G_i/G_{i-1}|$ and R is not p-solvable.

Consider the series

$$1 = G_0N/N < G_1N/N < \cdots < G_nN/N = G/N.$$

By the isomorphisms

$$G_iN/G_{i-1}N \simeq G_iNG_{i-1}/G_{i-1}N \simeq G_i/G_i \cap G_{i-1}N = G_i/G_{i-1}(G_i \cap N),$$

without any loss of generality of the proof, we may assume that this series is a composition series in G/N, where

$$|G_i : G_{i-1}| = |G_iN/N : G_{i-1}N/N|.$$

Let $(K/N, H/N)$ be a maximal pair in G/N and

$$G_{i-1}N/N \le K/N < H/N \le G_iN/N$$

for some i, where p divides $|G_iN/N : G_{i-1}N/N|$ and H/N is not a p-solvable group. We claim that M/N covers or isolates the pair $(K/N, H/N)$.

Note that $G_{i-1}N \le K < H \le G_iN$ and (K, H) is a maximal pair in G. Since $K = N(K \cap G_i)$ and $H = N(H \cap G_i)$, it follows that

$$\begin{aligned}|H : K| &= (|H \cap G_i||N|/|G_i \cap H \cap N|) : (|K \cap G_i||N|/|G_i \cap K \cap N|) \\ &= (|H \cap G_i||N|/|G_i \cap N|) : (|K \cap G_i||N|/|G_i \cap N|) = |H \cap G_i : K \cap G_i|.\end{aligned}$$

Hence $K \cap G_i \neq H \cap G_i$. We claim that $(K \cap G_i, H \cap G_i)$ is a maximal pair in G. Since $K \neq H$, it follows that $N(K \cap G_i) \neq N(H \cap G_i)$, i.e., N does not cover the pair $(K \cap G_i, H \cap G_i)$. Therefore, there is a maximal pair (L, T) in G such that

$$K \cap G_i \leq L < T \leq H \cap G_i,$$

and N does not cover (L, T). Indeed, if N covers every maximal pair (U, W) such that

$$K \cap G_i \leq U < W \leq H \cap G_i,$$

then N covers obviously the pair $(K \cap G_i, H \cap G_i)$; a contradiction. Since N is normal in G, it follows that N either covers or isolates every maximal pair in G. Therefore, N isolates the pair (L, T). In this case, by Lemma 2.1, (LN, TN) is a maximal pair in G and $|TN : LN| = |T : L|$. However,

$$K = N(K \cap G_i) \leq NL < NT \leq N(H \cap G_i) = H.$$

Hence $K \cap G_i = L$ and $H \cap G_i = T$. Therefore, $(K \cap G_i, H \cap G_i)$ is a maximal pair in G. It can readily be seen here that

$$G_{i-1} \leq K \cap G_i < H \cap G_i \leq G_i.$$

Since H/N is not a p-solvable group, it follows that $H \cap G_i$ is not p-solvable either, because

$$H/N = (H \cap G_i)N/N \simeq H \cap G_i / H \cap G_i \cap N.$$

Therefore, by the assumption of the theorem, M either covers or isolates $(K \cap G_i, H \cap G_i)$. If M covers $(K \cap G_i, H \cap G_i)$, then

$$MH = MN(G_i \cap H) = MN(G_i \cap K) = MK,$$

i.e., M covers (K, H). Then

$$(M/N)(H/N) = MH/N = MK/N = (M/N)(K/N),$$

i.e., M/N covers $(K/N, H/N)$. If M isolates $(K \cap G_i, H \cap G_i)$, then

$$\begin{aligned} M \cap H = M \cap N(G_i \cap H) &= N(M \cap G_i \cap H) = N(M \cap K \cap G_i) \\ &= M \cap N(K \cap G_i) = M \cap K, \end{aligned}$$

i.e., M isolates (K, H). Thus,

$$(M/N) \cap (H/N) = (M \cap H)/N = (M \cap K)/N = (M/N) \cap (K/N),$$

i.e., M/N isolates $(K/N, H/N)$. Hence the assumption of the theorem holds for G/N, and therefore, by the choice of the group G, the quotient group G/N is p-solvable. Since the class of all p-solvable groups is a saturated formation, it follows that N is a unique minimal normal subgroup of G, N is not Abelian, p divides $|N|$, and $N \nleq \Phi(G)$. Thus, $C_G(N) = 1$.

Let $N = N_1 \times N_2 \times \cdots \times N_t$ be a direct product of isomorphic simple groups. We claim that the group G has a maximal subgroup V such that p does not divide $|G : V|$, $NV = G$, and $N_i \neq V \cap N_i \neq 1$ for any $i = 1, \ldots, t$.

Let $N_p \leq P$, where N_p is a Sylow p-subgroup of N and P is a Sylow p-subgroup of G. Then $N \cap P = N_p$ is normal in P, and therefore $P \leq N_G(N_p)$. Hence G contains a maximal subgroup V such that $N_G(N_p) \leq V$. Then $G = NN_G(N_p) = NV$, and therefore $V_G = 1$. Since $N_G(N_p) \leq V$, it follows that $P \leq V$. Let P_i be a Sylow p-subgroup of N_i. Then $P_i \leq P^x$ for some $x \in G$. Since $G = NV$, it follows that $x = vn$, where $n \in N$ and $v \in V$. Therefore, $P_i \leq (P^v)^n$, where $P^v \leq V$. Hence $(P_i)^{n^{-1}} \leq V$. Since N_i is normal in N, it follows that $(P_i)^{n^{-1}} \leq N_i$. Therefore, $V \cap N_i \neq 1$ for any $i = 1, \ldots, t$. If $N_i \leq V$ for some i, then $N_i \leq V_G = 1$ by Lemma 2.6; a contradiction. Hence $V \cap N_i \neq N_i$ for any i.

Let $D = V \cap N_1$, and let M_1 be a maximal subgroup of V such that $D \leq M_1$ (in Case (4)) or $M_1 = V$ (in Case (3)). By the assumption of the theorem, the group G admits a composition series

$$1 = G_0 < G_1 < \cdots < G_n = G$$

such that M_1 either covers or isolates every maximal pair (K, H) in G such that $G_{i-1} \le K < H \le G_i$ for some i, where p divides $|G_i/G_{i-1}|$ and H is not p-solvable. Since G_1 is a minimal subnormal subgroup of G, it follows that $N \le N_G(G_1)$ by [2, A, Theorem 14.5]. Hence $G_1 \le N$, because, otherwise, we would have $NG_1 = N \times G_1$, and therefore $G_1 \le C_G(N) = 1$, which is impossible. Therefore, without loss of generality of the proof we may assume that $G_1 = N_1$. Then M_1 either covers or isolates every maximal pair (K, H) in N_1 such that H is not a p-solvable group. Hence, by Lemma 2.2, we see that $D = M_1 \cap N_1$ either covers or isolates every maximal pair (U, W) in N_1 such that W is not p-solvable. Therefore, by Lemma 2.4, there is a chain of subgroups

$$D = M_1 \cap N_1 = D_0 \le D_1 \le \dots \le D_{t-1} \le D_t = N_1$$

in N_1 such that either D_{i-1} is normal in D_i or $D_i/(D_{i-1})_{D_i}$ is p-solvable, $i = 1, \dots, t$. Since $N_1 = G_1$ and $1 \ne D \ne N_1$, it follows that D_{t-1} is not a normal subgroup of N_1. Thus, $N_1/(D_{t-1})_{N_1}$ is p-solvable and $(D_{t-1})_{N_1} = 1$, because G_1 is a simple group. Hence N_1 is p-solvable. The contradiction thus obtained completes the proof of the implications (3) ⇒ (1) and (4) ⇒ (1).

(5) ⇒ (1) Suppose that this is not the case. Let G be a counterexample of minimal order. Let P be a Sylow p-subgroup of G, and let

$$1 = G_0 < G_1 < \dots < G_n = G$$

be a composition series in G such that P either covers or isolates every maximal pair (K, H) in G such that $G_{i-1} \le K < H \le G_i$ for some i, where p divides $|G_i/G_{i-1}|$ and H is not p-solvable. Since G is not p-solvable, there is an index i such that G_i/G_{i-1} is a simple non-Abelian group and p divides $|G_i/G_{i-1}|$. Without loss of generality of the proof, we may assume that $i = 1$. Then $P \cap G_1 \ne G_1$. By Lemma 2.2, $P \cap G_1$ either covers or isolates every maximal pair (U, W) in G_1 such that W is not p-solvable. Then, by Lemma 2.4, there is a chain of subgroups

$$P \cap G_1 = P_0 \le P_1 \le \dots \le P_{t-1} \le P_t = G_1$$

in G_1 such that either P_{i-1} is normal in P_i or $P_i/(P_{i-1})_{P_i}$ is p-solvable, $i = 1, \dots, t$. Since G_1 is a simple group and $P \cap G_1 \ne G_1$, it follows that P_{t-1} is not normal in G_1. Thus, $G_1/(P_{t-1})_{G_1}$ is p-solvable. However, since $(P_{t-1})_{G_1} = 1$, it follows that G_1 is also p-solvable. The contradiction thus obtained completes the proof of the implication (5) ⇒ (1).

(1) ⇒ (6) Let G fail to be a primary group. Then G contains two maximal subgroups M_1 and M_2 such that $|G : M_1| = p^a$ for some $a \in \mathbb{N}$ and p does not divide $|G : M_2|$. Then $(|G : M_1|, |G : M_2|) = 1$. By (2), the group M_i is a weak CAP_p-subgroup of G. Therefore, (1) ⇒ (6).

(6) ⇒ (1) Let the group G contain two p-solvable maximal subgroups M_1 and M_2 such that $(|G : M_1|, |G : M_2|) = r^a q^b$ for some primes r and q and some $a, b \in \{0\} \cup \mathbb{N}$, and M_1 and M_2 are weak CAP_p-subgroups of G. We claim that G is p-solvable. Suppose that this is not the case. Let G be a counterexample of minimal order.

Let N be a minimal normal subgroup of G. Suppose that $N \le M_1 \cap M_2$. Then M_1/N and M_2/N are p-solvable maximal subgroups of G/N and

$$(|G/N : M_1/N|, |G/N : M_2/N|) = (|G : M_1|, |G : M_2|) = r^a q^b.$$

Moreover, M_1/N and M_2/N are weak CAP_p-subgroups of G/N (see the proof of the implication (3) ⇒ (1)). Thus, the condition of the theorem is satisfied for G/N. Therefore, by the choice of the group G, the quotient group G/N is p-solvable. On the other hand, if $N \nleq M_1 \cap M_2$, for example, $N \nleq M_1$, then $G/N = M_1N/N \simeq M_1/M_1 \cap N$ is p-solvable. Therefore, N is a unique minimal normal subgroup of G, $N \nleq \Phi(G)$, N is non-Abelian, and p divides $|N|$.

Let $\pi = \{p_1, p_2, \dots, p_t\}$ be the set of prime divisors of the order of N. Since N is not p-solvable, it follows that $t > 2$ and $G = NM_1 = NM_2$. On the other hand, since $(|G : M_1|, |G : M_2|) = r^a q^b$ for some primes r and q and $t > 2$, it follows that there is a $p_i \in \pi$ and a Sylow p_i-subgroup P_i in G such that either $P_i \le M_1$ or $P_i \le M_2$. Let $P_i \le M_1$, and let L be a minimal subnormal subgroup of G such that M_1 either covers or isolates every maximal pair (K, H) such that $K < H \le L$ and H is not p-solvable. As in the proof of the implication (3) ⇒ (1), one can show that $1 \ne M_1 \cap L \ne L$, which leads to a contradiction by Lemma 2.4.

(7) ⇒ (1) Suppose that this is not the case. Let G be a counterexample of minimal order. We claim that the condition of the theorem is satisfied for the subgroups of G. Let V be an arbitrary subgroup of G and M a nonsupersolvable Schmidt subgroup of V. Then, by the assumption of the theorem, M is a weak CAP_p-subgroup of G. Thus, there is a composition series

$$1 = G_0 < G_1 < \cdots < G_n = G$$

of G such that M either covers or isolates every maximal pair (K_1, H_1) of G such that

$$G_{i-1} \leq K_1 < H_1 \leq G_i$$

for some i, where p divides $|G_i/G_{i-1}|$ and H_1 is not p-solvable. Consider the series

$$1 = G_0 \cap V \leq G_1 \cap V \leq \cdots \leq G_n \cap V = V.$$

Let (K, H) be a maximal pair such that

$$G_{i-1} \cap V \leq K < H \leq G_i \cap V,$$

where p divides $|(G_i \cap V)/(G_{i-1} \cap V)|$ and H is not p-solvable. Then $KG_{i-1} \neq HG_{i-1}$ (otherwise

$$H = H \cap HG_{i-1} = H \cap KG_{i-1} = K(H \cap G_{i-1}) \leq K(V \cap G_{i-1}) \leq K,$$

which contradicts the choice of the pair (K, H)). Thus, G_{i-1} does not cover the pair (K, H). However, since G_{i-1} is a normal subgroup of G_i, it follows that G_{i-1} isolates (K, H). Hence $(G_{i-1}K, G_{i-1}H)$ is a maximal pair of G_i by Lemma 2.1. Moreover,

$$G_{i-1} \leq G_{i-1}K < G_{i-1}H \leq G_i.$$

Since

$$G_i \cap V/G_{i-1} \cap V \simeq (G_i \cap V)\,G_{i-1}/G_{i-1},$$

it follows that p divides $|G_i/G_{i-1}|$ and, since H is not p-solvable, it follows that $G_{i-1}H$ is not p-solvable either. Hence, by the assumption of the theorem, M either covers or isolates $(G_{i-1}K, G_{i-1}H)$. If M isolates $(G_{i-1}K, G_{i-1}H)$, then $M \cap G_{i-1}K = M \cap G_{i-1}H$, and therefore

$$\begin{aligned} M \cap K &= M \cap K(V \cap G_{i-1}) = M \cap V \cap G_{i-1}K \\ &= M \cap V \cap G_{i-1}H = M \cap H(V \cap G_{i-1}) = M \cap H, \end{aligned}$$

i.e., M isolates (K, H). If M covers $(G_{i-1}K, G_{i-1}H)$, we have $MG_{i-1}K = MG_{i-1}H$, and therefore

$$MH = M(V \cap G_{i-1}H) = V \cap MG_{i-1}H = V \cap MG_{i-1}K = M(G_{i-1}K \cap V) = MK,$$

i.e., M covers (K, H). Thus, the condition of the theorem holds for the subgroups of G. Hence, by the choice of the group G, all proper subgroups of G are p-solvable. It is clear that G is not q-nilpotent, where q stands for the least prime divisor of $|G|$, and therefore, by [12, IV, Theorem 5.4], G contains a q-closed Schmidt subgroup H. Let Q be the nilpotent coradical of H. By [1, Theorem 26.1], Q is a normal Sylow q-subgroup of H, and $Q/\Phi(Q)$ is a noncentral principal factor in H. If H is supersolvable, then $|Q/\Phi(Q)| = q$, and $|H/C_H(Q/\Phi(Q))|$ divides $q - 1$. Hence $C_H(Q/\Phi(Q)) = H$; a contradiction. Therefore, H is not supersolvable. Suppose that the group G is simple. Then G admits a unique composition series $1 < G$. By the assumption of Theorem (7), H either covers or isolates every maximal pair (U, W) in G such that W is not p-solvable, which leads to a contradiction by Lemma 2.4. Hence the group G is not simple.

Let M be a maximal normal subgroup of G such that G/M is non-Abelian and p divides $|G/M|$. Let L be a proper subnormal subgroup of G. Then $L \leq M$. Indeed, if $L \nleq M$, then $G = ML$ is p-solvable, which contradicts the choice of the group G. Suppose that $M \neq \Phi(G)$. Then G contains a maximal subgroup E such that $EM = G$. However, since the subgroups E and M are p-solvable, it follows that G is also p-solvable. This contradiction shows that $M = \Phi(G)$.

Let $H \leq E$, where E is a maximal subgroup of G. Since $M = \Phi(G)$, it follows from the assumption of the theorem that H either covers or isolates (E^x, G) for any $x \in G$. If H covers (E^x, G) for some x, then $HE^x = G$, and therefore $EE^x = G$, which contradicts Lemma 2.3. Hence H isolates the pair (E^x, G) for any $x \in G$, i.e., $H \leq E^x$ for any $x \in G$. Since $M = \Phi(G) \leq E$ and E_G is a maximal normal subgroup of G contained in E, it follows that $E_G = M$. Thus, $H \leq E_G = M = \Phi(G)$, and hence H is nilpotent, which contradicts the choice of the subgroup H. Therefore, (7) ⇒ (1). This completes the proof of the theorem. □

We say that a subgroup A of G is a *weak CAP-subgroup* of G if it is a weak CAP_p-subgroup of G for any prime divisor p of the order of G.

{cor3.2:u

Corollary 3.2. *Let G be a group. The following assertions are equivalent*:

(1) *G is solvable*;

(2) *every subgroup of G is a weak CAP-subgroup of G*;

(3) *every maximal subgroup of G is a weak CAP-subgroup of G*;

(4) *every 2-maximal subgroup of G is a weak CAP-subgroup of G*;

(5) *every Sylow subgroup of G is a weak CAP-subgroup of G*;

(6) *there is a solvable maximal subgroup M of G such that M is a weak CAP-subgroup of G*;

(7) *every nonsupersolvable Schmidt subgroup of G is a weak CAP-subgroup of G*;

(8) *every maximal subgroup of every Sylow subgroup of G is a weak CAP-subgroup of G.*

Proof. By Theorem 3.1, it suffices to prove the implications (6) $\Rightarrow$ (1) and (8) $\Rightarrow$ (1) only. Suppose that the implication (6) $\Rightarrow$ (1) fails to hold. Let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G. If $N \nleq M$, then $G = NM$, and therefore $G/N = NM/N \simeq M/M \cap N$ is solvable. Let $N \leq M$. Then M/N is a solvable maximal subgroup of G/N. As in the proof of the implication (3) $\Rightarrow$ (1) of Theorem 3.1, one can prove that M/N is a weak CAP-subgroup of G/N. Hence the assumption of the corollary holds for G/N, and therefore, by the choice of G, the quotient group G/N is solvable. Thus, N is a unique minimal normal subgroup of G, N is non-Abelian, and $G = NM$. If $N \cap M = 1$, then $G = [N]M$, and therefore N is an Abelian group by Lemma 2.8. In this case, G is solvable, which contradicts the choice of G. The contradiction thus obtained shows that $M \cap N \neq 1$, which leads to a contradiction by Lemma 2.4 (as in the proof of the implication (3) $\Rightarrow$ (1) in Theorem 3.1).

(8) $\Rightarrow$ (1) Suppose that this is not the case. Let G be a counterexample of minimal order. Let P be a Sylow p-subgroup of G, where p stands for the least prime divisor of $|G|$, and let V be a maximal subgroup of P. Let

$$1 = G_0 < G_1 < \cdots < G_n = G$$

be a composition series in G such that V either covers or isolates every maximal pair (K, H) in G such that $G_{i-1} \leq K < H \leq G_i$ for some i and H is not solvable. We assume first that G_1 is not Abelian. Then p divides $|G_1|$ and, for a Sylow p-subgroup W of G_1, we have $W \neq G_1$. Without loss of generality of the proof, we may assume that $V \cap G_1 \leq W$. If $V \cap W = 1$, then $|W| = p$ because V is maximal in P,. Therefore, G_1 is p-nilpotent by [12, V, Theorem 2.8], which contradicts the minimality of G_1. Hence $V \cap W \neq 1$, which leads to a contradiction by Lemmas 2.2 and 2.4. Thus, G_1 is a q-group for some prime q. Hence $O_q(G) \neq 1$ by [13]. If N is a minimal normal subgroup of G contained in $O_q(G)$, then, as in the proof of the implication (7) $\Rightarrow$ (1) in Theorem 3.1, one can prove that the assumption of the theorem is satisfied for G/N. Therefore, G/N is solvable by the choice of G. Then the group G is also solvable. The contradiction thus obtained completes the proof of the implication (8) $\Rightarrow$ (1). □

{cor3.3:u

Corollary 3.3 (Guo, Shum [10])**.** *A group G is solvable if and only if every maximal subgroup of G is a CAP-subgroup of G.*

{cor3.4:u

Corollary 3.4 (Fan, Guo, Shum [14])**.** *A group G is solvable if and only if every maximal subgroup of G is a partial CAP-subgroup of G.*

Let H be a subgroup of a group G; H is said to be *c-normal* [15] in G if there is a normal subgroup N of G for which $G = HN$ and $H \cap N \leq H_G$. It can readily be seen that every c-normal subgroup of G is a partial CAP-subgroup of G.

{cor3.5:u

Corollary 3.5 (Wang [15]). *A group G is solvable if and only if every maximal subgroup of G is c-normal in G.*

{cor3.6:u

Corollary 3.6 (Guo, Shum [10]). *If every 2-maximal subgroup of a group G is a CAP-subgroup of G, then G is solvable.*

{cor3.7:u

Corollary 3.7 (Fan, Guo, Shum [14]). *If every 2-maximal subgroup of G is a partial CAP-subgroup of G, then G is solvable.*

{cor3.8:u

Corollary 3.8 (Guo, Shum [10]). *A group G is solvable if and only if G admits a maximal subgroup M such that M is a solvable CAP-subgroup of G.*

{cor3.9:u

Corollary 3.9 (Wang [15]). *A group G is solvable if and only if G admits a maximal subgroup M such that M is a solvable c-normal subgroup of G.*

{cor3.10:

Corollary 3.10 (Guo, Shum [10]). *A group G is solvable if and only if every Sylow subgroup of G is a CAP-subgroup of G.*

{cor3.11:

Corollary 3.11 (Fan, Guo, Shum [14]). *A group G is solvable if and only if every Sylow subgroup of G is a partial CAP-subgroup of G.*

4. p-SUPERSOLVABILITY AND SUPERSOLVABILITY CRITERIA FOR A GROUP

{ssec4:u4

Let A, K, and H be subgroups of a group G and $K \leq H$. We say that A *conditionally* covers or isolates the pair (K, H) if there is an element $h \in H$ such that A covers or isolates the pair (K^h, H).

Ezquerro [11] obtained characterizations of p-supersolvable groups in terms of *CAP*-subgroups. In the present section, we give new characterizations of p-supersolvable, p-nilpotent, and supersolvable groups in terms of conditional covering and isolating maximal pairs.

{th4.1:u4

Theorem 4.1. *Let G be a group and p a prime. The following assertions are equivalent*:

(1) *G is p-supersolvable*;

(2) *every subgroup of G conditionally covers or isolates every maximal pair (K, H) in G such that p divides $|H : K|$*;

(3) *G is p-solvable and every subnormal subgroup of G covers or isolates every maximal pair (K, H) in G such that p divides $|Hp3 : p2K|$*;

(4) *G is p-solvable and every $\cap$-indecomposable subgroup in G conditionally covers or isolates every maximal pair (K, H) in G such that p divides $|H : K|$.*

Proof. (1) $\Rightarrow$ (2) Suppose that this is not the case. Let G be a counterexample of minimal order. Let A be a subgroup of G, and let (K, H) be a maximal pair in G such that p divides $|H : K|$. Then $|H : K| = p$. If $H < G$, then, by the choice of G, the subgroup $A \cap H$ conditionally covers or isolates (K, H), i.e., there is an $h \in H$ such that $A \cap H$ covers or isolates (K^h, H). If $A \cap H$ covers (K^h, H), then $K^h(A \cap H) = H(A \cap H) = H$, whence $K^h A = HA$; i.e., A covers (K^h, H). If $A \cap H$ isolates (K^h, H), then $(A \cap H) \cap H = (A \cap H) \cap K^h$, and therefore $A \cap H = A \cap K^h$; i.e., A isolates (K^h, H). Therefore, we may assume that $H = G$ and K is a maximal subgroup of G.

Assume first that $K_G = 1$. Then G is a primitive group. Let N be a minimal normal subgroup of G. Then $NK = G$, and therefore $|G : K| = p$ divides $|N|$. Since G is p-supersolvable, it follows that $|N| = p$. Moreover, $C_G(N) = N$ by [2, A, Theorem 15.2]. Hence

$$K \simeq NK/N = G/N = G/C_G(N) \leq \mathrm{Aut}(N),$$

where $|\mathrm{Aut}(N)| = p - 1$. Hence p does not divide $|K|$. Therefore, K is a p'-Hall subgroup of G. If p divides $|A|$, then

$$|AK| = |A||K|/|A \cap K| \geq |K|p = |G|.$$

Hence $AK = G$, i.e., A covers (K, G). If p does not divide $|A|$, then, by the Hall–Chunikhin theorem [12, VI, Theorem 1.7], there is an element $g \in G$ such that $A \leq K^g$, i.e., A conditionally isolates (K, G).

Suppose now that $K_G \neq 1$. In this case, by the choice of G, AK_G/K_G conditionally covers or isolates $(K/K_G, G/K_G)$. Hence there is a $gK_G \in G/K_G$ such that AK_G/K_G covers or isolates $((K/K_G)^{gK_G}, G/K_G)$. If AK_G/K_G covers $((K/K_G)^{gK_G}, G/K_G)$, then

$$(AK_G/K_G)(K/K_G)^{gK_G} = G/K_G.$$

Consequently, $AK_GK^g = AK^g = G$, and therefore A covers (K^g, G). If AK_G/K_G isolates

$$((K/K_G)^{gK_G}, G/K_G),$$

then

$$(AK_G/K_G) \cap (K/K_G)^{gK_G} = AK_G/K_G.$$

Hence $A \cap K^g = A$, i.e., A isolates (K^g, G). This shows that every subgroup of G conditionally covers or isolates every maximal pair (K, H) in G such that p divides $|H : K|$, which contradicts the choice of G. Thus, $(1) \Rightarrow (2)$.

$(2) \Rightarrow (1)$ Suppose that this is not the case. Let G be a counterexample of minimal order. Let us show first that the condition of the theorem is inherited by the quotient groups of G. Indeed, let N be an arbitrary minimal normal subgroup of G, let A/N be an arbitrary subgroup in G/N, and let $(K/N, H/N)$ be a maximal pair in G/N such that p divides $|H/N : K/N| = |H : K|$. In this case, by assumption, A conditionally covers or isolates pair (K, H), i.e., there is an $x \in H$ such that either $AK^x = AH$ or $A \cap K^x = A \cap H$. In the first case, we have

$$(A/N)(K/N)^{xN} = (A/N)(H/N) \quad (xN \in H/N),$$

i.e., A/N covers the pair $((K/N)^{xN}, H/N)$. In the other case, we obtain

$$(A/N) \cap (K/N)^{xN} = (A/N) \cap (H/N),$$

i.e., A/N isolates the pair $((K/N)^{xN}, H/N)$. Hence there is a unique minimal normal subgroup N of G, where $N \nleq \Phi(G)$ and N is a noncyclic p-group. Consequently, there is a maximal subgroup M of G such that $G = [N]M$. Let L be a subgroup of order p in N. Then it is clear that L does not isolate the maximal pair (M^x, G) for any $x \in G$. Since p divides $|G : M|$, there is an element $x \in G$ such that L covers the pair (M^x, G), and therefore $LM^x = G$. Hence

$$|G : M^x| = |G : M| = |L| = |N| = p.$$

This contradiction completes the proof of the implication $(2) \Rightarrow (1)$.

$(1) \Rightarrow (3)$ Let V be an arbitrary subnormal subgroup of G and (K, H) a maximal pair in G such that p divides $|H : K|$. Then $|H : K| = p$, and $V \cap H$ is subnormal in H. Hence, without loss of generality of the proof, we may assume that $H = G$. Then K is a maximal subgroup of G. Suppose that $V \nleq K$. If $K_G \neq 1$, then VK_G/K_G covers $(K/K_G, G/K_G)$ by induction. Therefore,

$$(K_GV/K_G)(K/K_G) = G/K_G,$$

and hence $VK = G$, i.e., V covers (K, G). Suppose now that $K_G = 1$. Then G is a primitive group. Let N be a minimal normal subgroup of G. Since $K_G = 1$, it follows that $G = NK$. Since G is p-supersolvable and p divides $|N|$, it follows that $|N| = p$. Since G is primitive, we have $C_G(N) = N$. Hence $G = [N]K$, and $K \simeq G/C_G(N)$ is an Abelian group whose exponent divides $p - 1$. Therefore, K is a p'-Hall subgroup of G, and $|G : K| = p$. If p does not divide $|V|$, then $V \subseteq O_{p'}(G)$, and therefore $O_{p'}(G) \nsubseteq K$. Then $G = KO_{p'}(G)$, and hence $|G : K| \neq p$; a contradiction. Therefore, p divides $|V|$. Thus,

$$|VK| = |V||K|/|V \cap K| \geq |K|p = |G|,$$

and hence $VK = G$, i.e., V covers (K, G). Thus, (1) ⇒ (3).

(3) ⇒ (1) Suppose that this is not the case. Let G be a counterexample of minimal order. It is clear that condition (3) is preserved for every quotient group of G (see the proof of the implication (2) ⇒ (1)). Therefore, the group G admits a unique minimal normal subgroup N, $N \nleq \Phi(G)$, and N is a noncyclic p-group. Hence $G = [N]M$ for some maximal subgroup M in G and $N = C_G(N) = O_p(G)$. Let L be a minimal normal subgroup in N. Then $L \neq N$ and L covers or isolates (M, G) by the condition of the theorem. Since $N \cap M = 1$, L does not isolate (M, G). Therefore, L covers (M, G). Hence $ML = G$, and therefore $|N| = |G : M| \leq |L| < |N|$; a contradiction. Hence G is p-supersolvable.

(4) ⇒ (1) Suppose that this is not the case. Let G be a counterexample of minimal order. Let E be a subgroup of G and V a ∩-indecomposable subgroup of E. Then, by Lemma 2.5, there is a ∩-indecomposable subgroup X in G such that $V = E \cap X$. Let (K, H) be a maximal pair in E such that p divides $|H : K|$. Then there is an element $h \in H$ for which X covers or isolates (K^h, H). If X covers the pair (K^h, H), then $XK^h = XH$, and therefore

$$K^hV = K^h(E \cap X) = E \cap XK^h = E \cap XH = H(E \cap X) = HV,$$

i.e., V covers (K^h, H). If X isolates the pair (K^h, H), i.e., $X \cap H \leq K^h$, then

$$V \cap H = X \cap E \cap H = X \cap H \leq K^h,$$

i.e., V isolates (K^h, H). Thus, the condition of the theorem holds for every proper subgroup of G. Hence all maximal subgroups of G are p-supersolvable by the choice of G.

Let N be an arbitrary minimal normal subgroup of G. It can readily be seen that the condition of the theorem is preserved for G/N. Hence G/N is p-supersolvable by the choice of G. Since the class of all p-supersolvable groups is a saturated formation, it follows that N is a unique minimal normal subgroup of G, $N \nleq \Phi(G)$, and N is a noncyclic p-group. Let M be a maximal subgroup of G such that $N \nleq M$. Then $G = [N]M$ and $M_G = 1$. Hence $N = C_G(N)$ by [2, A, Theorem 15.2], and M is p-supersolvable.

(a) N is a maximal subgroup of a Sylow p-subgroup P of G.

Let us show first that $N \neq P$. Suppose that $N = P$ and V is a maximal subgroup of N. Then V is a ∩-indecomposable subgroup of N, and therefore, by Lemma 2.5, there is a ∩-indecomposable subgroup X in G such that $V = X \cap N$. In this case, $N \nleq X$. By the condition of the theorem, there is an element $x \in G$ such that X covers or isolates (M^x, G). If X covers (M^x, G), i.e., $XM^x = G$, then $XM = G$ by Lemma 2.3. Since $N = P$, it follows that M is a p'-group, and therefore $P = N \leq X$. The contradiction thus obtained shows that X isolates the pair (M^x, G), i.e., $X \leq M^x$. Thus, $V \leq M^x$. Then $V = 1$, and thus N is a cyclic group. The contradiction thus obtained shows that $N \neq P$. Hence p divides $|M|$. Since M is p-supersolvable, it follows that M contains a maximal subgroup E such that $|M : E| = p$. Since $G = [N]M$, we clearly have $EN \neq G$. Hence EN is p-supersolvable. Moreover, $O_{p'}(EN) = 1$ because $C_G(N) = N$. By Lemma 2.7, the group EN is supersolvable. Hence, since $N = C_G(N)$, every Sylow p-subgroup P_1 in EN is normal in EN. It is also clear that P_1 is a maximal subgroup of some Sylow p-subgroup in G. Hence P_1 is normal in G, because $PE = G = P^xE$ for any $x \in G$ by Lemma 2.3. However, since $C_G(N) = N$ and $|O_p(G/N)| = |O_p(M)| = 1$, it follows that $N = P_1$ is a maximal subgroup of P.

(b) Every maximal subgroup V of N is normal in some Sylow p-subgroup of G.

Let X be a ∩-indecomposable subgroup of G such that $V = X \cap N$. In this case, by assumption, there is an element $x \in G$ such that X covers or isolates (M^x, G). If X covers (M^x, G), then $XM^x = G = XM$ by Lemma 2.3. By (a), we have $|M_p| = p$, where M_p is a Sylow p-subgroup of M. Hence every Sylow p-subgroup of X is a maximal subgroup of some Sylow p-subgroup of G. Let $V \leq X_p$, where X_p is a Sylow p-subgroup of X. Then X_p is a maximal subgroup of some Sylow p-subgroup G_p in G. Hence X_p is normal in G_p. Therefore, $V = N \cap X = N \cap X_p$ is normal in G_p. Finally, note that, since $V \neq 1$ and $V \leq X$, X cannot isolate the pair (M^x, G).

(c) Concluding contradiction.

Let E be a p'-Hall subgroup of M. Then $S = NE < G$ is p-supersolvable. Since $N = C_G(N)$, we have $O_{p'}(S) = 1$. Hence EN is supersolvable by Lemma 2.7. Therefore, some maximal subgroup V

in N is normal in S. Moreover, by (b), there is a Sylow p-subgroup G_p in G such that $G_p \leq N_G(V)$. Hence $G = SG_p \leq N_G(V)$, which contradicts the minimality of N. This completes the proof of the theorem. □

{cor4.2:u

Corollary 4.2. *Let G be a group, and let p be the least prime divisor of $|G|$. The group G is p-nilpotent if and only if every subgroup of G conditionally covers or isolates every maximal pair (K, H) in G such that p divides $|H : K|$.*

Proof. Since p is the least prime divisor of $|G|$, it follows that the group G is p-nilpotent if and only if G is p-supersolvable. Hence the assertion of the corollary follows immediately from Theorem 4.1. □

{cor4.3:u

Corollary 4.3. *Let G be a group. The following assertions are equivalent*:

(1) *G is supersolvable*;

(2) *every subgroup of G conditionally covers or isolates every maximal pair in G*;

(3) *every $\cap$-indecomposable subgroup of G conditionally covers or isolates every maximal pair in G*;

(4) *every cyclic subgroup of prime order or of order 4 in G conditionally covers or isolates every maximal pair in G*;

(5) *G is solvable and every subnormal subgroup in G covers or isolates every maximal pair in G.*

Proof. By Theorem 4.1, it suffices to prove only the implications (3) ⇒ (1) and (4) ⇒ (1).

(3) ⇒ (1) By induction, every maximal subgroup of G is supersolvable. Therefore, by [1, Theorem 26.3], the group G is solvable. Then, by Theorem 4.1, the group G is supersolvable.

(4) ⇒ (1) Suppose that this is not the case. Let G be a counterexample of minimal order. Obviously, the assumption of the theorem is satisfied for every subgroup of G. Hence G is a minimal nonsupersolvable group. Therefore, by [1, Theorem 26.3] the following assertions hold:

(a) G is solvable;

(b) $G^{\mathcal{U}}$ is a Sylow p-subgroup of G for some prime p dividing $|G|$;

(c) $G^{\mathcal{U}}/\Phi(G^{\mathcal{U}})$ is a noncyclic principal factor in G;

(d) if $p > 2$, then $G^{\mathcal{U}}$ is a group of exponent p and, if $p = 2$, then the exponent of $G^{\mathcal{U}}$ divides 4.

Let $P = G^{\mathcal{U}}$, and let $X/\Phi(P)$ be a subgroup of $P/\Phi(P)$ of order p. Let $x \in X \smallsetminus \Phi(P)$ and $L = \langle x \rangle$. Then either $|L| = p$ or $|L| = 4$. In this case, by assumption (4), we see that L conditionally covers or isolates every maximal pair in G. Since $\mathcal{U}$ is a saturated formation and $G/G^{\mathcal{U}}$ is supersolvable, it follows that $P \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $PM = G$. Then L conditionally covers or isolates the pair (M, G). Hence there is an element $h \in G$ such that L either covers or isolates (M^h, G). By [2, A, Theorem 9.2(e)], $\Phi(P) \leq \Phi(G)$. Hence $\Phi(P) \leq M^h$. In this case,

$$G/\Phi(P) = [P/\Phi(P)](M^h/\Phi(P)).$$

Since $L \nleq \Phi(P)$, it follows that $L \nleq M^h$. This shows that L does not isolate (M^h, G). Hence $LM^h = LM = G$. Then $|P/\Phi(P)| = |G : M| = p$, which contradicts the condition that $P/\Phi(P)$ is a noncyclic factor. Thus, (4) ⇒ (1). □

Following [16], we use the symbol $Z_{\mathcal{U}\Phi}(G)$ to denote the product of all normal subgroups of G all of whose non-Frattini G-principal factors are cyclic.

{th4.4:u4

Theorem 4.4. *Let $X \leq E$ be a solvable normal subgroup of G. Suppose that every maximal subgroup of every Sylow subgroup in X conditionally covers or isolates every maximal pair (M, G), where $MX = G$. If $X = E$ or $X = F(E)$, then $E \leq Z_{\mathcal{U}\Phi}(G)$.*

Proof. Assume first that $X = E$. Suppose that the theorem fails to hold in this case. Let (G, E) be a counterexample with the minimal product $|G||E|$. We claim first that $E/N \leq Z_{\mathcal{U}\Phi}(G/N)$ for every minimal normal subgroup N of G contained in E. Indeed, by the choice of the group (G, E), we are only to prove that the condition of the theorem remains valid for $(G/N, E/N)$. Let N be a p-group, let Q/N be a Sylow q-subgroup of E/N, and let V/N be a maximal subgroup of Q/N. Let T/N be a maximal subgroup of G/N such that $(T/N)(E/N) = G/N$. Then $TE = G$. Suppose that $q \neq p$. In this case, $V = NM$ and $Q = NP$, where M is a Sylow q-subgroup of V and P is a Sylow q-subgroup of Q which contains M. Then P is a Sylow q-subgroup of E, and therefore there is as element $x \in G$ such that M covers or isolates the pair (T^x, G). If $M \leq T^x$, then

$$V/N = NM/N \leq T^x/N = (T/N)^{xN}.$$

Otherwise $MT^x = G$, which yields $(M/N)(T/N)^{xN} = G/N$. If $q = p$, then one can similarly prove that V/N conditionally covers or isolates every maximal pair $(M/N, G/N)$, where $(M/N)(E/N) = G/N$. Therefore, $E/N \leq Z_{\mathcal{U}\Phi}(G/N)$ for every minimal normal subgroup N of G contained in E. Hence $N \nleq \Phi(G)$, and $|N| > p$ by the choice of (G, E).

Let M be a maximal subgroup of G such that $N \nleq M$. Then $G = [N]M$ and $E = [N](E \cap M)$. Let W be a Sylow p-subgroup in $E \cap M$ and V a maximal subgroup of NW containing W. Then, by the assumption of the theorem, V conditionally covers or isolates the pair (M, G). If $VM^x = G$ for some $x \in G$, then $VM = G$ by Lemma 2.3, and therefore

$$|G| = |VM| = |V||M| : |V \cap M| = |V||M| : |W| < |N||M| = |G|,$$

which is impossible. Hence $V \leq M^x$ for any $x \in G$. Thus, $V \leq M_G$, and therefore $V \cap N = 1$. Consequently, $|N| = p$; a contradiction. This contradiction shows that the theorem is true for $X = E$.

Assume now that $X = F(E)$. Suppose that the theorem fails to hold in this case, and let (G, E) be a counterexample with the minimal product $|G||E|$. Let $F = F(E)$ and P a Sylow p-subgroup in F, where p divides $|F|$.

(1) $P \leq Z_{\mathcal{U}\Phi}(G)$ and $E/P \nleq Z_{\mathcal{U}\Phi}(G/P)$.

Since P is a characteristic subgroup of F and F is a characteristic subgroup of E, it follows that P is normal in G. Hence, as in the case of $X = E$, we see that $P \leq Z_{\mathcal{U}\Phi}(G)$. Therefore, $E/P \nleq Z_{\mathcal{U}\Phi}(G/P)$, because otherwise we have $E \leq Z_{\mathcal{U}\Phi}(G)$, which contradicts the choice of (G, E).

(2) If L is a minimal normal subgroup of G and $L \leq P$, then $|L| > p$.

Suppose that $|L| = p$. Let $C_0 = C_E(L)$. Then the condition of the theorem is satisfied for $(G/L, C_0/L)$. Indeed, since $F \leq C_0$ and $L \leq Z(F)$, it follows that $F(C_0/L) = F/L$. Moreover, as in the case of $X = E$, one can prove that, if M/N is a maximal subgroup of G/L such that $(F/L)(M/L) = G/L$, Q/L is a Sylow q-subgroup of F/L, and V/L is a maximal subgroup of Q/L, then V/L conditionally covers or isolates $(M/L, G/L)$. Hence $C_0/L \leq Z_{\mathcal{U}\Phi}(G/L)$ by the choice of (G, E), and therefore, by the G-isomorphism $C_G(L)E/C_G(L) \simeq E/C_0$, we see that $E \leq Z_{\mathcal{U}\Phi}(G)$. The contradiction thus obtained shows that (2) holds.

(3) $\Phi(G) \cap P \neq 1$.

Suppose that $\Phi(G) \cap P = 1$. Let L be a minimal normal subgroup of G contained in P. Let M be a maximal subgroup of G such that $G = [L]M$. Let $P_1 = P \cap M$. Then $P = LP_1$ and $|P : P_1| = |N|$. Let V be a maximal subgroup of P containing P_1. Then $L \nleq V$, and V conditionally covers or isolates (M, G) by assumption. If $V \leq M^x$ for any $x \in G$, then $V \cap N = 1$, and therefore $|L| = p$, which contradicts (2). Hence $G = VM^x$ for any $x \in G$, and therefore $G = VM$ by Lemma 2.3. In this case,

$$|L| = |G : M| = |V||M| : |P_1||M| < |L|.$$

The contradiction shows that $\Phi(G) \cap P \neq 1$.

Concluding contradiction. By (3), G admits a minimal normal subgroup L such that $L \leq \Phi(G) \cap P$. Then $F(E/L) = F/L$ by [2, A, Theorem 9.3(c)]. Therefore, the condition of the theorem is satisfied for $(G/L, E/L)$, and hence $E/L \leq Z_{\mathcal{U}\Phi}(G/L)$ by the choice of G. Then $E \leq Z_{\mathcal{U}\Phi}(G)$ because $L \leq \Phi(G)$. This contradiction completes the proof of the theorem. □

{cor4.5:u

Corollary 4.5. *Let E be a solvable normal subgroup of a group G such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup in E conditionally covers or isolates every maximal pair (M, G), where $ME = G$, then G is supersolvable.*

{cor4.6:u

Corollary 4.6 (Ezquerro [11]). *Let E be a solvable normal subgroup of a group G such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup of E is a CAP-subgroup of E, then G is supersolvable.*

Proof. The proof follows from Corollary 4.5 and Lemma 2.10. □

{cor4.7:u

Corollary 4.7. *Let E be a solvable normal subgroup of G such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup in $F(E)$ conditionally covers or isolates every maximal pair (M, G), where $MF(E) = G$, then G is supersolvable.*

{cor4.8:u

Corollary 4.8 (Ezquerro [11]). *Let E be a solvable normal subgroup of G such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup in $F(E)$ is a CAP-subgroup of E, then G is supersolvable.*

ACKNOWLEDGMENTS

The authors express their sincere gratitude to the referee for useful remarks.

REFERENCES

1. L. A. Shemetkov, *Formations of Finite Groups*, in *Modern Algebra* (Nauka, Moscow, 1978) [in Russian].
2. K. Doerk and T. Hawkes, *Finite Soluble Groups*, in *de Gruyter Exp. Math.* (Walter de Gruyter, Berlin, 1992), Vol. 4.
3. O. Ore, "Contributions in the theory of groups of finite order," Duke Math. J. **5** (2), 431–460 (1939).
4. S. E. Stonehewer, "Permutable subgroups in Infinite Groups," Math. Z. **125**, 1–16 (1972).
5. A. Ballester-Bolinches, L. M. Ezquerro, and A. N. Skiba, "Local embeddings of some families of subgroups of finite groups," Acta Math. Sin. (Engl. Ser.) **25** (6), 869–882 (2009).
6. A. Ballester-Bolinches, L. M. Ezquerro, and A. N. Skiba, "Subgroups of finite groups with a strong cover-avoidance property," Bull. Aust. Math. Soc. **79** (3), 499–506 (2009).
7. D. L. Johnson, "A note on supersoluble groups," Canad. J. Math. **23**, 562–564 (1971).
8. H. G. Bray, W. E. Deskins, D. Johnson, J. F. Humphreys, B. M. Puttaswamaiah, P. Venzke, and G. L. Walls, *Between Nilpotent and Solvable*, ed. M. Weinstein (Polygonal Publ. House, Washington, NJ, 1982).
9. W. Guo, *The Theory of Classes of Groups*, in *Math. Appl.* (Kluwer Acad. Publ., Dordrecht, 2000), Vol. 505.
10. X. Guo and K. P. Shum, "Cover-avoidance properties and the structure of finite groups," J. Pure Appl. Algebra **181** (2-3), 297–308 (2003).
11. L. M. Ezquerro, "A contribution to the theory of finite supersolvable groups," Rend. Sem. Mat. Univ. Padova **89**, 161–170 (1993).
12. B. Huppert, *Endliche Gruppen*. I, in *Die Grundlehren Math. Wiss.* (Springer-Verlag, Berlin, 1967), Vol. 134.
13. H. Wielandt, *Subnormal Subgroups and Permutation Groups*, Lectures given at the Ohio State University (Columbus, Ohio, 1971).
14. Y. Fan, X. Y. Guo and K. P. Shum, "Remarks on two generalizations of normality of subgroups," Chinese J. Contemp. Math. **27** (2), 139–146 (2006).
15. Y. Wang, "c-normality of groups and its properties," J. Algebra **180** (3), 954–965 (1996).
16. L. A. Shemetkov and A. N. Skiba, "On the $\mathcal{X}\Phi$-hypercentre of finite groups," J. Algebra **322** (6), 2106–2117 (2009).