Siberian Mathematical Journal, Vol. 54, *No.* 1, *pp.* 65–73, 2013
Original Russian Text Copyright © 2013 *Kovaleva V.A. and Skiba A.N.*

FINITE SOLVABLE GROUPS WITH ALL n-MAXIMAL SUBGROUPS $\mathfrak{U}$-SUBNORMAL

V. A. Kovaleva and A. N. Skiba UDC 512.542

Abstract: We describe the finite solvable groups with all n-maximal subgroups $\mathfrak{U}$-subnormal.

DOI: 10.1134/S0037446613010011

Keywords: n-maximal subgroup, solvable group, supersolvable group, minimal nonsupersolvable group, $\mathfrak{U}$-subnormal subgroup

§ 1. Introduction

This article deals only with finite groups. The symbol $\pi(G)$ stands for the set of prime divisors of the order of a group G and the symbol $\mathfrak{U}$, for the class of all supersolvable groups.

Recall that a $\mathfrak{U}$-*residual* of G is the intersection of all normal subgroups N of G with $G/N \in \mathfrak{U}$; the $\mathfrak{U}$-residual of G is denoted by $G^{\mathfrak{U}}$.

Fix some ordering ϕ of primes. The record $p\phi q$ means that p precedes q in ϕ and $p \neq q$. Recall that a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ is called ϕ-*dispersive* whenever $p_1 \phi p_2 \phi \dots \phi p_n$ and for every i there is a normal subgroup of G of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$. Furthermore, if ϕ is such that $p\phi q$ always implies $p > q$ then every ϕ-dispersive group is called *Ore dispersive*.

Recall that a subgroup H of G is called a 2-*maximal* (*second maximal*) subgroup of G whenever H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define 3-maximal subgroups, and so on.

The interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its n-maximal subgroups (for $n > 1$). One of the earliest results in this direction is the article of Huppert [1] who established the supersolvability of the group whose all second maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of G are normal in G then the commutator subgroup G' of G is a nilpotent group and the principal rank of G is at most 2. These two results were developed by many authors. Among the recent results on n-maximal subgroups (for $n > 1$) we can mention [2], where the solvability of groups is established in which all 2-maximal subgroups enjoy the cover-avoidance property, and [3–5], where new characterizations of supersolvable groups in terms of 2-maximal subgroups were obtained. The classification of nonnilpotent groups whose all 2-maximal subgroups are TI-subgroups appeared in [6]. Description was obtained in [7] of groups whose every 3-maximal subgroup permutes with all maximal subgroups. The nonnilpotent groups are described in [8] in which every two 3-maximal subgroups are permutable. The groups are described in [9] whose all 3-maximal subgroups are S-quasinormal, that is, permute with all Sylow subgroups. Subsequently this result was strengthened in [10] to provide a description of the groups whose all 3-maximal subgroups are subnormal.

Despite of all these and many other known results about n-maximal subgroups, the fundamental work of Mann [11] still retains its value. It studied the structure of groups whose n-maximal subgroups are subnormal. Mann proved that if all n-maximal subgroups of a solvable group G are subnormal and $|\pi(G)| \geq n + 1$ then G is nilpotent; but if $|\pi(G)| \geq n - 1$ then G is ϕ-dispersive for some ordering ϕ of the set of all primes. Finally, in the case $|\pi(G)| = n$ Mann described G completely.

---

†) To the 70th anniversary of Victor Danilovich Mazurov.

Gomel. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 54, No. 1, pp. 86–97, January–February, 2013. Original article submitted October 15, 2012.

A supersolvable analog of a subnormal subgroup is the concept of $\mathfrak{U}$-subnormal subgroup. Recall that a subgroup H of a solvable group G is called $\mathfrak{U}$-*subnormal* in G whenever either $H = G$ or there is a chain $H = H_0 < \dots < H_n = G$ such that $|H_i/H_{i-1}|$ is a prime for each $i = 1, 2, \dots, n$.

The main goal of this article is to prove the following supersolvable analogs of Mann's theorems.

Theorem A. *If every n-maximal subgroup of a solvable group G is $\mathfrak{U}$-subnormal in G and $|\pi(G)| \geq n + 2$ then G is supersolvable.*

Theorem B. *Given a solvable group G with $|\pi(G)| \geq n + 1$, all n-maximal subgroups of G are $\mathfrak{U}$-subnormal in G if and only if G is a group of one of the following types:*

I. *G is supersolvable.*

II. *$G = A \rtimes B$, where $A = G^{\mathfrak{U}}$ and B are Hall subgroups of G, while G is Ore dispersive and satisfies the following:*

(1) *A is either of the form $N_1 \times \dots \times N_t$ with $t \geq 2$, where each N_i is a minimal normal subgroup of G, which is a Sylow subgroup of G for $i = 1, \dots, t$ or a Sylow p-subgroup of G of exponent p for some prime p; furthermore, the commutator subgroup, the Frattini subgroup, and the center of A coincide, every chief factor of G below $\Phi(A)$ is cyclic, while $P/\Phi(A)$ is a noncyclic chief factor of G;*

(2) *for every prime divisor p of the order of A every n-maximal subgroup H of G is supersolvable and induces on the Sylow p-subgroup of A an automorphism group which is an extension of some p-group by an abelian group of exponent dividing $p - 1$.*

Theorem C. *If every n-maximal subgroup of a solvable group G is $\mathfrak{U}$-subnormal in G and $|\pi(G)| \geq n$ then G is ϕ-dispersive for some ordering ϕ of the set of all primes.*

We use the standard notation which can be found in [12] if need be.

§ 2. Preliminary Results

Recall that a maximal subgroup H of a group G is called $\mathfrak{U}$-*normal* in G whenever $G/H_G \in \mathfrak{U}$; otherwise H is called $\mathfrak{U}$-*abnormal* in G. A subgroup H of a group G is called $\mathfrak{U}$-*subnormal* in G whenever either $H = G$ or there is a chain $H = H_0 < \dots < H_n = G$ such that H_{i-1} is a maximal $\mathfrak{U}$-normal subgroup of H_i for each $i = 1, 2, \dots, n$.

We use the following results.

Lemma 2.1. *Let G be a group with a $\mathfrak{U}$-subnormal subgroup H.*

(1) *If $K \leq G$ then $H \cap K$ is a $\mathfrak{U}$-subnormal subgroup of K* [13, *Lemma* 6.1.7(2)].

(2) *If N is a normal subgroup of G then HN/N is a $\mathfrak{U}$-subnormal subgroup of G/N* [13, *Lemma* 6.1.6(3)].

(3) *If $K \leq G$ is $\mathfrak{U}$-subnormal in H then K is $\mathfrak{U}$-subnormal in G* [13, *Lemma* 6.1.6(1)].

(4) *If K is a subgroup of G with $G^{\mathfrak{U}} \leq K$ then K is $\mathfrak{U}$-subnormal in G* [13, *Lemma* 6.1.7(1)].

Lemma 2.2 [12, Theorem 15.10]. *Let G be a group with supersolvable nilpotent residual. Let H and M be subgroups of G such that $H \in \mathfrak{U}$, $H \leq M$, and $HF(G) = G$. If H is $\mathfrak{U}$-subnormal in M then $M \in \mathfrak{U}$.*

Lemma 2.3 [12, Corollary 4.14.1]. *If a group G has four supersolvable subgroups of coprime indices in G then G is supersolvable.*

Lemma 2.4 [12, Chapter VI, Theorem 24.2]. *Let G be a solvable group. If $G^{\mathfrak{U}} \neq 1$ and every maximal $\mathfrak{U}$-abnormal subgroup of G belongs to $\mathfrak{U}$ then the following hold:*

(1) *$G^{\mathfrak{U}}$ is a p-group for some prime p;*

(2) *$G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})$ is a noncyclic chief factor of G;*

(3) *if $G^{\mathfrak{U}}$ is nonabelian then the center, commutator subgroup, and Frattini subgroup of G are of exponent p and coincide with one another;*

(4) *if $G^{\mathfrak{U}}$ is abelian then $G^{\mathfrak{U}}$ is elementary;*

(5) *if* $p > 2$ *then* $G^{\mathfrak{U}}$ *is of exponent* p*; for* $p = 2$ *the exponent of* $G^{\mathfrak{U}}$ *is at most* 4*;*

(6) *every pair of maximal* $\mathfrak{U}$*-abnormal subgroups of* G *are conjugate in* G*.*

Recall that a minimal nonsupersolvable group is a nonsupersolvable group whose all proper subgroups are supersolvable.

Lemma 2.5 [12, Chapter VI, Theorems 26.3 and 26.5]. *The following claims hold for every minimal nonsupersolvable group* G*:*

(1) G *is solvable and* $|\pi(G)| \leq 3$*;*

(2) *if* G *is not a Schmidt group then* G *is Ore dispersive;*

(3) $G^{\mathfrak{U}}$ *is a unique normal Sylow subgroup of* G*;*

(4) *If* S *is a complement to* $G^{\mathfrak{U}}$ *in* G *then* $S/S \cap \Phi(G)$ *is either a primary cyclic group or the Miller–Moreno group.*

§ 3. Groups with All Second Maximal Subgroups $\mathfrak{U}$-Subnormal

The proofs of Theorems A, B, and C rest on the properties of groups whose all 2-maximal subgroups are $\mathfrak{U}$-subnormal. In this section we describe these groups.

Recall that a *maximal chain of length* n (or an n*-maximal chain*) of G is a chain of the form

$$E_n < E_{n-1} < \dots < E_1 < E_0 = G,$$

where E_i is a maximal subgroup of E_{i-1} for $i = 1, \dots, n$. A subgroup H of G is called a *strictly* n*-maximal subgroup* of G whenever H is an n-maximal subgroup of G but not an n-maximal subgroup of each proper subgroup of G. A maximal chain $E_n < E_{n-1} < \dots < E_1 < E_0 = G$ is called *strictly* n*-maximal* whenever E_i is a strictly i-maximal subgroup of G for all $i = 1, \dots, n$.

Asaad [14], strengthening Huppert's results [1], studied the influence of strictly n-maximal subgroups on the structure of the group for $n = 2, 3, 4$. In particular, he proved that if all strictly 2-maximal subgroups are normal then the group is supersolvable. Spencer studied [15] the groups G whose every n-maximal chain includes at least one subnormal subgroup of G. In particular, he proved that if every maximal chain of length 2 in G includes a subnormal subgroup of G then G is a Schmidt group with abelian Sylow subgroups. In this section we study the groups whose every strictly 2-maximal chain includes a proper $\mathfrak{U}$-subnormal subgroup.

Observe that if G is supersolvable then every subgroup of G is $\mathfrak{U}$-subnormal.

Theorem 3.1. *The following conditions on a nonsupersolvable group* G *are equivalent:*

(1) $G = P \rtimes M$*, where* $P = G^{\mathfrak{U}}$ *is a minimal normal subgroup of* G *with* $|P| = p^{\alpha}$ *and* M *is a group of one of the following types:*

(i) M *is a cyclic group of order* q^a *for some prime* q *and* q^{a-1} *divides* $p - 1$*;*

(ii) $M = Q \rtimes R$*, where* $|Q| = q$ *divides* $p - 1$ *and* $|R| = r^b$ *divides* $p - 1$ *for some primes* q *and* r*, while* R *is a cyclic group, and* $Q \nleq C_G(P)$*.*

(2) *every* 2*-maximal subgroup of* G *is* $\mathfrak{U}$*-subnormal in* G*;*

(3) *every strictly maximal chain of length* 2 *in* G *includes a proper* $\mathfrak{U}$*-subnormal subgroup of* G*.*

Proof. (1) ⇒ (2) Suppose that $G = P \rtimes M$, where $P = G^{\mathfrak{U}}$ is a minimal normal subgroup of G with $|P| = p^{\alpha}$ and M is a group of the form (i). Then G includes precisely two classes of maximal conjugate subgroups, whose representatives are M and PM_1, where M_1 is a maximal subgroup of M. Since M is a cyclic group of order q^a and q^{a-1} divides $p-1$, it follows that M and PM_1 are supersolvable. Therefore, G is a minimal nonsupersolvable group and all subgroups of PM_1 are $\mathfrak{U}$-subnormal in PM_1. Since $|G : PM_1| = q$, it follows that PM_1 is $\mathfrak{U}$-normal in G. Therefore, by Lemma 2.1(3) all subgroups of PM_1 are $\mathfrak{U}$-subnormal in G. Take a maximal subgroup T of M. Then PT is a maximal normal subgroup of G; moreover, PT is supersolvable and $|G : PT| = q$. Consequently, all subgroups of PT are $\mathfrak{U}$-subnormal in G. Thus, T is $\mathfrak{U}$-subnormal in G.

Suppose now that M is a group of the form (ii). Then G includes precisely three classes of maximal conjugate subgroups, whose representatives are M, PR, and PQR_1, where R_1 is a maximal subgroup of R. It is clear that M, PR, and PQR_1 are supersolvable groups. Since $|G : PR| = q$ and $|G : PQR_1| = r$, it follows that PR and PQR_1 are $\mathfrak{U}$-normal in G. Therefore, all their maximal subgroups are $\mathfrak{U}$-subnormal in G. In addition, we infer as above that every maximal subgroup of M is $\mathfrak{U}$-subnormal in G.

The implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) To start with, verify that all maximal subgroups of nonprime index in G are supersolvable. Take a maximal subgroup M of G whose index $|G : M|$ is not a prime, and a maximal subgroup T of M. Suppose that T is not a strictly 2-maximal subgroup of G. This means that there exists at least one maximal chain of subgroups G_i of G (with $0 \leq i \leq n$) such that $T = G_r$ for $r \geq 3$ and the subgroup G_i is maximal in G_{i-1}. Among all these chains in G choose one of the greatest length:

$$T = G_r < \cdots < G_2 < G_1 < G_0 = G.$$

In this case G_2 is a strictly 2-maximal subgroup of G. Since $T \leq M$ and $T \leq G_2$, it follows that $T \leq G_2 \cap M$. If $G_2 \cap M = 1$ then $T = 1$ and $|M| = p$ for some prime p. Consequently, in this case we infer that M is a supersolvable group.

Suppose now that $G_2 \cap M \neq 1$ and G_1 is $\mathfrak{U}$-normal in G. If $T < G_1 \cap M$ then the maximality of T in M yields $G_1 \cap M = M$, whence $G_1 = M$. Since according to the assumption the subgroup G_1 is $\mathfrak{U}$-normal in G, it follows that the index $|G : M|$ is a prime, which contradicts the choice of M. Hence, $T = G_1 \cap M$. Consequently, T is $\mathfrak{U}$-subnormal in M by Lemma 2.1(1), so that $|M : T|$ is a prime. But if G_1 is not a $\mathfrak{U}$-normal subgroup of G then according to the condition the subgroup G_2 is $\mathfrak{U}$-subnormal in G. If $T < G_2 \cap M$ then $G_2 \cap M = M$ by the maximality of T in M; therefore, $G_2 = M$, which contradicts the maximality of M. Hence, $T = G_2 \cap M$; moreover, T is $\mathfrak{U}$-normal in M by Lemma 2.1(1) and $|M : T|$ is a prime. Since T is arbitrary, we infer that all maximal subgroups of M are of prime indices in M; thus, M is a supersolvable group. Consequently, all maximal subgroups of G of nonprime index are supersolvable.

Verify now that G is a solvable group. Assuming the contrary, consider a counterexample G of minimal order. Take a minimal normal subgroup N of G and a strictly 2-maximal chain $H/N < T/N < G/N$ in G/N. Then $H < T < G$ is a strictly 2-maximal chain in G; therefore, according to the condition either H or T is $\mathfrak{U}$-subnormal in G. Consequently, by Lemma 2.1(2) either H/N or T/N is $\mathfrak{U}$-subnormal in G/N. Hence, G/N is a solvable group by the choice of G. Therefore, N is the unique minimal normal subgroup of G; moreover, $N \nleq \Phi(G)$, and N is not abelian. Consequently, by Burnside's theorem on the solvability of biprimary groups there is a prime divisor p of the order of N with $p \geq 5$. Take an arbitrary Sylow p-subgroup N_p of N. Then some Sylow p-subgroup P of G satisfies $N_p \leq P$, which yields $N_p = P \cap N$. Hence, $P \leq N_G(N_p)$. Since N is a nonabelian group, G includes a maximal subgroup M with $N_G(N_p) \leq M$ and $N_p \leq P \leq M$, whence $G = NN_G(N_p) = NM$, and so $N \nleq M$.

Suppose that $N \cap M \nleq \Phi(M)$. Then there is a maximal subgroup T of M with $(N \cap M)T = M$. Therefore,

$$G = NM = N(N \cap M)T = NT.$$

Suppose that T is not a strictly 2-maximal subgroup of G. Then G includes a maximal subgroup V distinct from M such that T is a proper nonmaximal subgroup of V. Take a maximal subgroup L of V such that $T \leq L$ and L is a strictly 2-maximal subgroup of G. Suppose that V is $\mathfrak{U}$-normal in G. Then $G/V_G \in \mathfrak{U}$, whence $V_G \neq 1$. Therefore, $N \leq V_G$ and $G = NV \leq V$, which contradicts the choice of V. Hence, L is $\mathfrak{U}$-subnormal in G, which, as above, leads to a contradiction. Consequently, T is a strictly 2-maximal subgroup of G.

By the hypotheses of the theorem either T or M is $\mathfrak{U}$-subnormal in G. In both cases, as above, we arrive at a contradiction. Hence, $N \cap M \leq \Phi(M)$; therefore, $N \cap M$ is a nilpotent subgroup. Consequently, $N_N(N_p) = N_G(N_p) \cap N \leq M \cap N$, which implies the nilpotency of $N_N(N_p)$. But then $O^p(N) \neq N$ by Theorem 8.13 of [16, Chapter X]. Consequently, N has an abelian composition factor, and so N is an abelian group. The resulting contradiction completes the proof of the solvability of G.

Since G is a solvable group, Lemma 2.4 implies the following:

(a) $P = G^{\mathfrak{U}}$ is an s-group for some prime divisor s of the order of G;

(b) $P/\Phi(P)$ is a chief factor of G and $|P/\Phi(P)| > s$;

(c) all maximal subgroups of nonprime index of G are conjugate in G.

Take a maximal subgroup M of G of nonprime index. Then M is not $\mathfrak{U}$-subnormal in G. Therefore, $P \nleq M$ by Lemma 2.1(4), which yields $G = PM$. Since $|G : M| = |P/\Phi(P)|$ is not a prime, it follows, as we showed above, that M is a supersolvable group. Suppose that $\Phi(P) \neq 1$. Since M is supersolvable, it includes a maximal subgroup T with $|M : T| = s$. As in the proof of the solvability of G, we can show that T is a strictly 2-maximal subgroup of G. By the hypotheses of the theorem T is $\mathfrak{U}$-subnormal in G. Consequently, G includes a maximal subgroup L with $T \leq L$ and $G/L_G \in \mathfrak{U}$. But then $P = G^{\mathfrak{U}} \leq L_G$, which yields $G = PT = L$. The resulting contradiction shows that $\Phi(P) = 1$; therefore, using (b) we infer that P is a minimal normal subgroup of G and $G = P \rtimes M$. Take a maximal subgroup D of G with $P \leq D$. Then $D = P \rtimes (D \cap M)$, where $D \cap M$ is a maximal subgroup of M. As above, we can show that $D \cap M$ is a strictly 2-maximal subgroup of G; therefore, $D \cap M$ is $\mathfrak{U}$-subnormal in G. Since M is supersolvable, so is $D \cap M$, and Lemma 2.2 implies that so is D. Consequently, all maximal subgroups of G are supersolvable. Hence, G is a minimal nonsupersolvable group. Thus, claims (1)(i) and (1)(ii) follow from Lemma 2.5.

The proof of the theorem is complete.

It is not difficult to verify that the results of Asaad and Spencer follow from Theorem 3.1.

§ 4. Proofs of Theorems A, B, and C

Proof of Theorem A. Assume the theorem false and consider some counterexample G of minimal order. Take a maximal subgroup M of G. Then by the hypotheses all $(n-1)$-maximal subgroups of M are $\mathfrak{U}$-subnormal in G, and so they are $\mathfrak{U}$-subnormal in M by Lemma 2.1(1). Since the solvability of G implies that either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$, it follows that M is supersolvable by the choice of G. Hence, G is a minimal nonsupersolvable group. Then Lemma 2.5 yields $|\pi(G)| = 3$. Consequently, all maximal subgroups of G are $\mathfrak{U}$-normal; therefore, G is a supersolvable group, which contradicts our choice of it.

The proof of the theorem is complete.

Proof of Theorem B. *Necessity.* To start with, we verify that the hypotheses of the theorem are preserved for G/N, where N is a minimal normal subgroup of G. Indeed, if N is not a Sylow subgroup of G then $|\pi(G/N)| = |\pi(G)|$. Moreover, if H/N is an n-maximal subgroup of G/N then H is an n-maximal subgroup of G; therefore, H is $\mathfrak{U}$-subnormal in G. Consequently, H/N is $\mathfrak{U}$-subnormal in G/N by Lemma 2.1(2). But if G/N includes no n-maximal subgroups then by the solvability of G the trivial subgroup of G/N is $\mathfrak{U}$-subnormal in G/N and is a unique i-maximal subgroup of G/N for some $i < n$ with $i < |\pi(G/N)|$. Thus, in this case as well the hypotheses of the theorem are fulfilled for G/N. Finally, consider the case that N is a Sylow p-subgroup of G. Then by the Schur–Zassenhaus theorem G has a Hall p'-subgroup E. It is clear that $|\pi(E)| = |\pi(G)| - 1$ and E is a maximal subgroup of G. Therefore, all $(n-1)$-maximal subgroups of E are $\mathfrak{U}$-subnormal in E by Lemma 2.1(1). Thus, by induction on $|G|$ we may assume that G/N is either a supersolvable group or a group satisfying condition II.

Suppose that G is not supersolvable and verify that in this case G is a group satisfying condition II. Assume the contrary and consider a counterexample G of minimal order. Observe aforehand that $|\pi(G)| > 2$. Indeed, if $|\pi(G)| = 2$ then by assumption all maximal subgroups of G are $\mathfrak{U}$-normal, which implies the supersolvability of G.

Put $A = G^{\mathfrak{U}}$.

(a) *G is an Ore dispersive group and A is a nilpotent group.*

Take a minimal normal subgroup N of G. Then G/N is an Ore dispersive group and $(G/N)^{\mathfrak{U}}$ is a nilpotent group. It is known that the class of all Ore dispersive groups is a saturated formation; therefore, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Consequently, G includes a maximal subgroup L with $G = N \rtimes L$ and $L_G = 1$. Therefore, $C_G(N) = N$.

Verify firstly that G is an Ore dispersive group. Since G is solvable, G includes a maximal normal subgroup M with $|G : M| = p$ for some prime p and either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$. Since all $(n-1)$-maximal subgroups of M are $\mathfrak{U}$-subnormal in M by Lemma 2.1(1), the hypotheses of the theorem hold for M. Therefore, M is an Ore dispersive group by the choice of G. Denote by q the greatest number in $\pi(M)$. Take a Sylow q-subgroup M_q of M and a Sylow p-subgroup P of G. Since M_q is a characteristic subgroup of M, it follows that M_q is normal in G. Consider the case $|\pi(M)| = |\pi(G)|$ first. Then q is the greatest prime divisor of the order of G and $M_q \neq 1$. Therefore, G/M_q is an Ore dispersive group, and by the maximality of q so is G. Suppose now that $|\pi(M)| = |\pi(G)| - 1$. If $q > p$ then, as above, we conclude that G is an Ore dispersive group as well. Consequently, $p > q$, and so p is the greatest prime divisor of the order of G. Since $M_q \neq 1$, it follows that $N \leq M_q$. Consequently, N is a q-group. In addition, since $|\pi(G)| > 2$ as we have established already, $q \neq r \neq p$ for some prime divisor r of the order of G. Take a Hall r'-subgroup W of G. Then $PN \leq W$. Moreover, $|\pi(W)| = |\pi(G)| - 1$ and every $(n-1)$-maximal subgroup of W is $\mathfrak{U}$-subnormal in W. Consequently, W is an Ore dispersive group by the choice of G. Hence, P is normal in G, and so $P \leq C_G(N) \leq N$. The resulting contradiction shows that G is an Ore dispersive group.

Verify that A is a nilpotent group. If $|\pi(G)| = 3$ then by assumption either all maximal subgroups of G or all its 2-maximal subgroups are $\mathfrak{U}$-subnormal in G. In the first case we infer that G is a supersolvable group, which contradicts our assumption concerning G. Hence, all 2-maximal subgroups of G are $\mathfrak{U}$-subnormal, and so Theorem 3.1 implies that G is a minimal nonsupersolvable group with supersolvable abelian residual A. Thus, since $|\pi(G)| > 2$, it only remains to consider the case $|\pi(G)| \geq 4$. Assume that N is a p-group, and take a Sylow p-subgroup P of G. Observe that if $N \neq P$ then by Theorem A the subgroup L is supersolvable, and so A is nilpotent. Hence, $N = P$.

Suppose that $|\pi(G)| = 4$. Since G is not supersolvable, either all 2-maximal subgroups of G or all its 3-maximal subgroups are $\mathfrak{U}$-subnormal in G. Therefore, all second maximal subgroups of G are supersolvable. Consequently, L is either a supersolvable group or a minimal nonsupersolvable group. But in the first case the $\mathfrak{U}$-residual of G is nilpotent. Therefore, it suffices to consider the case that L is a minimal nonsupersolvable group.

In this case $L = Q \rtimes (R \rtimes T)$, where $Q = L^{\mathfrak{U}}$ is a noncyclic q-group which is a minimal normal subgroup of L, while R is a group of prime order r dividing $q - 1$, and $Q \nleq C_G(P)$. Put $V = PQR$. Theorem 3.1 implies that V cannot be a minimal nonsupersolvable group. Hence, V is supersolvable. Observe that $F(V)$ is a characteristic subgroup of V and V is a normal subgroup of G. Consequently, $F(V)$ is a normal subgroup of G, and so every Sylow subgroup of $F(V)$ is normal in G. But N is the unique minimal normal subgroup of G. Therefore, $F(V) = N = P$. Hence, $V/P \simeq QR$ is an abelian group, and so $R \leq C_L(Q)$. This contradiction shows that for $|\pi(G)| = 4$ the subgroup $G^{\mathfrak{U}}$ is nilpotent.

Consider, finally, the case $|\pi(G)| > 4$. If $\pi(L) = \{p_1, \dots, p_t\}$ then $t > 3$. Denote a Hall p_i'-subgroup of G by E_i. Then E_i is either a supersolvable group or a minimal nonsupersolvable group. In the second case $|\pi(E_i)| \leq 3$, and so Lemma 2.5 yields $|\pi(G)| \leq 4$, which contradicts the case under consideration. Hence, E_i is a supersolvable group, and Lemma 2.3 implies that so is G.

(b) *A is a Hall subgroup of G.*

According to claim (a) the group G is Ore dispersive. Consequently, for the greatest prime divisor r of the order of G the Sylow r-subgroup R is normal in G.

Since by claim (a) A is nilpotent, $A \leq F(G)$. Suppose that G includes two minimal normal subgroups H and K such that H is a p-group and K is a q-group, where $p \neq q$. Without loss of generality we may assume that $H \leq A$. As we showed above, the hypotheses of the theorem are preserved for G/K and

$$(G/K)^{\mathfrak{U}} = G^{\mathfrak{U}}K/K = AK/K;$$

therefore, AK/K is a Hall subgroup of G/K. Denote by A_p a Sylow p-subgroup of A. Then KA_p/K is a Sylow p-subgroup of AK/K; therefore, KA_p/K is a Sylow p-subgroup of G/K. Consequently, A_p is a Sylow p-subgroup of G. Suppose that $A_p \neq A$ and take a Sylow t-subgroup A_t of A with $t \neq p$. Considering the quotient G/H, we see that A_t is a Sylow t-subgroup of G. Therefore, A is a Hall subgroup of G. Consider now the case that all minimal normal subgroups of G are p-groups and $p = r$. Then $F(G) = P$ is a Sylow p-subgroup of G, and so $A \leq P$. If $H \neq A$ then, using the same arguments, we see that A is a Sylow p-subgroup of G. Consequently, we can put $H = A$. If $\Phi = \Phi(P) \neq 1$ then $\Phi A/\Phi = \Phi G^{\mathfrak{U}}/\Phi = (G/\Phi)^{\mathfrak{U}}$ is a Hall subgroup of G/Φ. If $H \leq \Phi$ then G/Φ is a nilpotent group. But P is normal in G, and so $\Phi \leq \Phi(G)$. This shows that G is a nilpotent group; therefore, $H = A = G^{\mathfrak{U}} = 1$, which contradicts our assumption concerning G. Consequently, $H \nleq \Phi$, and so $H\Phi/\Phi$ is a nontrivial p-group. Hence, $H\Phi = P$, and so $H = P$. The resulting contradiction shows that $\Phi(P) = 1$. By Maschke's theorem $P = N_1 \times \cdots \times N_k$ is the direct product of minimal normal subgroups of G. If $N_1 \neq P$ then G/N_1 and G/N_2 are supersolvable groups. Consequently, so is G. This contradiction shows that $A = G^{\mathfrak{U}}$ is a Hall subgroup of G.

(c) *The subgroup A is either of the form $N_1 \times \cdots \times N_t$ with $t \geq 2$, where N_i is a minimal normal subgroup of G, which amounts to a Sylow subgroup of G for $i = 1, \ldots, t$, or a Sylow p-subgroup of G of exponent p for some prime divisor p of the order of G. Furthermore, the commutator subgroup, the Frattini subgroup, and the center of A coincide, every chief factor of G below $\Phi(A)$ is cyclic, and $P/\Phi(A)$ is a noncyclic chief factor of G.*

Take a Sylow p-subgroup P of A, where p divides $|A|$. Claims (a) and (b) imply that P is a normal Sylow subgroup of G. Suppose that N is a minimal normal subgroup of G with $N \leq P$. Suppose firstly that $N \leq \Phi(G)$, and take a maximal subgroup M of G with $P \nleq M$. Then M is supersolvable by Theorem A. Therefore, $G/P \simeq M/M \cap P$ is a supersolvable group. In this case $A = P$ and every maximal subgroup of G which fails to include P is supersolvable. Observe also that every maximal subgroup of G which includes P is $\mathfrak{U}$-normal. Thus, by Lemma 2.4 the subgroup $A = G^{\mathfrak{U}}$ satisfies condition II(1).

Suppose that every minimal normal subgroup of G included into A is not included into $\Phi(G)$. If $N \neq P$ then G/N is supersolvable by Theorem A. Therefore, $A \leq N$, which contradicts claim (b). Consequently, all Sylow subgroups of A are minimal normal subgroups of G. Therefore, $A = N_1 \times \cdots \times N_t$ with $t \geq 2$, where N_i is a minimal normal subgroup of G for $i = 1, \ldots, t$.

(d) *For every prime divisor p of the order of A every n-maximal subgroup H of G is supersolvable and induces on the Sylow p-subgroup of A an automorphism group which is an extension of a p-group by an abelian group of exponent dividing $p - 1$.*

Take an n-maximal subgroup H of G. Suppose that H is a maximal subgroup of a subgroup V of G such that V is an $(n-1)$-maximal subgroup of G. Then all maximal subgroups of V are $\mathfrak{U}$-subnormal in G. Consequently, V is a supersolvable group. Thus, so is H.

There exist maximal subgroups M_1 and M_2 of G such that $H \leq M_1 \cap M_2$, while M_2 is $\mathfrak{U}$-normal in G and H is an $(n-1)$-maximal subgroup of M_1. It is clear that the claims of the theorem hold for M_1 and M_2. Consequently, if V_i is an $(n-1)$-maximal subgroup of M_i and P is a Sylow p-subgroup of $M_i^{\mathfrak{U}}$ then $V_i/C_{V_i}(P)$ is an extension of a p-group by an abelian group of exponent dividing $p - 1$.

Suppose that M_2 is a supersolvable group. Then $M_2/C_{M_2}(P)$ is an extension of a p-group by an abelian group of exponent dividing $p - 1$. Consequently,

$$H/C_H(P) = H/C_{M_2}(P) \cap H \simeq C_{M_2}(P)H/C_{M_2}(P)$$

is an extension of a p-group by an abelian group of exponent dividing $p - 1$ (for $i = 1, 2$).

Suppose that M_2 is not a supersolvable group. By Theorem A this means that M_2 is a Hall q'-subgroup of G, where $|G : M_2| = q$. Consequently, $|M_1 : M_1 \cap M_2| = q$, which yields $M_1^{\mathfrak{U}} \leq M_1 \cap M_2$.

Suppose that $\Phi(A) = 1$. In this case A is an abelian group. Put $E = M_1^{\mathfrak{U}}H$ and take a chief factor T/L of E, where $T \leq M_1^{\mathfrak{U}} \cap H$ and $|T/L| = p^a$. Take a Sylow p-subgroup P of $M_1^{\mathfrak{U}}$. Then $T \leq P$ and $H/C_H(P)$ is an extension of a p-group by an abelian group of exponent dividing $p - 1$. Therefore, $H/C_H(T/L)$ is an extension of a p-group by an abelian group of exponent dividing $p-1$. The isomorphism $AM_1/A \simeq M_1/M_1 \cap A$ yields

$$M_1^{\mathfrak{U}} \leq G^{\mathfrak{U}} = A,$$

and so $M_1^{\mathfrak{U}}$ is an abelian group. Consequently,

$$C_E(T/L) = M_1^{\mathfrak{U}}(C_E(T/L) \cap H) = M_1^{\mathfrak{U}}C_H(T/L).$$

Therefore,

$$E/C_E(T/L) = M_1^{\mathfrak{U}}H/M_1^{\mathfrak{U}}C_H(T/L) \simeq H/C_H(T/L)(H \cap M_1^{\mathfrak{U}})$$

is an extension of a p-group by an abelian group of exponent dividing $p-1$. Therefore, $|T/L| = p$. Furthermore, since H is a supersolvable group, so is E. It is clear that so is $E/M_1^{\mathfrak{U}}$. Thus, H is an $(a_1 + \dots + a_t)$-maximal subgroup of $M_1 \cap M_2$, where $p_1^{a_1} \dots p_t^{a_t}$ is the prime decomposition of the index $|M_1 \cap M_2 : H|$. Thus,

$$n \leq |M_1 \cap M_2 : H| + 1.$$

Consequently, M_2 includes an $(n-1)$-maximal subgroup of V with $H \leq V$. Thus, if P is a Sylow p-subgroup of $M_2^{\mathfrak{U}}$ then $V/C_V(P)$ is an extension of a p-group by an abelian group of exponent dividing $p-1$. But if $\Phi(A) \neq 1$ then M_1 is a supersolvable group, and in this case we can prove the required claim in the same fashion.

Sufficiency. It is obvious that if G is a supersolvable group then every subgroup of G is $\mathfrak{U}$-subnormal. Suppose that G is a group of type II. Take an n-maximal subgroup H of G. Since by the hypotheses of the theorem every n-maximal subgroup H of G is supersolvable and, for each $p \in \pi(G^{\mathfrak{U}})$, induces on the Sylow p-subgroup of $G^{\mathfrak{U}}$ an automorphism group which is an extension of a p-group by an abelian group of exponent dividing $p-1$, it follows that $G^{\mathfrak{U}}H$ is supersolvable (see claim (d) in the proof of necessity). Therefore, H is $\mathfrak{U}$-subnormal in $G^{\mathfrak{U}}H$. In addition, since $G^{\mathfrak{U}} \leq G^{\mathfrak{U}}H$, Lemma 2.1(4) implies that $G^{\mathfrak{U}}H$ is $\mathfrak{U}$-subnormal in G.

Therefore, Lemma 2.1(3) implies that H is $\mathfrak{U}$-subnormal in G. The proof of the theorem is complete.

Proof of Theorem C. Suppose that $|\pi(G)| = 2$. Then by assumption either all maximal subgroups of G or all its 2-maximal subgroups are $\mathfrak{U}$-subnormal in G. Therefore, every maximal subgroup of G is supersolvable. Consequently, G is either a supersolvable group or a minimal nonsupersolvable group. Therefore, Lemma 2.5 implies that G is ϕ-dispersive for some ordering ϕ of the set of all primes. Thus, we may assume that $|\pi(G)| > 2$.

The hypotheses of the theorem hold for G/N, where N is a minimal normal subgroup of G (see claim (a) in the proof of Theorem B). Consequently, by induction G/N is ϕ-dispersive for some ordering ϕ of the set of all primes. We may assume that N is not a Sylow subgroup of G. Moreover, since the class of all ϕ-dispersive groups is a saturated formation (see [12, p. 35]), it follows that $N \not\leq \Phi(G)$. Therefore (see claim (a) in the proof of Theorem B), G includes a maximal subgroup M such that $G = N \rtimes M$, all $(n-1)$-maximal subgroups of M are $\mathfrak{U}$-subnormal in M, and $|\pi(M)| = |\pi(G)|$. But then Theorem B implies that $G/N \simeq M$ is an Ore dispersive group. Thus, we may assume that N is the unique minimal normal subgroup of G, which implies $C_G(N) = N$.

Suppose that N is a p-group, and take a prime divisor q of the order of G distinct from p. Take a Hall q'-subgroup E of G. Then $N \leq E$ and the hypotheses of the theorem hold for E. Consequently, by induction a Sylow subgroup R of E is normal in E. Furthermore, if $N \not\leq R$ then $R \leq C_G(N) = N$. Hence, R is a Sylow p-subgroup of E. It is clear also that R is a Sylow p-subgroup of G and $(|G : N_G(R)|, r) = 1$ for every prime $r \neq q$. Since we consider the case $|\pi(G)| > 2$, this implies that R is normal in G.

The proof of the theorem is complete.

In closing note that the restrictions on $|\pi(G)|$ in Theorems A, B, and C cannot be weakened. For Theorem A this follows from the description of minimal nonsupersolvable groups (see Lemma 2.5). This also applies to Theorem C, as we can see from the example of the symmetric group of degree 4. Now take primes p, q, and r with $p > q > r$ such that r divides $q-1$, while both q and r divide $p-1$. Take a nonabelian group $Q \rtimes R$ of order qr, and a simple faithful $\mathbb{F}_pQR$-module P_1. Finally, put

$$G = (P_1 \rtimes (Q \rtimes R)) \times P_2,$$

where P_2 is a group of order p. Then the supersolvable residual $G^{\mathfrak{U}} = P_1$ of G is not a Hall subgroup of G. Furthermore, it is easy to verify that all 3-maximal subgroups of G are $\mathfrak{U}$-subnormal in G.

The authors are grateful to the referee for useful remarks.

References

1. *Huppert B.,* “Normalteiler and maximal Untergruppen endlicher gruppen,” Math. Z., Bd 60, 409–434 (1954).
2. *Guo X. Y. and Shum K. P.*, “Cover-avoidance properties and the structure of finite groups,” J. Pure Appl. Algebra, **181**, 297–308 (2003).
3. *Guo W., Shum K. P., and Skiba A. N.*, “X-Semipermutable subgroups of finite groups,” J. Algebra, **315**, 31–41 (2007).
4. *Li Baojun and Skiba A. N.*, “New characterizations of finite supersoluble groups,” Sci. China Ser. A: Math., **50**, No. 1, 827–841 (2008).
5. *Guo W. and Skiba A. N.*, “Finite groups with given s-embedded and n-embedded subgroups,” J. Algebra, **321**, 2843–2860 (2009).
6. *Li Shirong*, “Finite non-nilpotent groups all of whose second maximal subgroups are TI-groups,” Math. Proc. of the Royal Irish Academy, **100A**, No. 1, 65–71 (2000).
7. *Guo Wenbin, Legchekova H. V., and Skiba A. N.,* “Finite groups in which every 3-maximal subgroup permutes with all maximal subgroups,” Math. Notes, **86**, No. 3, 325–332 (2009).
8. *Guo W., Lutsenko Yu. V., and Skiba A. N.*, “On nonnilpotent groups with every two 3-maximal subgroups permutable,” Siberian Math. J., **50**, No. 6, 988–997 (2009).
9. *Lutsenko Yu. V. and Skiba A. N.,* “Structure of finite groups with S-quasinormal third maximal subgroups,” Ukrainian Math. J., **61**, No. 12, 1915–1922 (2009).
10. *Lutsenko Yu. V. and Skiba A. N.,* “Finite groups with subnormal second and third maximal subgroups,” Math. Notes, **91**, No. 5, 680–688 (2012).
11. *Mann A.*, “Finite groups whose n-maximal subgroups are subnormal,” Trans. Amer. Math. Soc., **132**, 395–409 (1968).
12. *Shemetkov L. A.*, Formations of Finite Groups [in Russian], Nauka, Moscow (1978).
13. *Ballester-Bolinches A. and Ezquerro L. M.*, Classes of Finite Groups, Springer-Verlag, Dordrecht (2006).
14. *Asaad M.,* “Finite groups some whose n-maximal subgroups are normal,” Acta Math. Hung., **54**, No. 1–2, 9–27 (1989).
15. *Spencer A. E.*, “Maximal nonnormal chains in finite groups,” Pacific J. Math., **27**, No. 1, 167–173 (1968).
16. *Huppert B. and Blackburn N.,* Finite Groups. III, Springer-Verlag, Berlin and New York (1982).

V. A. Kovaleva; A. N. Skiba
Skorina Gomel State University, Gomel, Belarus
E-mail address: vika.kovalyova@rambler.ru; alexander.skiba49@gmail.com