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On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups

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Abstract. Let G be a finite group. A subgroup A of G is said to be S-quasinormal in G if $AP = PA$ for all Sylow subgroups P of G. The symbol H_{sG} denotes the subgroup generated by all those subgroups of H which are S-quasinormal in G. A subgroup H is said to be S-supplemented in G if G has a subgroup T such that $T \cap H \leqslant H_{sG}$ and $HT = G$; see [24].

Theorem A. *Let E be a normal subgroup of a finite group G. Suppose that for every non-cyclic Sylow subgroup P of E, either all maximal subgroups of P or all cyclic subgroups of P of prime order and order* 4 *are S-supplemented in G. Then each G-chief factor below E is cyclic.*

Theorem B. *Let $\mathscr{F}$ be any formation and G a finite group. If $E \lhd G$ and $F^*(E) \leqslant Z_{\mathscr{F}}(G)$, then $E \leqslant Z_{\mathscr{F}}(G)$.*

These theorems give positive answers to two questions of Shemetkov and strengthen results of various authors.

1 Introduction

Throughout this paper, all groups considered are finite.

An interesting question in finite group theory is to determine the influence of the embedding properties of members of some distinguished families of subgroups on the structure of the group. The present paper adds some results to this line of research.

Recall that a subgroup A of a group G is said to be S-quasinormal, S-permutable, or $\pi(G)$-permutable in G (Kegel [17]) if $AP = PA$ for all Sylow subgroups P of G; the subgroup A is said to be c-normal in G (Wang [26]) if G has a normal subgroup T such that $AT = G$ and $A \cap T \leqslant A_G$; and A is said to be c-supplemented in G (Ballester-Bolinches, Wang and Guo [7]) if G has a subgroup T such that $AT = G$ and $A \cap T \leqslant A_G$, where A_G is the largest normal subgroup of G contained in A. Buckley [8] obtained a description of nilpotent groups of odd order all of whose subgroups of prime order are normal. As a consequence, he also proved that a group of odd order is supersoluble if all subgroups of prime order are normal. Applying the description of minimal non-supersoluble groups due to Huppert [15] and Doerk [9], we can go further and prove that a group is supersoluble if all cyclic subgroups of prime order and order 4 are normal. Later, Srinivasan [25] proved that a group G is supersoluble if every maximal subgroup of every Sylow subgroup of G is normal in G. These results have been developed in various directions, especially in the framework of formation theory.

Recall that a formation $\mathscr{F}$ is a class of groups which is closed under taking homomorphic images and such that each group G has a smallest normal subgroup (denoted by $G^{\mathscr{F}}$) whose quotient is in $\mathscr{F}$. A formation $\mathscr{F}$ is said to be saturated if $G \in \mathscr{F}$ for any group G with $G/\Phi(G) \in \mathscr{F}$. If $\mathscr{F}$ is a saturated formation containing all supersoluble groups and G is a group with a normal subgroup E, then the following results are true.

(1) If $G/E \in \mathscr{F}$ and the cyclic subgroups of E of prime order and order 4 are either all S-quasinormal (Ballester-Bolinches and Pedraza-Aguilera [5], Asaad and Csörgő [2]), or all c-normal (Ballester-Bolinches and Wang [6]), or all c-supplemented (Ballester-Bolinches, Wang and Guo [7], Wang and Li [28]) in G, then $G \in \mathscr{F}$.

(2) If $G/E \in \mathscr{F}$ and the cyclic subgroups of every Sylow subgroup of $F^*(E)$ of prime order and order 4 are either all S-quasinormal (Li and Wang [18]), or all c-normal (Wei, Wang and Li [31]), or all c-supplemented (Wang, Wei and Li [29], Wei, Wang and Li [32]) in G, then $G \in \mathscr{F}$.

(3) If $G/E \in \mathscr{F}$ and the maximal subgroups of every Sylow subgroup of E are either all S-quasinormal (Asaad [1]) or all c-normal (Wei [30]) or all c-supplemented (Ballester-Bolinches and Guo [4]) in G, then $G \in \mathscr{F}$.

(4) If $G/E \in \mathscr{F}$ and the maximal subgroups of every Sylow subgroup of $F^*(E)$ are either all S-quasinormal (Li and Wang [19]), or all c-normal (Wei, Wang and Li [31]), or all c-supplemented (Wei, Wang and Li [29]) in G, then $G \in \mathscr{F}$.

In these results $F^*(E)$ denotes the generalized Fitting subgroup of E, that is, the product of all normal quasinilpotent subgroups of E; see [16, Chapter X].

Bearing in mind the above results L. A. Shemetkov asked in 2004 at the Gomel Algebraic Seminar the following two questions:

(I) Can the above-mentioned results be strengthened to assert that every G-chief factor below E is cyclic?

(II) Does the conclusion that G-chief factors below E are cyclic still hold if we omit the hypothesis that $G/E \in \mathscr{F}$?

A partial solution of these problems was obtained in [23, Theorem 1.4]. Our main purpose here is to answer these questions completely.

We shall use the notion of S-quasinormal embedding introduced in [24]: a subgroup H of a group G is said to be S-supplemented in G if G has a subgroup T such that $G = HT$ and $T \cap H \leqslant H_{sG}$, where H_{sG} is the subgroup generated by all subgroups of H which are S-quasinormal in G.

We prove:

Theorem A. *Let E be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of E, either all maximal subgroups of P or all cyclic subgroups of P of prime order and order 4 are S-supplemented in G. Then each G-chief factor below E is cyclic.*

Theorem B. *Let $\mathscr{F}$ be any formation and G a group. If $E \lhd G$ and $F^*(E) \leqslant Z_{\mathscr{F}}(G)$, then $E \leqslant Z_{\mathscr{F}}(G)$.*

Here $Z_{\mathscr{F}}(G)$ denotes the product of all normal subgroups N of G such that $H/K \rtimes (G/C_G(H/K)) \in \mathscr{F}$ for each G-chief factor H/K of N; see [10, p. 389].

Corollary 1.1. *Let E be a normal subgroup of a group G. If every G-chief factor below $F^*(E)$ is cyclic, then every G-chief factor below E is cyclic.*

It is rather clear that if $\mathscr{F}$ is a saturated formation containing all supersoluble groups and G is a group with a cyclic normal subgroup E such that $G/E \in \mathscr{F}$, then $G \in \mathscr{F}$. Hence Theorem A and Corollary 1.1 allow us to give affirmative answers to Questions I and II. Finally, in view of Corollary 1.1, Theorem A not only generalizes the results in [1], [2], [4]–[8], [18], [19], [25]–[32] mentioned above but also gives shorter proofs of many of them.

2 Preliminaries

We write $\mathscr{U}$ to denote the class of all supersoluble groups. The symbol $\mathscr{A}(p-1)$ denotes the formation of all abelian groups of exponent dividing $p-1$; see [22].

The following lemma is well known (see, for example, [33, Chapter I, Theorem 1.4]).

Lemma 2.1. *Let p be a prime, and H/K a p-chief factor of a group G. Then $|H/K| = p$ if and only if $G/C_G(H/K) \in \mathscr{A}(p-1)$.*

Lemma 2.2. *Let E be a normal p-subgroup of a group G. If $E \leqslant Z_{\mathscr{U}}(G)$, then $(G/C_G(E))^{\mathscr{A}(p-1)} \leqslant O_p(G/C_G(E))$.*

Proof. Let $1 = E_0 < E_1 < \cdots < E_t = E$ be a chief series of G below E. Let $C_i = C_G(E_i/E_{i-1})$ and $C = C_1 \cap C_2 \cap \cdots \cap C_t$. Then $C_G(E) \leqslant C$ and by [12, Corollary 5.3.3], $C/C_G(E)$ is a p-group. On the other hand, since $|E_i/E_{i-1}| = p$, we have $G/C \in \mathscr{A}(p-1)$. Hence $(G/C_G(E))^{\mathscr{A}(p-1)} \leqslant O_p(G/C_G(E))$. □

Recall that the product $\mathscr{M}\mathscr{H}$ of the formations $\mathscr{M}$ and $\mathscr{H}$ is the class $(G \mid G^{\mathscr{H}} \in \mathscr{M})$. It is well known that the product of any two formations is also a formation.

Lemma 2.3 ([13, Lemma 3.1]). *Let $\mathcal{F} = \mathcal{N}_p\mathcal{H}$, where $\mathcal{H}$ is some formation of nilpotent groups and $\mathcal{N}_p$ is the class of all p-groups for some prime p. Suppose that $G = AB$, where A, B are normal in G. If $A, B \in \mathcal{F}$ and $(|G : A|, |G : B|) = 1$, then $G \in \mathcal{F}$.*

Lemma 2.4. *Let A, B, E be normal subgroups of a group G. Suppose that $G = AB$. If $E \leqslant Z_{\mathfrak{U}}(A) \cap Z_{\mathfrak{U}}(B)$ and $(|G : A|, |G : B|) = 1$, then $E \leqslant Z_{\mathfrak{U}}(G)$.*

Proof. Assume that (G, E) is a counter-example with $|G||E|$ minimal. Then $E \neq 1$. Let L be a minimal normal subgroup of G contained in E. Since $Z_{\mathfrak{U}}(A)$ is soluble, L is a p-group for some prime p. Let $C = C_G(L)$, $C_1 = C \cap A$, $C_2 = C \cap B$. Since $E \leqslant Z_{\mathfrak{U}}(A)$, Lemma 2.2 implies that A has a normal subgroup V_1 such that $C_1 \leqslant V_1$, V_1/C_1 is a p-group and $A/V_1 \in \mathcal{A}(p-1)$. Similarly, B has a normal subgroup V_2 such that $C_2 \leqslant V_2$, V_2/C_2 is a p-group and $B/V_2 \in \mathcal{A}(p-1)$. Since $CA/C \simeq A/C_1$, $CB/C \simeq B/C_2$ and $(|G : A|, |G : B|) = 1$, it follows that $G/C \in \mathcal{A}(p-1)$ by Lemma 2.3 and [33, Appendices, Corollary 6.4]. Hence $|L| = p$ by Lemma 2.1. It is clear also that the hypothesis is still true for G/L. Thus $E/L \leqslant Z_{\mathfrak{U}}(G/L) = Z_{\mathfrak{U}}(G)/L$ by the choice of (G, E). Therefore $E \leqslant Z_{\mathfrak{U}}(G)$. This contradiction completes the proof. □

Lemma 2.5 ([11, Theorem 2.4]). *Let P be a p-group and α a p′-automorphism of P.*
(1) *If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$.*
(2) *If $[\alpha, \Omega_1(P)] = 1$ and either p is odd or P is abelian, then $\alpha = 1$.*

Lemma 2.6. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and let G be a group with soluble $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p-group for some prime p. In addition, if all cyclic subgroups of P of prime order and order 4 are S-supplemented in G, then $|P/\Phi(P)| = p$. In particular, p is not the smallest prime dividing $|G|$.*

Proof. See the proof of [24, Lemma 2.12]. □

Lemma 2.7 ([2, Lemma 4]). *Let P be a p-subgroup of a group G, where $p > 2$. Suppose that all subgroups of P of order p are S-quasinormal in G. If a is a p′-element of $N_G(P) \setminus C_G(P)$, then a induces in P a fixed-point free automorphism.*

Lemma 2.8 ([17]). *Let G be a group and $H \leqslant K \leqslant G$.*
(1) *If H is S-quasinormal in G, then H is S-quasinormal in K.*
(2) *Suppose that H is normal in G. Then K/H is S-quasinormal in G/H if and only if K is S-quasinormal in G.*
(3) *If H is S-quasinormal in G, then H is subnormal in G.*
(4) *If A and B are S-quasinormal subgroups of G, then so is $A \cap B$.*

The following observation is well known (see, for example, [20, Lemma A]).

Lemma 2.9. *If H is S-quasinormal in G and H is a p-group for some prime p, then $O^p(G) \leqslant N_G(H)$.*

Lemma 2.10. *Let P be a Sylow p-subgroup of a group G and E a normal subgroup of P. Then $E_{sG} = E_G$.*

Proof. By Lemma 2.9 we have $O^p(G) \leqslant N_G(E_{sG})$. On the other hand, for any $x \in P$ we have $(E_{sG})^x \leqslant E$. Moreover $(E_{sG})^x$ is S-quasinormal in G since E_{sG} is S-quasinormal in G. Hence $(E_{sG})^x = E_{sG}$, so E_{sG} is normal in G. □

The following lemma is a corollary of Shemetkov's general results on radicals (see [21], [22, Theorem 15.11]). For the reader's convenience, we give a direct proof.

Lemma 2.11. *$F^*(G) \leqslant C_G(H/K)$ for any abelian chief factor H/K of the group G.*

Proof. By the Jordan–Hölder theorem, we may suppose that $H \leqslant F(G)$. Then by [33, Appendices, Theorem 2.5], we have $F(G) \leqslant C_G(H/K)$. On the other hand, $F^*(G) = E(G)F(G)$ and $[E(G), F(G)] = 1$, where $E(G)$ is the layer of G; see [16, p. 128]. Hence $F^*(G) \leqslant C_G(H/K)$. □

3 Proofs of Theorems A and B

Proof of Theorem A. Suppose that the theorem is false and consider a counterexample (G, E) for which $|G|\,|E|$ is minimal. Let P be a Sylow p-subgroup of E, where p is the smallest prime dividing $|E|$ and $C = C_G(P)$. If P is not a non-abelian 2-group we use Ω to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega = \Omega_2(P)$.

(1) *If X is a Hall subgroup of E, the hypothesis is still true for (X, X). If, in addition, X is normal in G, then the hypothesis also holds for (G, X) and for $(G/X, E/X)$.*
This follows directly from [24, Lemma 2.10].

(2) *If X is a non-identity normal Hall subgroup of E, then $X = E$.*
Since X is a characteristic subgroup of E, it is normal in G and by (1) the hypothesis is still true for $(G/X, E/X)$ and for (G, X). If $X \neq E$, the minimal choice of (G, E) implies that $E/X \leqslant Z_{\mathcal{U}}(G/X)$ and $X \leqslant Z_{\mathcal{U}}(G)$. Hence $E \leqslant Z_{\mathcal{U}}(G)$, a contradiction.

(3) *If $E \neq P$, then E is not p-nilpotent.*
Indeed, if E is p-nilpotent, then by (2), p does not divide $|E|$, contrary to the choice of p.

(4) *P is not cyclic.*
This follows from (3) and [12, Theorem 7.6.1].

(5) *If $E \neq P$, then every maximal subgroup of P is S-supplemented in G.*
Suppose that this is false. Then for all $x \in G$ all cyclic subgroups of P^x of prime order and order 4 are S-supplemented in G. By (3), E is not p-nilpotent, so it has a p-closed Schmidt subgroup by [14, Chapter IV, Satz 5.4], which in view of [24, Lemma 2.10] contradicts Lemma 2.6.

(6) *If* $E = G$, *then* $P_G \neq 1$.

Suppose that $E = G$ and $P_G = 1$. Then by [24, Lemma 2.5 (6)] and Lemma 2.8 (3), we have $P_{sG} = 1$. Hence by (5) every maximal subgroup of P has a complement T in G. Since for a Sylow p-subgroup T_p of T we have $|T_p| = p$, the hypothesis is true for (T, T). Therefore T is supersoluble, and so by [24, Lemma 2.2], G is q-closed, where q is the largest prime divisor of $|G|$. But this contradicts (2).

(7) $E \neq G$.

Suppose that $E = G$. Then $P_G \neq 1$ by (6). Moreover $E \neq P$. Hence by (5) and [24, Lemma 2.10] the hypothesis holds for G/N for every minimal normal subgroup N of G contained in P. Therefore G/N is supersoluble by the choice of (G, E). Thus $N \not\leqslant \Phi(G)$ and N is the only minimal normal subgroup of G contained in P. Let M be a maximal subgroup of G such that $G = N \rtimes M$.

For every maximal subgroup A of P containing N we have $AM = G$, so $M \simeq G/N$ is a supersoluble supplement of A in G. Thus by (2) and [24, Lemma 2.2] some maximal subgroup V of P neither contains N nor has a supersoluble supplement in G. Let $L = V_{sG}$ and let T be a subgroup of G such that $VT = G$ and $T \cap V \leqslant L$. By Lemma 2.10, $L = V_G$. Suppose that $L = 1$. Then T is supersoluble (see the proof of (6)), contrary to the choice of V. Thus $L \neq 1$, so $N \leqslant L \leqslant V$. This contradiction completes the proof of (7).

(8) $E = P$ *is not a minimal normal subgroup of* G.

Suppose that $E \neq P$. By (1) the hypothesis holds for (E, E), so E is supersoluble by (7) and the choice of (G, E). Hence a Hall p'-subgroup H of E is normal in E and $H \neq E$, which contradicts (2). Therefore $E = P$. Suppose that P is a minimal normal subgroup of G. Suppose also that some proper subgroup $V \neq 1$ of P has a proper supplement T in G. Then $P = V(P \cap T)$, $1 \neq P \cap T \neq P$ and $P \cap T \lhd G$, which contradicts the minimality of P. Hence either every minimal subgroup or every maximal subgroup of P is S-quasinormal in G, which in view of [24, Lemma 2.11] contradicts the minimality of P.

(9) *Every cyclic subgroup of* P *of prime order and order* 4 *is* S*-supplemented in* G.

Suppose that this is false. By hypothesis every maximal subgroup of P is S-supplemented in G. Hence by [24, Lemma 2.10] the hypothesis holds for G/N for any minimal normal subgroup N of G contained in P, so $P/N \leqslant Z_{\mathfrak{U}}(G/N)$ by the choice of $(G, E) = (G, P)$. Therefore N is the only minimal normal subgroup of G contained in P and $|N| > p$.

We show that $\Phi(P) \neq 1$. Indeed, suppose that $\Phi(P) = 1$. Then P is an elementary abelian p-group. Let N_1 be any maximal subgroup of N. We show that N_1 is S-quasinormal in G. Let B be a complement of N in P and $V = N_1 B$. Then V is S-supplemented in G. Let T be a subgroup of G such that $G = TV$ and $T \cap V \leqslant V_{sG}$. If $T = G$, then $V = V_{sG}$ is S-quasinormal in G and hence the subgroup

$$V \cap N = V_{sG} \cap N = N_1 B \cap N = N_1(B \cap N) = N_1$$

is S-quasinormal in G by Lemma 2.8 (4). Let $T \neq G$. Then $1 \neq T \cap P < P$. Since $G = PT$ and P is abelian, $T \cap P$ is normal in G and hence $N \leqslant T \cap P$. Therefore

$N_1 \leqslant T$, which implies $N_1 \leqslant V \cap T \leqslant V_{sG}$. Obviously $N \cap V = N_1$, so $N \cap V_{sG} = N_1$ is S-quasinormal in G. Therefore every maximal subgroup of N is S-quasinormal in G. Hence some maximal subgroup of N is normal in G by [24, Lemma 2.11], a contradiction. Therefore $\Phi(P) \neq 1$.

Since $P/N \leqslant Z_{\mathfrak{U}}(G/N)$ we have $P/\Phi(P) \leqslant Z_{\mathfrak{U}}(G/\Phi(P))$. Therefore by Lemma 2.2, $(G/C_G(P/\Phi(P)))^{\mathscr{A}(p-1)}$ is a p-group. Hence $(G/C)^{\mathscr{A}(p-1)}$ is a p-group by [12, Theorem 5.1.4]. Thus $G/C_G(N) \in \mathscr{A}(p-1)$ since $O_p(G/C_G(N)) = 1$ by [33, Appendices, Corollary 6.4]. Therefore $|N| = p$ by Lemma 2.1, a contradiction.

(10) *G has a normal subgroup $1 \neq R \leqslant P$ such that P/R is a non-cyclic chief factor of G, $R \leqslant Z_{\mathfrak{U}}(G)$ and $V \leqslant R$ for any normal subgroup $V \neq P$ of G contained in P.*

Let P/R be a chief factor of G. Then $R \neq 1$ by (8). Moreover by (9) the hypothesis holds for (G, R), so $R \leqslant Z_{\mathfrak{U}}(G)$ and P/R is not cyclic by the choice of $(G, P) = (G, E)$. Now let V be any normal subgroup of G with $V < P$. Then $V \leqslant Z_{\mathfrak{U}}(G)$. If $V \nleqslant R$, then from the G-isomorphism $P/R = VR/R \simeq V/(V \cap R)$ we deduce that $P \leqslant Z_{\mathfrak{U}}(G)$, contrary to the choice of (G, P). Hence $V \leqslant R$.

(11) $P \leqslant O^p(G)$.

Suppose that $P \nleqslant O^p(G)$. Then in view of the G-isomorphism

$$O^p(G)P/O^p(G) \simeq P/(O^p(G) \cap P),$$

G has a cyclic chief factor of the form P/V, where $O^p(G) \cap P \leqslant V$, which contradicts (10).

(12) *G has no normal maximal subgroup M such that $|G : M| = p$ and $MP = G$.*

Otherwise, from the G-isomorphism $G/M \simeq P/(M \cap P)$ we deduce that $P/(M \cap P)$ is a cyclic chief factor of G, contrary to (10).

(13) *If L is a cyclic subgroup of P and either $|L|$ is a prime or $|L| = 4$, then L is S-quasinormal in G.*

Suppose that L is not S-quasinormal in G. Then by (9), G has a subgroup T such that $LT = G$ and $L \cap T \leqslant L_{sG} \neq L$. Therefore $T \neq G$ and either $|G : T| = p$ or $L \cap T = 1$ and $|G : T| = 4$. Obviously $T < N_G(P \cap T)$, so in view of (12), we have $P \cap T \lhd G$. But $P = P \cap LT = L(P \cap T)$ and $P/(P \cap T) \simeq L/(L \cap P \cap T)$, so G has a cyclic chief factor of the form P/R. This contradicts (10) and completes the proof of (13).

(14) *If $O^p(G) = G$, then $\Omega \leqslant Z_{\mathfrak{U}}(G)$.*

In view of (10) we may suppose that $\Omega = P$. Therefore for some cyclic subgroup L of P of prime order or order 4 we have $L \nleqslant R$. But from $O^p(G) = G$, Claim (13) and Lemma 2.9 we deduce that $L \lhd G$, so $L \leqslant R$ by (10). This contradiction shows that $\Omega \leqslant Z_{\mathfrak{U}}(G)$.

(15) *There is a prime $q \neq p$ such that q divides $|G : C|$.*

Otherwise, any G_p-chief factor of P, where G_p is a Sylow p-subgroup of G, is a chief factor of G, which implies $P \leqslant Z_{\mathfrak{U}}(G)$.

(16) *$C_G(\Omega)/C$ is a p-group.*
This follows from Lemma 2.5.

(17) *There is a prime $q \neq p$ such that $O^q(G) \neq G$.*
Assume that $O^q(G) = G$ for all primes $q \neq p$. Then for every G-chief factors H/K of order p we have $C_G(H/K) = G$. In particular, for a minimal normal subgroup L of G contained in R we have $L \leqslant Z(G)$.

If $\Omega < P$ then $\Omega \leqslant R \leqslant Z_{\mathfrak{U}}(G)$ by (10). Hence $\Omega \leqslant Z_{\infty}(G)$ and by [12, Corollary 5.3.3], $G/C_G(\Omega)$ is a p-group. Therefore by (16), G/C is a p-group, which contradicts (15).

Hence $\Omega = P$. Thus $P/R = (V_1R/R)(V_2R/R)\dots(V_tR/R)$, where V_i is a cyclic group of order p or order 4 and V_iR/R is a cyclic group of order p. By (13), V_i is S-quasinormal in G, so if Q is a Sylow subgroup of G, then V_i is subnormal in V_iQ by Lemma 2.8 (3). Suppose that $p = 2$. Then V_iQ is nilpotent, so $Q \leqslant C_G(V_i)$. Therefore $O^p(G) \leqslant C_G(P/V)$, which implies that $C_G(P/V) = G$, a contradiction. Thus $p > 2$. But in this case G/C is a p-group by Lemma 2.7 and (13), which contradicts (15). Hence we have (17).

(18) *$O^p(G) = G$.*
By (11), $P \leqslant O^p(G)$. Hence the hypothesis holds for $(O^p(G), P)$. Suppose that $O^p(G) \neq G$. Then $P \leqslant Z_{\mathfrak{U}}(O^p(G))$ by the choice of (G, P). On the other hand, there is a prime $q \neq p$ such that $O^q(G) \neq G$, which implies that $P \leqslant Z_{\mathfrak{U}}(O^q(G))$. Hence $P \leqslant Z_{\mathfrak{U}}(G)$ by Lemma 2.4, a contradiction.

The final contradiction. By (14) and (18), $\Omega \leqslant Z_{\mathfrak{U}}(G)$. Hence by Lemma 2.2, $(G/C_G(\Omega))^{\mathscr{A}(p-1)}$ is a p-group. Thus $G/C_G(P/R) \in \mathscr{A}(p-1)$ by (16), so $|P/R| = p$. This contradiction completes the proof of Theorem A. □

Proof of Theorem B. Suppose that the theorem is false and consider a counterexample (G, E) for which $|G|\,|E|$ is minimal.

Let $F^* = F^*(E)$ and $F = F(E)$. First suppose that $F \neq 1$. Let L be a minimal normal subgroup of G contained in F and $C = C_G(L) \cap E$. We shall show that the hypothesis is still true for $(G/L, C/L)$. Indeed, clearly $L \leqslant Z(C)$. Moreover $F^* \leqslant C$ by Lemma 2.11. Therefore $F^*(C/L) = F^*/L$ by [16, Chapter X, (13.6)]. Hence the hypothesis is still true for $(G/L, C/L)$, so $C/L \leqslant Z_{\mathscr{F}}(G/L)$ by the choice of (G, E). Since $L \leqslant F \leqslant F^* \leqslant Z_{\mathscr{F}}(G)$, it follows that $C \leqslant Z_{\mathscr{F}}(G)$. On the other hand, it follows from the G-isomorphism $E/C \simeq C_G(L)E/C_G(L)$ and [10, Chapter IV, Proposition 1.5] that $E/C \leqslant Z_{\mathscr{F}}(G/C)$. Thus $E \leqslant Z_{\mathscr{F}}(G)$, a contradiction. Hence $F = 1$, so in view of [16, Chapter X, (13.6)], $F^* = N_1 \times N_2 \times \cdots \times N_t$, where N_i is a minimal normal subgroup of G. Therefore $C_G(F^*) = C_G(N_1) \cap C_G(N_2) \cap \cdots \cap C_G(N_t)$. But $G/C_G(N_i) \in \mathscr{F}$ for $i = 1, 2, \dots, t$ and hence $G/C_G(F^*) \in \mathscr{F}$. On the other hand, by [16, Chapter X, (13.12)] we have $C_G(F^*) \cap E \leqslant F = 1$. Therefore from the G-isomorphism $EC_G(F^*)/C_G(F^*) \simeq E/(C_G(F^*) \cap E)$ we deduce that $E \leqslant Z_{\mathscr{F}}(G)$. This contradiction completes the proof. □

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References

[1] M. Asaad. On maximal subgroups of finite group. *Comm. Algebra* **26** (1998), 3647–3652.
[2] M. Asaad and P. Csörgő. Influence of minimal subgroups on the structure of finite group. *Arch. Math.* (*Basel*) **72** (1999), 401–404.
[3] A. Ballester-Bolinches and L. M. Ezquerro. *Classes of finite groups* (Springer-Verlag, 2006).
[4] A. Ballester-Bolinches and X. Y. Guo. On complemented subgroups of finite groups. *Arch. Math.* (*Basel*) **72** (1999), 161–166.
[5] A. Ballester-Bolinches and M. C. Pedraza-Aguilera. On minimal subgroups of finite groups. *Acta Math. Hungar.* **73** (1996), 335–342.
[6] A. Ballester-Bolinches and Y. Wang. Finite groups with some C-normal minimal subgroups. *J. Pure Appl. Algebra* **153** (2000), 121–127.
[7] A. Ballester-Bolinches, Y. Wang and X. Y. Guo. c-supplemented subgroups of finite groups. *Glasgow Math. J.* **42** (2000), 383–389.
[8] J. Buckley. Finite groups whose minimal subgroups are normal. *Math. Z.* **15** (1970), 15–17.
[9] K. Doerk. Minimal nicht überauflösbare, endliche Gruppen. *Math. Z.* **91** (1966), 198–205.
[10] K. Doerk and T. Hawkes. *Finite soluble groups* (Walter de Gruyter, 1992).
[11] T. M. Gagen. *Topics in finite groups* (Cambridge University Press, 1976).
[12] D. Gorenstein. *Finite groups* (Harper & Row Publishers, 1968).
[13] W. Guo and A. N. Skiba. On finite quasi-$\mathfrak{F}$-groups. *Comm. Algebra* **37** (2009), 470–481.
[14] B. Huppert. *Endliche Gruppen*, vol. 1 (Springer-Verlag, 1967).
[15] B. Huppert. Normalteiler and maximale Untergruppen endlicher Gruppen. *Math. Z.* **60** (1954), 409–434.
[16] B. Huppert and N. Blackburn. *Finite Groups*, vol. 3 (Springer-Verlag, 1982).
[17] O. Kegel. Sylow-Gruppen and Subnormalteiler endlicher Gruppen. *Math. Z.* **78** (1962), 205–221.
[18] Y. Li and Y. Wang. The influence of minimal subgroups on the structure of a finite group. *Proc. Amer. Math. Soc.* **131** (2002), 337–341.
[19] Y. Li and Y. Wang. The influence of π-quasinormality of some subgroups of a finite group. *Arch. Math.* (*Basel*). **81** (2003), 245–252.
[20] P. Schmid. Subgroups permutable with all Sylow subgroups. *J. Algebra* **82** (1998), 285–293.
[21] L. A. Shemetkov. On $\mathfrak{F}$-radicals of finite groups. *Dokl. Akad. Nauk. BSSR.* **25** (1981), 869–872.
[22] L. A. Shemetkov and A. N. Skiba. *Formations of algebraic systems* (Nauka, 1989).
[23] L. A. Shemetkov and A. N. Skiba. On the $\mathfrak{X}\Phi$-hypercentre of finite groups. *J. Algebra* **322** (2009), 2106–2117.
[24] A. N. Skiba. On weakly s-permutable subgroups of finite groups. *J. Algebra* **315** (2007), 192–209.
[25] S. Srinivasan. Two sufficient conditions for supersolvability of finite groups. *Israel J. Math.* **35** (1980), 210–214.

[26] Y. Wang. *c*-normality of groups and its properties. *J. Algebra* **180** (1996), 954–965.
[27] Y. Wang. Finite groups with some subgroups of Sylow subgroups *c*-supplemented. *J. Algebra.* **224** (2000), 467–478.
[28] Y. Wang, Y. Li and J. Wang. Finite groups with *c*-supplemented minimal subgroups. *Algebra Colloq.* **10** (2003), 413–425.
[29] Y. Wang, H. Wei and Y. Li. A generalization of Kramer's theorem and its applications. *Bull. Austral. Math. Soc.* **65** (2002), 467–475.
[30] H. Wei. On *c*-normal maximal and minimal subgroups of Sylow subgroups of finite groups. *Comm. Algebra* **29** (2001), 2193–2200.
[31] H. Wei, Y. Wang and Y. Li. On *c*-normal maximal and minimal subgroups of Sylow subgroups of finite groups, II. *Comm. Algebra* **31** (2003), 4807–4816.
[32] H. Wei, Y. Wang and Y. Li. On *c*-supplemented maximal and minimal subgroups of Sylow subgroups of finite groups. *Proc. Amer. Math. Soc.* **132** (2004), 2197–2204.
[33] M. Weinstein. *Between nilpotent and solvable* (Polygonal Publishing House, 1982).

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