J. Group Theory **18** (2015), 905–912
DOI 10.1515/jgth-2015-0023

Finite groups with complemented subgroups of prime orders

Victor S. Monakhov and Viktoryia N. Kniahina

Communicated by Robert M. Guralnick

Abstract. In this paper, we study finite groups with some complemented subgroups of prime orders. We prove the p-supersolvability of a group with complemented subgroups of order p, where p is the second smallest prime divisor of the group order.

1 Introduction

Let H be a subgroup of a group G. A subgroup K of G is a complement for H in G if $G = HK$ and $H \cap K = 1$. In 1937, P. Hall [4] established that *finite groups, in which all subgroups are complemented, are exhausted by supersolvable groups with elementary abelian Sylow subgroups*. Later Gorchakov [5] showed that the complementability of all subgroups is equivalent to the complementability of prime order subgroups. Ballester-Bolinche and Guo [3] proved that a finite group G is supersolvable if it contains a normal group N such that G/N is supersolvable and all subgroups of N of prime order are complemented in G. This subject matter is still under development at present, e.g. see [2, 6, 12].

In this paper, we obtain a "p-local" analog of Gorchakov's result [5], which is used for describing groups with a lower number of complemented subgroups. In particular, we prove the p-supersolvability of a group with complemented subgroups of order p, where p is the second smallest in ascending order prime divisor of the group order.

2 Notation and preliminary results

We consider only finite groups. We use standard terminology and notation in finite group theory which correspond to [8, 9].

Let π be a set of prime numbers. Denote by π' the complement to π in the set of all prime numbers. The symbol π is also used to denote the function defined on the set of positive integers as follows: $\pi(a)$ is the set of primes dividing a positive integer a. For a group G, define $\pi(G) = \pi(|G|)$. Cyclic and dihedral groups of order n are denoted by Z_n and D_n respectively. An elementary abelian group of order p^m is denoted E_{p^m}. The notation $G = [A]B$ stands for a semidirect product of some subgroups A and B with the normal subgroup A. The Frattini and Fitting subgroups of a group G are denoted by $\Phi(G)$ and $F(G)$ respectively, while A_n and S_n stand for the alternating and symmetric groups of degree n respectively. A group G is said to be A_4-free if it does not have subgroups X and Y with the following properties: Y is a normal subgroup of X and X/Y is isomorphic to A_4.

Choose some set of primes π. If $\pi(m) \subseteq \pi$, then a positive integer m is said to be a π-number. A group G is called a π-group if $\pi(G) \subseteq \pi$ and a π'-group if $\pi(G) \subseteq \pi'$. A finite collection of normal subgroups G_i of a group G is a normal series for G provided that

$$1 = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G. \tag{2.1}$$

The groups G_{i+1}/G_i are called the quotient groups of the series (2.1).

Fix some set of prime numbers π. A group G is π-solvable if it has a normal series (2.1) where each factor is either a π'-group or a solvable π-group. If G is π-solvable, then the π-length of G, denoted by $l_\pi(G)$, is the minimum possible number of factors that are π-groups in any normal series of type (2.1) for G. Since all π-factors of the series (2.1) for a π-solvable group G are solvable, every π-solvable group has a normal series where each π-factor is nilpotent. The nilpotent π-length $l_\pi^n(G)$ of a π-solvable group G is defined as the least number of nilpotent π-factors among all such normal series of G. For the case $\pi(G) \subseteq \pi$, any π-solvable group G becomes solvable, and the nilpotent π-length $l_\pi^n(G)$ coincides with the nilpotent length $n(G)$. If each factor of the series (2.1) is either a cyclic π-group or a π'-group, then G is called a π-supersolvable group.

To prove the theorems we need the following lemmas.

Lemma 1. *The following statements hold.*

(1) *Let A and H be subgroups of a group G with $A \subseteq H$. If A is complemented in G, and all complements for A in H are complemented in G, then H is complemented in G.*

(2) *Let H be a subgroup of a group G. If every prime order subgroup of H is complemented in G, then H is complemented in G.*

Proof. (1) Suppose that K complements A in G. We have

$$G = AK, \quad A \cap K = 1, \quad H = A(H \cap K),$$

hence $H \cap K$ is a complement for A in H. By hypothesis, G contains a subgroup L such that

$$G = (H \cap K)L \quad \text{and} \quad H \cap K \cap L = 1.$$

We deduce that
$$\begin{aligned}|H(K \cap L)| &= |H||K \cap L| \\ &= |A||H \cap K||K \cap L| \\ &= \frac{|G|}{|K|}\frac{|G|}{|L|}|K \cap L| \\ &= \frac{|G|^2}{|KL|} \geq |G|.\end{aligned}$$
Thus $G = H(K \cap L)$, so $K \cap L$ complements H in G.

(2) Work by induction on $|G| + |H|$. Let $A \subseteq H$ be a subgroup of prime order. By hypothesis, H has a subgroup B such that $G = AB$ and $A \cap B = 1$. By Dedekind's identity, $H = A(H \cap B)$. If $H \cap B = 1$, then $H = A$, and so H is complemented in G. Now let $H \cap B \neq 1$. By assumption, every prime order subgroup of $H \cap B$ is complemented in G. Since $|G| + |H \cap B| < |G| + |H|$, we can apply the inductive hypothesis to $H \cap B$, so $H \cap B$ is complemented in G. Now assertion (1) of the lemma implies that H is complemented in G. Thus the lemma is proved. □

Lemma 2. *Let G be a group, $p \in \pi(G)$. Suppose that every subgroup of G of order p is complemented in G. Then:*

(1) *If H is a subgroup of G, then every subgroup of H of order p is complemented in H; in particular, a Sylow p-subgroup of G is elementary abelian and complemented in G.*

(2) *If N is a normal p'-subgroup of G, then every subgroup of G/N of order p is complemented in G/N.*

(3) *If N is a normal subgroup of G and $N \subseteq \Phi(G)$, then every subgroup of G/N of order p is complemented in G/N.*

(4) *If N is a normal subgroup of G and G is p-solvable, then every subgroup of G/N of order p is complemented in G/N.*

Proof. It is clear that only the proofs for (3) and (4) are required.

(3) All nonidentity subgroups of $\Phi(G)$ are not complemented in G. So $\Phi(G)$ is a p'-subgroup. Then (3) follows from (2).

(4) It suffices to prove this claim in the case when N is a minimal normal subgroup of G. By (2), we can assume that N is an elementary abelian p-subgroup. Since $N \not\subseteq \Phi(G)$, it follows that G contains a maximal subgroup M such that $G = [N]M$. By (1), every subgroup of M of order p is complemented. Since $M \cong G/N$, this is (4). The proof of the lemma is complete. □

Lemma 3. *Suppose that a simple group G is the product of a biprimary subgroup A with a p-subgroup B for some prime p. Then G is isomorphic to one of the following groups:*

(1) $\mathrm{PSL}(2,5) = A_4 Z_5$,

(2) $\mathrm{PSL}(2,7) = S_4 Z_7 = ([Z_7]Z_3)D_8$,

(3) $SL(2,8) = ([E_{2^3}]Z_7)Z_9$,

(4) $\mathrm{PSL}(3,3) = AZ_{13}$, $|A| = 2^4 \cdot 3^3$.

Proof. This lemma follows from Guralnick's theorem ([7], [10, Theorem 2]). □

3 Main results

Theorem 1. *The following statements hold.*

(1) *Let G be a group, $p \in \pi(G)$. Every subgroup of order p is complemented in a p-solvable group G if and only if G is a p-supersolvable group with elementary abelian Sylow p-subgroups.*

(2) *Let G be a group, $p \in \pi(G)$. Suppose that every subgroup of order p is complemented in G. If G is not p-solvable, then nonabelian composition pd-factors are isomorphic to one of the following groups:*

  (a) $\mathrm{PSL}(2,7)$*, and* $p = 7$*,*

  (b) $\mathrm{PSL}(2,11)$*, and* $p = 11$*,*

  (c) $\mathrm{PSL}(2,2^t)$*, and* $p = 2^t + 1 > 3$ *is a Fermat prime,* M_{11}*, and* $p = 11$*,*

  (d) M_{23}*, and* $p = 23$*,*

  (e) A_p*, and* $p \geq 5$*,*

  (f) $\mathrm{PSL}(n,q)$*,* $n \geq 3$ *is prime,* $(n, q-1) = 1$*, and* $p = (q^n - 1)/(q-1)$.

(3) *Let G be a group with $\pi(G) = \{p_1, p_2, \dots, p_n\}$, $p_1 < p_2 < \dots < p_n$. Fix $p \in \{p_1, p_2\}$. Every subgroup of order p is complemented in G if and only if G is p-supersolvable and every Sylow p-subgroup of G is elementary abelian.*

Proof. (1) By Lemma 2 (1), every Sylow p-subgroup of G is elementary abelian and complemented in G. Let us verify that G is p-supersolvable. We use induction on the order of G. By Lemma 2 (4), the conditions of the theorem hold for all quotient groups, so we may assume that

$$O_{p'}(G) = \Phi(G) = 1, \quad N = F(G) = O_p(G) = C_G(F(G)),$$

and that N is a minimal normal subgroup of G. It follows that G has a maximal subgroup M with $G = [N]M$. Let X be a subgroup of N of prime order p. By the hypotheses of the theorem, there is a subgroup Y of G such that $G = XY$ and $Y \cap X = 1$. The subgroup $N_1 = N \cap Y$ is normal in Y and centralized by X. Hence N_1 is normal in G. But N is a minimal normal subgroup of G. Therefore, $N_1 = 1$, N has prime order p, so G is p-supersolvable. Thus necessity is proved.

We now prove sufficiency. Let G be a p-supersolvable group with elementary abelian Sylow p-subgroups. Fix an arbitrary subgroup A of prime order p. Suppose that there exists a normal subgroup $N \neq 1$ such that A is not contained in N. Then $A \cap N = 1$ and by induction, the subgroup AN/N is complemented in G/N. Suppose that B/N is a complement for AN/N in G/N. Then

$$G/N = (AN/N)B/N, \quad AN/N \cap B/N = N/N, \quad G = (AN)B = AB,$$

and $A \cap B = 1$ since $|G : B| = p$. So we may assume that every normal subgroup of the group G contains A. Since every minimal normal subgroup of a p-supersolvable group either has order p or is a p'-subgroup, it follows that A is the only minimal normal subgroup of G. But A is complemented in the Sylow p-subgroup since the latter is elementary abelian. Gaschutz's theorem [8, Theorem I.17.4] implies that the subgroup A is complemented in G. Thus (1) of Theorem 1 is established.

(2) By hypothesis, the group G is not p-solvable, so there is a nonabelian composition pd-factor H/K. Lemma 2 (1) implies that in the subgroup H, all subgroups of order p are complemented. Hence H/K has a subgroup of index p. Thus, H/K is a simple nonabelian group and it has a subgroup of index p. Using the classification of finite simple groups, the authors in [7] and [1, Corollary 5.3] listed all simple groups with subgroups of prime power index. By choosing the groups with a subgroup of prime index, we obtain the required list.

(3) By (1), it suffices to prove the p-solvability of G with complemented subgroups of order p. Applying Lemma 2 (1) and the inductive hypothesis, we infer that all proper subgroups of G are p-solvable. Take a maximal subgroup M of G and a minimal normal subgroup N of G. Suppose that $N \neq G$. Then $MN \neq G$ since the subgroup MN is p-solvable. Hence $MN = M$ and $N \subseteq M$. Since M is an arbitrarily selected subgroup, we have $N \subseteq \Phi(G)$. By Lemma 2 (3) and the inductive hypothesis, the quotient group G/N is p-solvable. So the group G is p-solvable. Hence, the assumption is false, $N = G$, so G is a simple group. Now let P be a subgroup of prime order p, and let H be a complement for P in G. Then $|G : H| = p$. If $p = p_1$, then the subgroup H is normal in G. If $p = p_2$, then, by considering the representation of G on the cosets of H, we deduce that G has order $p_1^a p_2$, so again G is not simple. Thus the theorem is proved. □

Example 1. For the third smallest prime divisor of the order of G the complementability of subgroups of order p does not imply the p-supersolvability of G. The example is a simple group of order 60, all of whose subgroups of order 5 are complemented.

Example 2. In the group $G = A_5 \times A_5$, all subgroups of order 5 are complemented.

Theorem 2. *Let G be a group, $r \in \pi(G)$, $\pi = \pi(G) \setminus \{r\}$. Suppose that every subgroup of order p is complemented in G for each prime $p \in \pi$. Then the following holds:*

(1) *G is solvable and π-supersolvable.*

(2) *$l_\pi(G) \leq 1$, $l_\pi^n(G) \leq 2$, $l_r(G) \leq 2$, and $n(G) \leq 4$.*

Proof. (1) Let $q \in \pi$ be the smallest integer. By Theorem 1 (2), the group G is q-supersolvable. Thus there is a q'-Hall subgroup $H = G_{q'}$. By Lemma 1 (1), the subgroup H inherits the hypothesis of the theorem. By induction the subgroup H is solvable and π-supersolvable. Therefore, G is solvable and π-supersolvable.

(2) Let us verify that $l_\pi(G) \leq 1$. By induction and using [8, Theorems VI.6.4 and VI.6.9], we have

$$O_{\pi'}(G) = \Phi(G) = 1, \quad N = F(G) = C_G(N) = O_p(G), \quad p \in \pi,$$

and N is the only minimal normal subgroup of G. Since G is π-supersolvable, it implies that the subgroup N has prime order p. So G/N is a cyclic group of order dividing $p - 1$, and we conclude that $l_\pi(G) \leq 1$.

The group G has a normal series $1 \subseteq G_1 \subseteq G_2 \subseteq G$ such that G_2/G_1 is isomorphic to a π-Hall subgroup of G, while G_1 and G/G_2 are r-subgroups. So $l_r(G) \leq 2$. Since a π-Hall subgroup of G is a supersolvable subgroup with elementary abelian Sylow subgroups, it follows that $n(G_2/G_1) \leq 2$ and $l_\pi^n(G) \leq 2$.

The factors G_1 and G/G_2 of the normal series $1 \subseteq G_1 \subseteq G_2 \subseteq G$ are nilpotent, while G_2/G_1 is supersolvable. Since the derived subgroup of a supersolvable group is nilpotent, we have $n(G) \leq 4$. Thus Theorem 2 is proved. □

Theorem 3. *Let G be a group, and let $r, t \in \pi(G)$, $\pi = \pi(G) \setminus \{r, t\}$. Suppose that every subgroup of order p is complemented in G for each prime $p \in \pi$. Then the following hold:*

(1) *If $\{r, t\} \neq \{2, 3\}$, then G is solvable.*

(2) *If G is a nonsolvable group, then its nonabelian composition factors are isomorphic to one of the following groups:* PSL(2, 5), PSL(2, 7), PSL(3, 3).

(3) *If G is A_4-free, then G is solvable.*

Proof. (1) If $r = t$, then the group G is solvable by Theorem 2. If $2 \in \pi$, then the group G is solvable by Theorem 1. Therefore, we can assume that $r = 2$, $t > 2$. Suppose that G is a nonsolvable group. By induction on $|G|$, we see that every proper subgroup of G is solvable. So $G/\Phi(G)$ is a simple group. Lemma 1 (3) implies that the quotient group $G/\Phi(G)$ satisfies the hypothesis of the theorem, thus we obtain $\Phi(G) = 1$, so G is a simple group isomorphic by Thompson's theorem [8, Theorem II.7.5] to one of the groups $\mathrm{Sz}(2^p)$, $\mathrm{PSL}(2, p)$, $\mathrm{PSL}(2, 2^p)$, $\mathrm{PSL}(2, 3^p)$, $\mathrm{PSL}(3, 3)$ for some prime p. By [11, Theorem 1.3], the group $Sz(2^p)$ cannot be a product of two of its own subgroups, so it is excluded. The orders of the other groups are divisible by 3. If $3 \in \pi$, then G is not simple by Theorem 1 (2), a contradiction. Thus, $3 \in \{r, t\}$, and $\{r, t\} = \{2, 3\}$. However, this is impossible by the hypothesis of the theorem.

(2) By assumption, the group G is nonsolvable, so G has some simple nonabelian composition factor H/K. By Lemma 2 (1), in the nonsolvable subgroup H, all subgroups of order p are complemented for each prime $p \in \pi \cap \pi(H)$. If the set $\{r, t\}$ is not contained in $\pi(H)$, then $|\pi(H) \cap \{r, t\}| \leq 1$, Theorem 1 applies to H, and we conclude that H is solvable; this is a contradiction. Thus $\{r, t\}$ is contained in $\pi(H)$ and $\{r, t\} = \{2, 3\}$ by claim 1. Since H is nonsolvable, we have $|\pi(H)| \geq 3$, and $|\pi \cap \pi(H)| \geq 1$. So H contains a subgroup of index p for each prime $p \in \pi \cap \pi(H)$, and thus H/K also has a subgroup of index p. Therefore, the group H/K is simple nonabelian, and it has a subgroup of index p for every $p \in \pi \cap \pi(H/K)$. Since a simple group cannot contain two subgroups of distinct prime indices, we have $\pi \cap \pi(H/K) = \{p\}$. Lemma 1 (2) implies that any Sylow p-subgroup of H/K is complemented in H/K, and by Lemma 3, we have $H/K \in \{\mathrm{PSL}(2, 5), \mathrm{PSL}(2, 7), \mathrm{PSL}(3, 3)\}$.

(3) If G is nonsolvable, then by (2) of the theorem there is a factor isomorphic to $\mathrm{PSL}(2, 5)$, $\mathrm{PSL}(2, 7)$ or $\mathrm{PSL}(3, 3)$. But these groups contain a subgroup isomorphic to A_4, a contradiction. Thus Theorem 3 is proved. □

Example 3. In $G = \mathrm{PSL}(2, 5) \times \mathrm{PSL}(2, 7) \times \mathrm{PSL}(3, 3)$, all subgroups of order p are complemented for each prime $p \in \{5, 7, 13\}$, and $\pi(G) = \{2, 3, 5, 7, 13\}$.

Bibliography

[1] Z. Arad and E. Fisman, On finite factorizable groups, *J. Algebra* **86** (1984), 522–548.

[2] M. Asaad, Finite groups with certain subgroups of Sylow subgroups complemented, *J. Algebra* **323** (2010), 1958–1965.

[3] A. Ballester-Bolinches and X. Guon, On complemented subgroups of finite groups, *Arch. Math.* **72** (1999), 161–166.

[4] P. Hall, Complemented groups, *J. Lond. Math. Soc.* **12** (1937), 201–204.

[5] Y. M. Gorchakov, Primitively factorizable groups (in Russian), *Sci. Notes Perm. Univ.* **17** (1960), 15–31.

[6] X. Guo, K. P. Shum and A. Ballester-Bolinches, On complemented minimal subgroups of finite groups, *J. Group Theory* **6** (2003), 159–167.

[7] R. Guralnick, Subgroups of prime power index in a simple groups, *J. Algebra* **81** (1983), 304–311.

[8] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.

[9] I. M. Isaacs, *Finite Group Theory*, American Mathematical Society, Providence, 2008.

[10] V. S. Monakhov, The product of a biprimary group with a 2-decomposable group (in Russian), *Math. Notes* **23** (1978), 641–649.

[11] V. S. Monakhov, The product of a supersolvable group with a cyclic or a primary group (in Russian), in: *Finite Groups*, Nauka and Tekhnika, Minsk (1978), 50–63.

[12] S. Qiao, I. M. Isaacs and Y. Li, Complemented subgroups and p-nilpotence in finite groups, *Arch. Math.* **98** (2012), 403–411.

Received January 6, 2014; revised June 3, 2015.

Author information

Victor S. Monakhov, Gomel F. Scorina State University,
Sovetskaya str. 104, 246019 Gomel, Belarus.
E-mail: Victor.Monakhov@gmail.com

Viktoryia N. Kniahina, Gomel Engineering Institute of Emergency Ministry of the Republic of Belarus, Retchitskoe Shosse 35a, 246035 Gomel, Belarus.
E-mail: knyagina@inbox.ru