Finite groups with $\mathbb{P}$-subnormal 2-maximal subgroups

V. N. Kniahina and V. S. Monakhov

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Abstract

A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G if either $H = G$ or there is a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $|H_i : H_{i-1}|$ is prime for $1 \leq i \leq n$. In this paper we study the groups all of whose 2-maximal subgroups are $\mathbb{P}$-subnormal.

Keywords: finite group, $\mathbb{P}$-subnormal subgroup, 2-maximal subgroup.

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1 Introduction

We consider finite groups only. A subgroup K of a group G is called 2-maximal in G if K is a maximal subgroup of some maximal subgroup M of G.

Let H be a subgroup of a group G and n is a positive integer. If there is a chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = G,$$

such that H_i is a maximal subgroup of H_{i+1}, $i = 0, 1, \ldots, n-1$, then H is called n-maximal in G.

For example, in the symmetric group S_4 the subgroup I of order 2 from S_3 is 2-maximal in the chain of subgroups $I \subset S_3 \subset S_4$ and 3-maximal in the chain of subgroups $I \subset Z_4 \subset D_8 \subset S_4$. Here, Z_4 is the cyclic group of order 4 and D_8 is the dihedral group of order 8. For any $n \geq 3$, there exists a group in which some 2-maximal subgroup is n-maximal, see Example 1 below.

A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov in [1] introduced the following definition. Let $\mathbb{P}$ be the set of all prime numbers. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G if either $H = G$ or there is a chain

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

of subgroups such that $|H_i : H_{i-1}|$ is prime for $1 \le i \le n$. In [1], [2] studied groups with $\mathbb{P}$-subnormal Sylow subgroups.

In [1] proposed the following problem:

Describe the groups in which all 2-maximal subgroups are $\mathbb{P}$-subnormal.

This problem is solved in the article. The following theorem is proved.

Theorem. *Every 2-maximal subgroup of a group G is $\mathbb{P}$-subnormal in G if and only if $\Phi(G^{\mathfrak{U}}) = 1$ and every proper subgroup of G is supersolvable.*

Here, $G^{\mathfrak{U}}$ is the smallest normal subgroup of G such that the corresponding quotent group is supersolvable, $\Phi(G^{\mathfrak{U}})$ is the Frattini subgroup of $G^{\mathfrak{U}}$.

2 Preliminary results

We use the standart notation of [3]. The set of prime divisors of $|G|$ is denoted $\pi(G)$. We write $[A]B$ for a semidirect product with a normal subgroup A. If H is a subgroup of a group G, then $\bigcap_{x \in G} x^{-1}Hx$ is called the core of H in G, denoted H_G. If a group G contains a maximal subgroup M with trivial core, then G is said to be primitive and M is its primitivator. We will use the following notation: S_n and A_n are the symmetric and alternating groups of degree n, E_{p^t} is the elementary abelian group of order p^t, Z_m is the cyclic group of order m. Let $|G| = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_1 > p_2 > \dots > p_k$. We say that G has an ordered Sylow tower of supersolvable type if there exist normal subgroups G_i with

$$1 = G_0 \le G_1 \le G_2 \le \dots \le G_{k-1} \le G_k = G,$$

and, where each factor G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for all $i = 1, 2, \dots, k$.

Lemma 1. [4, Theorem IX.8.3] *Let a, n be integers greater than 1. Then except in the cases $n = 2$, $a = 2^b - 1$ and $n = 6$, $a = 2$, there is a prime q with the following properties:*

1) *q divides $a^n - 1$;*

2) *q does not divide $a^i - 1$ whenever $0 < i < n$;*

3) *q does not divide n.*

Example 1. For every $n \ge 3$ there exists a group in which some 2-maximal subgroup is n-maximal. Let $n = 3$. In the symmetric group S_4 the subgroup I of order 2 from S_3 is 2-maximal in the chain of subgroups $I \subset S_3 \subset S_4$ and 3-maximal in the chain of subgroups $I \subset E_4 \subset D_8 \subset S_4$. Now let $n > 3$ and $a = 5$. By Lemma 1, there exists a prime q such that q divides $5^{n-1} - 1$ and q does not divide $5^i - 1$ for all $i \in \{1, 2, \dots, n-2\}$. Hence

$GL(n-1,5)$ contains a subgroup Z of order q which acts irreducibly on the elementary abelian group $E_{5^{n-1}}$ of order 5^{n-1}. In the group $X=[E_{5^{n-1}}]Z$ the identity subgroup 1 is 2-maximal in the chain of subgroups $1 \subset Z \subset X$ and n-maximal in the chain of subgroups $1 \subset E_5 \subset E_{5^2} \subset \ldots \subset E_{5^{n-1}} \subset X$.

Recall that a Schmidt group is a finite non-nilpotent group in which every proper subgroup is nilpotent.

Example 2. Let $S=[P]Q$ be a Schmidt group of order $2^{11}11$, $A=\Phi(P)$, $|A|=2$. Then $A \times Q$ is maximal in S, A is 2-maximal in S, and A is 10-maximal in S because $A=A_0 \subset A_1 \subset \ldots \subset A_9 = P \subset S$, $|A_i : A_{i-1}|=2$, $1 \leq i \leq 9$, $|S:P|=11$.

Lemma 2. [1, Lemma 2.1] *Let N be a normal subgroup of a group G, H an arbitrary subgroup of G. Then the following hold:*

1) *if H is $\mathbb{P}$-subnormal in G, then $(H \cap N)$ is $\mathbb{P}$-subnormal in N, and HN/N is $\mathbb{P}$-subnormal in G/N;*

2) *if $N \subseteq H$ and H/N is $\mathbb{P}$-subnormal in G/N, then H is $\mathbb{P}$-subnormal in G;*

3) *if H is $\mathbb{P}$-subnormal in K, K is $\mathbb{P}$-subnormal in G, then H $\mathbb{P}$-subnormal in G;*

4) *if H is $\mathbb{P}$-subnormal in G, then H^g is $\mathbb{P}$-subnormal in G for each element $g \in G$.*

Example 3. In the alternating group $G=A_5$ the subgroup $H=A_4$ is $\mathbb{P}$-subnormal. If $x \in G \setminus H$, then H^x is $\mathbb{P}$-subnormal in G. The subgroup $D=H \cap H^x$ is a Sylow 3-subgroup of the group G and D is not $\mathbb{P}$-subnormal in H. Therefore an intersection of two $\mathbb{P}$-subnormal subgroups is not $\mathbb{P}$-subnormal. Moreover, if a subgroup H is $\mathbb{P}$-subnormal in a group G and K is an arbitrary subgroup of G, in general, their intersection $H \cap K$ is not $\mathbb{P}$-subnormal in K.

Lemma 3. *Let H be a subgroup of a solvable group G, and assume that $|G:H|$ is a prime number. Then G/H_G is supersolvale.*

Proof. By hypothesis, $|G:H|=p$, where p is a prime number. If $H=H_G$, then G/H is cyclic of prime order p, and thus G/H_G is supersolvable, as required. Assume now that $H \neq H_G$, i.e. H is not normal in G. Then G/H_G contains a maximal subgroup H/H_G with trivial core. Therefore G/H_G is primitive and its Fitting subgroup F/H_G has prime order p. Since $F/H_G = C_{G/H_G}(F/H_G)$, it follows that $(G/H_G)/(F/H_G) \simeq H/H_G$ is isomorphic to a cyclic group of order dividing $p-1$. Thus G/H_G is supersolvable.

Lemma 4. *Let p be the largest prime divisor of $|G|$, and suppose that P is a Sylow p-subgroup of G. Assume that P is not normal in G, and that $H, K \subseteq G$ are subgroups with $N_G(P) \subseteq K \subseteq H$. Then $|H:K|$ is not prime.*

Proof. It is clear that $N_G(P)=N_K(P)=N_H(P)$, and P is a Sylow p-subgroup of K

and of H. By the lemma on indexes, we have

$$|H : N_H(P)| = |H : K|\ |K : N_K(P)|,$$

and, by the Sylow theorem,

$$|H : N_H(P)| = 1 + hp,\ |K : N_K(P)| = 1 + kp,\ h,\ k \in \mathbb{N} \cup \{0\}.$$

Let $|H : K| = t$. Now,

$$1 + hp = t(1 + kp), \quad t = 1 + (h - tk)p.$$

We see that p divides $t-1$, and thus $t > p$. If t is prime, this contradicts the maximality of p.

Lemma 5. 1. *A group is supersolvable if and only if the index of every of its maximal subgroup is prime.*

2. *Every subgroup of a supersolvable group is $\mathbb{P}$-subnormal.*

3. *A group is supersolvable if and only if the normalizers of all of its Sylow subgroups are $\mathbb{P}$-subnormal.*

Proof. 1. This is Huppert's classic theorem, see [3, Theorem VI.9.5].

2. The statement follows from (1) of the lemma.

3. If a group is supersolvable, then all of its subgroups are $\mathbb{P}$-subnormal, see (2).

Conversely, suppose that the normalizer of every Sylow subgroup of a group G is $\mathbb{P}$-subnormal. By Lemma 4, for the largest $p \in \pi(G)$ a Sylow p-subgroup P of G is normal. It is easy to check that the conditions of the lemma are inherited by all quotient groups and so G/P is supersolvable. In particular, G has an ordered Sylow tower of supersolvable type. Since the class of all supersolvable groups is a saturated formation, we can assume, by the inductive hypothesis, that G is primitive, in particular, $G = [P]M$, where M is a maximal subgroup with trivial core. Since M is supersolvable, it follows that $M = N_G(Q)$ for the largest $q \in \pi(M)$. It is obvious that $p \neq q$ and $M = N_G(Q)$ is $\mathbb{P}$-subnormal in G, by the condition of the lemma. Therefore $|P| = p$ and, by Lemma 3, G is supersolvable.

Lemma 6. [5, Theorem 22], [6] *Let G be a minimal non-supersolvable group. We have:*

1) *G is solvable and $|\pi(G)| \leq 3$.*

2) *If G is not a Schmidt group, then G has an ordered Sylow tower of supersolvable type.*

3) *G has a unique normal Sylow subgroup P and $P = G^{\mathfrak{U}}$.*

4) *$|P/\Phi(P)| > p$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(G)$.*

5) *The Frattini subgroup $\Phi(P)$ of P is supersolvable embedded in G, i.e., there exists a series*

$$1 \subset N_0 \subset N_1 \ldots \subset N_n = \Phi(P)$$

such that N_i is a normal subgroup of G and $|N_i/N_{i-1}| \in \mathbb{P}$ for $1 \le i \le n$.

6) *Let Q be a complement to P in G. Then $Q/Q \cap \Phi(G)$ is a minimal non-abelian group or a cyclic group of prime power order.*

7) *All maximal subgroups of non-prime index are conjugate in G, and moreover, they are conjugate to $\Phi(P)Q$.*

3 Main results

Theorem. *Every 2-maximal subgroup of a group G is $\mathbb{P}$-subnormal in G if and only if $\Phi(G^{\mathfrak{U}}) = 1$ and every proper subgroup of G is supersolvable.*

Proof. Suppose that all 2-maximal subgroups of a group G are $\mathbb{P}$-subnormal. We proceed by induction on $|G|$. Show first that G has an ordered Sylow tower of supersolvable type. By Lemma 2, the conditions of the theorem are inherited by all quotient groups of G.

(1) G has an ordered Sylow tower of supersolvable type.

Let P be a Sylow p-subgroup of G, where p is the largest prime divisor of $|G|$. Suppose that P is not normal in G. It follows that $N_G(P)$ is a proper subgroup of G. If $N_G(P)$ is not maximal in G, then there exists a 2-maximal subgroup A containing $N_G(P)$. By the condition of the theorem, A is $\mathbb{P}$- subnormal in G, and so A is contained in a subgroup of prime index. This contradicts Lemma 4. Therefore $N_G(P)$ is maximal in G and $|G : N_G(P)| \notin \mathbb{P}$ by Lemma 4. If $N_G(P) = P$, then G is solvable by Theorem IV.7.4 [3]. It follows that $N_G(P) \neq P$ and $N_G(P)$ has a maximal subgroup B which contains P. We see that B is 2-maximal in G and, by the condition of the theorem, B is $\mathbb{P}$-subnormal. Hence there exists a chain of subgroups

$$P \subseteq B = B_0 \subset B_1 \subset \ldots \subset B_{t-1} = V \subset B_t = G,\ |B_i : B_{i-1}| \in \mathbb{P},\ 1 \le i \le t.$$

The subgroup V is maximal in G and V different from $N_G(P)$, because $|G : N_G(P)|$ is not a prime number, whereas $|G : V|$ is prime. Besides, $t \ge 3$. Thus $V \cap N_G(P) = B$ and $N_V(P) = V \cap N_G(P) = B = N_{B_1}(P)$. We have $|B_1 : N_{B_1}(P)| \in \mathbb{P}$, this contradicts Lemma 4. Therefore the assumption is false and P is normal in G. By induction on $|G|$, every proper subgroup of G/P is supersolvable, and by Lemma 6, G/P has an ordered Sylow tower of supersolvable type. Thus G has an ordered Sylow tower of supersolvable type, in particular, G is solvable.

(2) Every proper subgroup of G is supersolvable.

Suppose that G contains a non-supersolvable maximal subgroup H. Then, by Lemma 5, H contains a maximal subgroup K of non-prime index. Since K is 2-maximal in G, there exists a chain of subgroups

$$K = K_0 \subset K_1 \subset \ldots \subset K_{n-1} = T \subset K_n = G$$

such that $|K_i : K_{i-1}| \in \mathbb{P}$ for all $i = 1, 2, \ldots, n$. It is clear that $H \neq T$ and $H \cap T = K$.

Assume that $G = HT$. In this case,

$$|G : T| = |H : H \cap T| = |H : K| \in \mathbb{P},$$

this is a contradiction. Hence $G \neq HT$. Since H and T are distinct maximal subgroups of G, and G is solvable, by Theorem II.3.9 [3], we have $T = H^g$ for some $g \in G$. Since $H \neq T$, we see that H is a non-normal maximal subgroup of prime index in G. By Lemma 3, the quotient group G/H_G is supersolvable. Since

$$H_G \subseteq H \cap H^g = H \cap T = K,$$

we have K/H_G is maximal in H/H_G. By Lemma 5,

$$|H : K| = |H/H_G : K/H_G| \in \mathbb{P},$$

this is a contradiction. Therefore the assumption is false and every proper subgroup of G is supersolvable.

(3) $\Phi(G^{\mathfrak{U}}) = 1$

If G is supersolvavle, then $G^{\mathfrak{U}} = 1$, it follows that $\Phi(G^{\mathfrak{U}}) = 1$. Assume now that G is non-supersolvable. Then G has the properties listed in Lemma 6. We keep the notation of that lemma. Now $G^{\mathfrak{U}} = P$ and $[\Phi(P)]Q$ is maximal in G.

Suppose that $\Phi(P) \neq 1$. Assume that $A = N_{m-1}$ is a maximal subgroup of $\Phi(P)$, and that A is normal in G. Then $[A]Q$ is a 2-maximal subgroup of G. By the condition of the theorem, $[A]Q$ is $\mathbb{P}$-subnormal in G. Hence, there exists a chain of subgroups $[A]Q \subseteq B \subseteq G$ such that $|G : B| \in \mathbb{P}$. Since $G = [P]Q$ and $Q \subseteq B$, by the Dedekind identity, we have $B = (B \cap P)Q$, and $B \cap P$ is maximal in P. Therefore $\Phi(P) \subseteq B \cap P$ and $\Phi(P)Q$ is conained in B, where $\Phi(P)Q$ is maximal in G. Thus $B = \Phi(P)Q$ and $p = |G : B| = |P : \Phi(P)|$, this contradicts Lemma 6. Therefore our assumption is false and $\Phi(P) = 1$. The necessity is proved.

Prove the sufficiency. Assume that every proper subgroup of G is supersolvable and $\Phi(G^{\mathfrak{U}}) = 1$. If a group is supersolvable, then every its maximal subgroup has a prime index, it follows that every 2-maximal subgroup of a supersolvable group is $\mathbb{P}$-subnormal. Let G be non-supersolvable. Then G is minimal non-supersolvable and the structure of G is described in Lemma 6. We keep for G the notation of that lemma, in particular, we have:

$P = G^{\mathfrak{U}}$, $\Phi(P) = 1$ and Q is a maximal subgroup of G. Let H be an arbitrary 2-maximal subgroup of the group G. If $H \subseteq M$, where M is a maximal subgroup of G and $|G : M| \in \mathbb{P}$, then H is $\mathbb{P}$-subnormal in G, because M is supersolvable. If $H \subseteq K$, where K is a maximal subgroup of the group G and $|G : K| \notin \mathbb{P}$, then, by Lemma 6, the subgroup H contained in Q^g for some $g \in G$. Therefore PH is a proper subgroup of G, thus PH is supersolvable, and H is $\mathbb{P}$-subnormal in PH. Let T be a maximal subgroup of G containing PH. Since T is supersolvable and $|G : T| \in \mathbb{P}$, we see that PH is $\mathbb{P}$-subnormal in G. Using Lemma 2, we deduce that H is $\mathbb{P}$-subnormal in G. The theorem is proved.

Corollary. *Suppose that every 2-maximal subgroup of a group G is $\mathbb{P}$-subnormal. If $|\pi(G)| \geq 4$, then G is supersolvable.*

PROOF. Let every 2-maximal subgroup of a group G be $\mathbb{P}$-subnormal. Suppose that G is not supersolvable. By the previous theorem, G is a minimal non-supersolvable group. By Lemma 6, the order of G has at most three prime divisors, i.e. $|\pi(G)| \leq 3$, which is a contradiction. Therefore, our assumption is false and G is supersolvable.

The following examples show that for $|\pi(G)| = 2$ and for $|\pi(G)| = 3$ there exist non-supersolvable groups in which every 2-maximal subgroup is $\mathbb{P}$-subnormal.

Example 4. There are three non-isomorphic minimal non-supersolvable groups of order 400:

$$[E_{5^2}](< a >< b >), \ |a| = |b| = 4.$$

Numbers of these groups in the library of SmallGroups [7] are [400,129], [400,130], [400,134]. The Sylow 2-subgroups of these groups are non-abelian and have the form: $[Z_4 \times Z_2]Z_2$ and $[Z_4]Z_4$. Suppose that G is one of these groups. Then $G^{\mathfrak{U}} = [E_{5^2}]$ and $\Phi(G^{\mathfrak{U}}) = 1$. All subgroups of the group G are $\mathbb{P}$-subnormal, except the maximal subgroup $< a >< b >$.

Example 5. The general linear group $GL(2, 7)$ contains the symmetric group S_3 which acts irreducibly on the elementary abelian group E_{7^2} of order 49. The semidirect product $[E_{7^2}]S_3$ is a minimal non-supersolvable group, it has subgroups of orders 14 and 21. Therefore, in the group $[E_{7^2}]S_3$, every 2-maximal subgroup is $\mathbb{P}$-subnormal.

References

[1] *Vasilyev A. F., Vasilyeva T. I., Tyutyanov V. N.* On finite groups similar to supersoluble groups // Problems of physics, mathematics and technics. 2010. No. 2 (3). P. 21–27.

[2] *Vasilyev A. F., Vasilyeva T. I., Tyutyanov V. N.* On the finite groups of supersoluble type // Siberian Mathem. J. 2010. Vol. 51, No. 6. P. 1004–1012.

[3] *Huppert B.* Endliche Gruppen I. Berlin, Heidelberg, New York. 1967. 792 p.

[4] *Huppert B., Blackburn N.* Finite groups, II. Berlin–Heidelberg–New York: Springer. 1982.

[5] *Huppert B.* Normalteiler und maximale Untergruppen endlicher Gruppen // Mathematische Zeitschrift. 1954. Vol. 60. P. 409–434.

[6] *Doerk K.* Minimal nicht überauflösbare, endliche Gruppen // Mathematische Zeitschrift. 1966. Vol. 91. P. 198–205.

[7] GAP (2009) Groups, Algorithms, and Programming, Version 4.4.12. *www.gap-system.org.*

V. N. KNIAHINA

Gomel Engineering Institute, Gomel 246035, BELARUS

E-mail address: knyagina@inbox.ru

V. S. MONAKHOV

Department of mathematics, Gomel F. Scorina State University, Gomel 246019, BELARUS

E-mail address: Victor.Monakhov@gmail.com