Finite groups with $\mathbb{P}$-subnormal primary cyclic subgroups

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Abstract

A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G whenever either $H = G$ or there is a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for all i. In this paper, we study the groups in which all primary cyclic subgroups are $\mathbb{P}$-subnormal.

Keywords: finite group, supersolvable group, primary subgroup, cyclic subgroup
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Introduction

We consider finite groups only. A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov in [1] introduced the following definition. Let $\mathbb{P}$ be the set of all prime numbers. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G whenever either $H = G$ or there is a chain

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

of subgroups such that $|H_i : H_{i-1}|$ is prime for all i.

Let $|G| = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1 > p_2 > \ldots > p_k$, $a_i \in \mathbb{N}$. We say that G has an ordered Sylow tower of supersolvable type if there exist normal subgroups G_i with

$$1 = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_{k-1} \subset G_k = G,$$

and where each factor G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for all i. We denote by $\mathfrak{D}$ the class of all groups which have an ordered Sylow tower of supersolvable type. It is well known that $\mathfrak{D}$ is a hereditary saturated formation.

In [1] finite groups with $\mathbb{P}$-subnormal Sylow subgroups were studied. A group G is called w-supersolvable if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G. Denote by w$\mathfrak{U}$ the class of all w-supersolvable groups. Observe that the class $\mathfrak{U}$ of all supersolvable groups is included into w$\mathfrak{U}$. In [1], the authors proved that the class w$\mathfrak{U}$ is a saturated hereditary formation; every group in w$\mathfrak{U}$ possesses an ordered Sylow tower of supersolvable type; all metanilpotent and all biprimary subgroups in w$\mathfrak{U}$ are supersolvable.

In [2] the following problem was proposed.

To describe the groups in which all primary cyclic subgroups are $\mathbb{P}$-subnormal.

In this note we solve this problem. Denote by $\mathfrak{X}$ the class of groups whose primary cyclic subgroups are all $\mathbb{P}$-subnormal. It is easy to verify that $\mathfrak{U} \subset$ w$\mathfrak{U} \subset \mathfrak{X}$.

Theorem. 1. *A group $G \in$ w$\mathfrak{U}$ if and only if G possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of G is supersolvable.*

2. *The class $\mathfrak{X}$ is a hereditary saturated formation.*

3. *A group $G \in \mathfrak{X}$ if and only if G possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of G with cyclic Sylow subgroup is supersolvable.*

4. *Every minimal non-$\mathfrak{X}$-group is a biprimary minimal non-supersolvable group whose non-normal Sylow subgroup is cyclic.*

1 Preliminary results

We use the standart notation of [3]. The set of all prime divisors of $|G|$ is denoted $\pi(G)$. We write $[A]B$ for a semidirect product with a normal subgroup A. If H is a subgroup of G, then $\mathrm{Core}_G H = \bigcap_{x\in G} x^{-1}Hx$ is called the core of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be primitive and M is its primitivator. We will use the following notation: S_n and A_n are the symmetric and the alternating groups of degree n, E_{p^t} is the elementary abelian group of order p^t, Z_m is the cyclic group of order m, D_8 is the dihedral group of order 8, $Z(G)$, $\Phi(G)$, $F(G)$, G' is the center, the Frattini subgroup, the Fitting subgroup and the derived subgroup of G respectively.

Lemma 1. *Let H be a subgroup of a solvable group G, and assume that $|G : H|$ is a prime number. Then $G/\mathrm{Core}_G H$ is supersolvable.*

Proof. By hypothesis, $|G : H| = p$, where p is a prime number. If $H = \mathrm{Core}_G H$, then G/H is cyclic of order p and $G/\mathrm{Core}_G H$ is supersolvable, as required. Assume now that $H \neq \mathrm{Core}_G H$, i. e., H is not normal in G. It

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follows that $G/\mathrm{Core}_G H$ contains a maximal subgroup $H/\mathrm{Core}_G H$ with trivial core. Hence, $G/\mathrm{Core}_G H$ is primitive and the Fitting subgroup $F/\mathrm{Core}_G H$ of $G/\mathrm{Core}_G H$ has prime order p. Since

$$F/\mathrm{Core}_G H = C_{G/\mathrm{Core}_G H}(F/\mathrm{Core}_G H),$$

it follows that

$$(G/\mathrm{Core}_G H)/(F/\mathrm{Core}_G H) \simeq H/\mathrm{Core}_G H$$

is isomorphic to a cyclic group of order dividing $p-1$. Thus, $G/\mathrm{Core}_G H$ is supersolvable.

Lemma 2. 1. *A group G is supersolvable if and only if all of its maximal subgroups have prime indices.*

2. *Every subgroup of a supersolvable group is $\mathbb{P}$-subnormal.*

Proof. 1. This is Huppert's classic theorem, see [3, Theorem VI.9.5].

2. The statement follows from 1 of the lemma.

Immediately, using the definition of $\mathbb{P}$-subnormality, we deduce the following properties.

Lemma 3. *Suppose that H is a subgroup of G, and let N be a normal subgroup of G. Then the following hold:*

1) *if H is $\mathbb{P}$-subnormal in G, then $(H \cap N)$ is $\mathbb{P}$-subnormal in N, and HN/N is $\mathbb{P}$-subnormal in G/N;*

2) *if $N \subseteq H$ and H/N is $\mathbb{P}$-subnormal in G/N, then H is $\mathbb{P}$-subnormal in G;*

3) *if H is $\mathbb{P}$-subnormal in K, and K is $\mathbb{P}$-subnormal in G, then H is $\mathbb{P}$-subnormal in G;*

4) *if H is $\mathbb{P}$-subnormal in G, then H^g is $\mathbb{P}$-subnormal in G for each element $g \in G$.*

Example 1. The subgroup $H = A_4$ of the alternating group $G = A_5$ is $\mathbb{P}$-subnormal. If $x \in G \setminus H$, then H^x is $\mathbb{P}$-subnormal in G. The subgroup $D = H \cap H^x$ is a Sylow 3-subgroup of G and D is not $\mathbb{P}$-subnormal in H. Therefore, an intersection of two $\mathbb{P}$-subnormal subgroups is not $\mathbb{P}$-subnormal. Moreover, if H is $\mathbb{P}$-subnormal in G and K is an arbitrary subgroup of G, in general, their intersection $H \cap K$ is not $\mathbb{P}$-subnormal in K.

However, this situation is impossible if G is a solvable group.

Lemma 4. *Let G be a solvable group. Then the following hold:*

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1) *if H is $\mathbb{P}$-subnormal in G, and K is a subgroup of G, then $(H \cap K)$ is $\mathbb{P}$-subnormal in K;*

2) *if H_i is $\mathbb{P}$-subnormal in G, $i = 1, 2$, then $(H_1 \cap H_2)$ is $\mathbb{P}$-subnormal in G.*

Proof. 1. It is clear that in the case $H = G$ the statement is true. Let $H \neq G$. According to the definition of $\mathbb{P}$-subnormality, there exists a chain of subgroups

$$H = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

such that $|H_i : H_{i-1}|$ is a prime number for any i. We will use induction by n.

Consider the case when $n = 1$. In this situation, $H = H_{n-1}$ is a maximal subgroup of prime index in G. By Lemma 1, G/N is supersolvable, $N = \mathrm{Core}_G H$. Since, by Lemma 2 (2), every subgroup of a supersolvable group is $\mathbb{P}$-subnormal, we have

$$H/N \cap KN/N = N(H \cap K)/N$$

is $\mathbb{P}$-subnormal in KN/N. Lemma 3 (2) implies that $N(H \cap K)$ is $\mathbb{P}$-subnormal in KN. It means that there exists a chain of subgroups

$$N(H \cap K) = A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m = NK$$

such that $|A_i : A_{i-1}| \in \mathbb{P}$ for all i. Since

$$N(H \cap K) \subseteq A_i \subseteq NK,$$

we have $A_i = N(A_i \cap K)$ and $H \cap K \subseteq A_i \cap K$ for all i. We introduce the notation $B_i = A_i \cap K$. It is clear that

$$B_{i-1} \subseteq B_i, \ A_i = N(A_i \cap K) = NB_i, \ N \cap B_i = N \cap A_i \cap K = N \cap K$$

for all i. Since $N \subseteq H$, we have

$$B_0 = A_0 \cap K = N(H \cap K) \cap K = (N \cap K)(H \cap K) = H \cap K,$$

$$B_m = A_m \cap K = KN \cap K = K.$$

Moreover,

$$|A_i : A_{i-1}| = \frac{|NB_i|}{|NB_{i-1}|} = \frac{|N||B_i||N \cap B_{i-1}|}{|N \cap B_i||N||B_{i-1}|} = \frac{|B_i : B_{i-1}|}{|N \cap B_i : N \cap B_{i-1}|} =$$

$$= \frac{|B_i : B_{i-1}|}{|N \cap K : N \cap K|} = |B_i : B_{i-1}|.$$

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Now we have a chain of subgroups

$$H \cap K = B_0 \subset B_1 \subset \ldots \subset B_{m-1} \subset B_m = K,\ |B_i : B_{i-1}| \in \mathbb{P},\ 1 \leqslant i \leqslant m,$$

which proves that the subgroup $H \cap K$ is $\mathbb{P}$-subnormal in K.

Let $n > 1$. Since H_{n-1} is a maximal subgroup of prime index in G, and G is solvable, thus, as it was proved, $H_{n-1} \cap K$ is $\mathbb{P}$-subnormal in K. The subgroup H is $\mathbb{P}$-subnormal in the solvable group H_{n-1} and the induction is applicable to them. By induction,

$$H \cap (H_{n-1} \cap K) = H \cap K$$

is $\mathbb{P}$-subnormal in $H_{n-1} \cap K$. By Lemma 3 (3), $H \cap K$ is $\mathbb{P}$-subnormal in K.

2. Let H_i is $\mathbb{P}$-subnormal in G, $i = 1, 2$. It follows from 1 of the lemma, that $(H_1 \cap H_2)$ is $\mathbb{P}$-subnormal in H_2. Now by Lemma 3 (3), we obtain that $(H_1 \cap H_2)$ is $\mathbb{P}$-subnormal in G.

Lemma 5. *Let H be a subnormal subgroup of a solvable group G. Then H is $\mathbb{P}$-subnormal in G.*

Proof. Since H is subnormal in G, and G is solvable, then there exists a series

$$H = H_0 \subset H_1 \subset \ldots \subset H_{n-1} \subset H_n = G,$$

such that H_i is normal in H_{i+1} for all i. Working by induction on $|G|$, we can assume that H is $\mathbb{P}$-subnormal in H_{n-1}. Since G/H_{n-1} is solvable, the composition factors of G/H_{n-1} have prime orders. Thus, there is a chain of subgroups

$$H_{n-1} = G_0 \subset G_1 \subset \ldots \subset G_{m-1} \subset G_m = G$$

such that G_j is normal in G_{j+1} and $|G_{j+1}/G_j| \in \mathbb{P}$ for all j. This means that $H_{n-1} = G_0$ is $\mathbb{P}$-subnormal in G. Using Lemma 3 (3), we deduce that H is a $\mathbb{P}$-subnormal in G.

Example 2. The subgroup $Z(SL(2, 13))$ of the non-solvable group $SL(2, 13)$ is normal, but is not $\mathbb{P}$-subnormal. This follows from the fact that the identity subgroup is not $\mathbb{P}$-subnormal in $PSL(2, 13) = SL(2, 13)/Z(SL(2, 13))$.

Lemma 6. *Let A be a p-subgroup of a group G. Then A is subnormal in G if and only if $A \subseteq O_p(G)$.*

Proof. The statement follows from Theorem 2.2 [4].

Lemma 7. *Let A be a p-subgroup of a group G. If $|G : N_G(A)| = p^\alpha$, $\alpha \in \mathbb{N}$, then A is subnormal in G.*

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PROOF. Let P be a Sylow p-subgroup of G with the property that P contains A. Then

$$G = N_G(A)P,\ A^G = A^{N_G(A)P} = A^P \subseteq P,$$

so $A^G \subseteq O_p(G)$. It is clear that A is subnormal in G.

Lemma 8. *Let p be the largest prime divisor of $|G|$, and let A be a p-subgroup of G. If A is $\mathbb{P}$-subnormal in G, then A is subnormal in G.*

PROOF. Let $|A| = p^\alpha$. Since A is $\mathbb{P}$-subnormal in G, then there exists a series

$$A = A_0 \subset A_1 \subset \ldots \subset A_{t-1} \subset A_t = G,\ |A_i : A_{i-1}| \in \mathbb{P},\ 1 \leqslant i \leqslant t.$$

Since $|A_1 : A_0| \in \mathbb{P}$, we have

$$|A_1| = p^{1+\alpha} \ \ or \ \ |A_1| = p^\alpha q, \ \ p \neq q.$$

If $|A_1| = p^{1+\alpha}$, then A is a normal subgroup of A_1. If $|A_1| = p^\alpha q$, then $p > q$ and again A is normal in A_1. Suppose we already know that A is subnormal in A_j. Using Lemma 6 we have, $A \subseteq O_p(A_j)$. Since $|A_{j+1} : A_j| \in \mathbb{P}$, we obtain

$$|A_{j+1}| = p|A_j| \ \ or \ \ |A_{j+1}| = q|A_j|, \ \ p \neq q.$$

If $|A_{j+1}| = p|A_j|$, then, by Lemma 7, $O_p(A_j) \subseteq O_p(A_{j+1})$, and A is subnormal in A_{j+1}. If $|A_{j+1}| = q|A_j|$, $p \neq q$, then $p > q$. Consider the set of left cosets of A_j in A_{j+1}. We know that $A_{j+1}/\mathrm{Core}_{A_{j+1}}A_j$ is isomorphic to a subgroup of the symmetric group S_q and any Sylow p-subgroup of A_{j+1} is contained in $\mathrm{Core}_{A_{j+1}}A_j$. Since A is subnormal in A_j, so A is subnormal in $\mathrm{Core}_{A_{j+1}}A_j$. Since $\mathrm{Core}_{A_{j+1}}A_j$ is normal in A_{j+1}, it follows that A is subnormal in A_{j+1}. Therefore, A is subnormal in A_i for all i. This implies that A is subnormal in G.

Corollary. ([1, Proposition 2.8]) *Every w-supersolvable group possesses an ordered Sylow tower of supersolvable type.*

PROOF. Use induction on $|G|$. Let G be a w-supersolvable group, and assume that p is the largest prime divisor of $|G|$. Let P be a Sylow p-subgroup of G. By Lemma 8, P is normal in G. It follows by Lemma 3 (1), that any quotient group of a w-supersolvable group is w-supersolvable. Working by induction on $|G|$, we deduce that G/P possesses an ordered Sylow tower of supersolvable type, so G possesses an ordered Sylow tower of supersolvable type. The corollary is proved.

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Recall that a Schmidt group is a finite non-nilpotent group all of whose proper subgroups are nilpotent. Given a class $\mathfrak{F}$ of groups. By $\mathcal{M}(\mathfrak{F})$ we denote the class of all minimal non-$\mathfrak{F}$-groups. A group G is a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but all proper subgroups of G belong to $\mathfrak{F}$. Clearly, the class $\mathcal{M}(\mathfrak{N})$ consists of Schmidt groups. Here $\mathfrak{N}$ denotes the class of all nilpotent groups. We will need the properties of groups from $\mathcal{M}(\mathfrak{N})$ and $\mathcal{M}(\mathfrak{U})$.

Lemma 9. ([5], [6]) *Let $S \in \mathcal{M}(\mathfrak{N})$. Then the following statements hold:*

1) *$S = [P]\langle y\rangle$, where P is a normal Sylow p-subgroup, and $\langle y\rangle$ is a non-normal cyclic Sylow q-subgroup, p and q are distinct primes, $y^q \in Z(S)$;*

2) *$|P/P'| = p^m$, where m is the order of p modulo q;*

3) *if P is abelian, then P is an elementary abelian p-group of order p^m and P is a minimal normal subgroup of S;*

4) *if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^m$;*

5) *$Z(S) = \Phi(S) = \Phi(P) \times \langle y^q\rangle$; $S' = P$, $P' = (S')' = \Phi(P)$;*

6) *if N is a proper normal subgroup of S, then N does not contain $\langle y\rangle$ and either $P \subseteq N$ or $N \subseteq \Phi(S)$.*

Lemma 10. ([7]) *Let $G \in \mathcal{M}(\mathfrak{U})$. Then the following statements hold:*

1) *G is solvable and $|\pi(G)| \leqslant 3$;*

2) *if G is not a Schmidt group, then G possesses an ordered Sylow tower of supersolvable type;*

3) *G has a unique normal Sylow subgroup P and $P = G^{\mathfrak{U}}$;*

4) *$|P/\Phi(P)| > p$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(G)$;*

5) *the Frattini subgroup $\Phi(P)$ of P is supersolvable embedded in G, i.e., there exists a series*

$$1 \subset N_0 \subset N_1 \ldots \subset N_m = \Phi(P)$$

such that N_i is a normal subgroup of G and $|N_i/N_{i-1}| \in \mathbb{P}$ for all i;

6) *let Q be a complement to P in G, then $Q/Q \cap \Phi(G)$ is a minimal non-abelian group or a cyclic group of prime power order;*

7) *all maximal subgroups of non-prime index are conjugate in G, and moreover, they are conjugate to $\Phi(P)Q$.*

We now present new properties of w-supersolvable groups.

Lemma 11. 1. *If $G \in \mathcal{M}(\mathfrak{U})$ and $|\pi(G)| = 3$, then G is* w*-supersolvable.*

2. *$\mathcal{M}(\mathfrak{U}) \setminus w\mathfrak{U} = \{G \in \mathcal{M}(\mathfrak{U}) \mid |\pi(G)| = 2\}$.*

3. *If $G \in \mathrm{w}\mathfrak{U}$, then the derived length of $G/\Phi(G)$ is at most $|\pi(G)|$.*

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PROOF. 1. Let $G \in \mathcal{M}(\mathfrak{U})$ and $|\pi(G)| = 3$. By Lemma 10, $G = [P]([Q]R)$, where P, Q and R are Sylow subgroups of G. The subgroup P is normal in G, and using Lemma 5, we see that P is $\mathbb{P}$-subnormal in G. The subgroup PQ is normal in G, and by Lemma 5, PQ is $\mathbb{P}$-subnormal in G. Since PQ is supersolvable, it follows by Lemma 2 (2), that Q is $\mathbb{P}$-subnormal in PQ. By Lemma 3 (3), Q is $\mathbb{P}$-subnormal in G. Since $G/P \simeq QR$, so G/P is supersolvable and PR/P is $\mathbb{P}$-subnormal in G/P by Lemma 2 (2). Hence PR is $\mathbb{P}$-subnormal in G by Lemma 3 (2). Since PR is supersolvable, we see that R is $\mathbb{P}$-subnormal in PR by Lemma 2 (2). Hence R is $\mathbb{P}$-subnormal in G by Lemma 3 (3). We conclude that all Sylow subgroups of G are $\mathbb{P}$-subnormal in G. Therefore, G is w-supersolvable.

2. If $G \in \mathcal{M}(\mathfrak{U}) \setminus \mathrm{w}\mathfrak{U}$, then $|\pi(G)| = 2$ by assertion 1 of the lemma. Conversely, let $G \in \{\mathcal{M}(\mathfrak{U}) \mid |\pi(G)| = 2\}$. Suppose that $G \in \mathrm{w}\mathfrak{U}$. Then by Theorem 2.13 (2) [1], the group G is supersolvable, this is a contradiction.

3. By theorem 2.13 (3) [1], $G/F(G)$ has only abelian Sylow subgroups. By theorem VI.14.16 [3], the derived length of $G/F(G)$ is at most $|\pi(G/F(G))|$. Since G has an ordered Sylow tower of supersolvable type, so $|\pi(G/F(G))| \leqslant |\pi(G)|-1$. But if G is solvable, then the quotient group $F(G)/\Phi(G)$ is abelian, and we conclude that the derived length of $G/\Phi(G)$ does not exceed $|\pi(G)|$.

2 Finite groups with $\mathbb{P}$-subnormal primary cyclic subgroups

Example 3. There are three non-isomorphic minimal non-supersolvable groups of order 400:

$$[E_{5^2}](< a >< b >),\ |a| = |b| = 4.$$

Numbers of these groups in the library of SmallGroups [8] are [400,129], [400,130], [400,134]. Sylow 2-subgroups of these groups are non-abelian and have the form: $[Z_4 \times Z_2]Z_2$ and $[Z_4]Z_4$. Let G be one of these groups. All subgroups of G are $\mathbb{P}$-subnormal except the maximal subgroup $< a >< b >$. Therefore, these groups belong to the class $\mathfrak{X}$.

Example 4. The general linear group $GL(2,7)$ contains the symmetric group S_3 which acts irreducibly on the elementary abelian group E_{7^2} of order 49. The semidirect product $[E_{7^2}]S_3$ is a minimal non-supersolvable group, it has subgroups of orders 14 and 21. Every primary cyclic subgroup of the group $[E_{7^2}]S_3$ is $\mathbb{P}$-subnormal. Therefore, these group belong to the class $\mathfrak{X}$.

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Example 5. Non-supersolvable Schmidt groups do not belong to the class $\mathfrak{X}$. We verify this fact. Let $S = [P]Q$ be a non-supersolvable Schmidt group. Suppose that $S \in \mathfrak{X}$. It follows that Q is $\mathbb{P}$-subnormal in S and Q is contained in some subgroup M of prime index. Therefore, $M = P_1 \times Q$, where P_1 is a normal subgroup of S with the property $|P/P_1| = p$. By the properties of Schmidt groups, see Lemma 9, we have $|P/\Phi(P)| > p$ and $P/\Phi(P)$ is a chief factor of S. We have a contradiction.

Lemma 12. *Suppose that all cyclic p-subgroups of a group G are $\mathbb{P}$-subnormal and let N be a normal subgroup of G. Then all cyclic subgroups of N and G/N are $\mathbb{P}$-subnormal.*

Proof. Lemma 3 (1) implies that all cyclic p-subgroups of the normal group N are $\mathbb{P}$-subnormal in N. Let A/N be a cyclic p-subgroup of G/N and assume that $a \in A \setminus N$. Let P be a Sylow p-subgroup of $\langle a \rangle$. By hypothesis, P is $\mathbb{P}$-subnormal in G. Since $PN/N = AN/N$, it follows by Lemma 3 (1), that A/N is $\mathbb{P}$-subnormal in G/N.

Lemma 13. 1. *If every primary cyclic subgroup of a group G is $\mathbb{P}$-subnormal, then G possesses an ordered Sylow tower of supersolvable type.*

2. $\mathfrak{U} \subset \mathrm{w}\mathfrak{U} \subset \mathfrak{X} \subset \mathfrak{D}$.

Proof. 1. Let P be a Sylow p-subgroup of G, where p is the largest prime divisor of $|G|$. If $a \in P$, then by hypothesis, the subgroup $\langle a \rangle$ is $\mathbb{P}$-subnormal in G. By Lemma 8, the subgroup $\langle a \rangle$ is subnormal in G, and by Lemma 6, $\langle a \rangle \subseteq O_p(G)$. Since a is an arbitrary element of P, we see that $P \subseteq O_p(G)$, and hence G is p-closed. By Lemma 12, the conditions of the lemma are inherited by all quotient groups of G. Applying induction on $|G|$, we see that G/P possesses an ordered Sylow tower of supersolvable type, and thus G possesses an ordered Sylow tower of supersolvable type.

2. By Lemma 2 (2), we have the inclusion $\mathfrak{U} \subseteq \mathrm{w}\mathfrak{U}$. It follows from Example 4, that $[E_{7^2}]S_3$ is non-supersolvable and $[E_{7^2}]S_3 \in \mathrm{w}\mathfrak{U} \setminus \mathfrak{U}$. Therefore, $\mathfrak{U} \subset \mathrm{w}\mathfrak{U}$.

We verify the inclusion $\mathrm{w}\mathfrak{U} \subseteq \mathfrak{X}$. Suppose that $G \in \mathrm{w}\mathfrak{U}$, and let A be an arbitrary primary cyclic subgroup of G. Then A is a p-subgroup for some $p \in \pi(G)$. By Sylow's theorem, A is contained in some Sylow p-subgroup P of the group G. Since $G \in \mathrm{w}\mathfrak{U}$, it follows that P is $\mathbb{P}$-subnormal in G. By Lemma 2 (2), A is $\mathbb{P}$-subnormal in P, and by Lemma 3 (3), A is $\mathbb{P}$-subnormal in G. Therefore, $G \in \mathfrak{X}$. The group $[E_{5^2}]Q$ from Example 3 is a biprimary minimal non-supersolvable group, Q is non-cyclic. The group $[E_{5^2}]Q \in \mathfrak{X} \setminus \mathrm{w}\mathfrak{U}$, therefore, $\mathrm{w}\mathfrak{U} \subset \mathfrak{X}$.

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By the above assertion of the lemma, $\mathfrak{X} \subseteq \mathfrak{D}$. Since there exist non-supersolvable Schmidt groups which have an ordered Sylow tower of supersolvable type (for example, $[E_{5^2}]Z_3$), and they do not belong to the class $\mathfrak{X}$, it follows that $[E_{5^2}]Z_3 \in \mathfrak{D} \setminus \mathfrak{X}$. Therefore, $\mathfrak{X} \subset \mathfrak{D}$.

Lemma 14. *Let G be a minimal non-supersolvable group. The group $G \notin \mathfrak{X}$ if and only if G is a biprimary group whose non-normal Sylow subgroup is cyclic.*

PROOF. Let $G \in \mathcal{M}(\mathfrak{U}) \setminus \mathfrak{X}$. If $|\pi(G)| = 3$, then by Lemma 11 (1), $G \in \mathrm{w}\mathfrak{U}$. Since $\mathrm{w}\mathfrak{U} \subset \mathfrak{X}$, we have $G \in \mathfrak{X}$, which contradicts the choice of G. So, if $G \in \mathcal{M}(\mathfrak{U}) \setminus \mathfrak{X}$, then $|\pi(G)| = 2$ and $G = [P]Q$, where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G. Suppose that Q is non-cyclic, and let $a \in Q$. Since $P\langle a\rangle$ is a proper subgroup of G, we deduce that $P\langle a\rangle$ is supersolvable. Lemma 2 (2) implies that $\langle a\rangle$ is a $\mathbb{P}$-subnormal subgroup of $P\langle a\rangle$. Since $P\langle a\rangle$ is subnormal in G, it follows by Lemma 5, that $P\langle a\rangle$ is a $\mathbb{P}$-subnormal subgroup of G. Now by Lemma 3 (3), we deduce that $\langle a\rangle$ is $\mathbb{P}$-subnormal in G. Applying Lemma 3 (4), we can conclude that all cyclic q-subgroups of G are $\mathbb{P}$-subnormal in G. Lemma 5 implies that all cyclic p-subgroups of G are $\mathbb{P}$-subnormal in G. Thus $G \in \mathfrak{X}$. We have a contradiction. Therefore, the assumption is false and Q is cyclic.

Conversely, let $G \in \mathcal{M}(\mathfrak{U})$, $|\pi(G)| = 2$ and a non-normal Sylow subgroup Q of G is cyclic. Assume that $G \in \mathfrak{X}$. This implies that Q is $\mathbb{P}$-subnormal in G, and so both Sylow subgroups of the group G are $\mathbb{P}$-subnormal. Now, by Theorem 2.13 (2) [1], G is supersolvable, which is a contradiction.

Lemma 15. ([9]) *If P is a normal Sylow subgroup of a group G, then $\Phi(P) = \Phi(G) \cap P$.*

Theorem. 1. *A group $G \in \mathrm{w}\mathfrak{U}$ if and only if G possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of G is supersolvable.*

2. *The class $\mathfrak{X}$ is a hereditary saturated formation.*

3. *A group $G \in \mathfrak{X}$ if and only if G possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of G with cyclic Sylow subgroup is supersolvable.*

4. *Every minimal non-$\mathfrak{X}$-group is a biprimary minimal non-supersolvable group whose non-normal Sylow subgroup is cyclic.*

PROOF. 1. If a group $G \in \mathrm{w}\mathfrak{U}$, then G possesses an ordered Sylow tower of supersolvable type by the corollary of Lemma 8, and every biprimary subgroup

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of G is w-supersolvable by Lemma 4 (1). We conclude by Lemma 10 (2), that every biprimary subgroup of G is supersolvable.

Conversely, suppose that a group G possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of G is supersolvable. Assume that G is not w-supersolvable. Let us choose among all such groups a group with the smallest possible $|\pi(G)|$. Then $|\pi(G)| \geqslant 3$ and G contains a Sylow r-subgroup R such that R is not $\mathbb{P}$-subnormal in G. Let $p \in \pi(G)$, where p is the largest prime divisor of $|G|$ and let P be a Sylow p-subgroup of G. Since G possesses an ordered Sylow tower of supersolvable type, we deduce that P is normal in G. By hypothesis, the subgroup PR is supersolvable, and we deduce by Lemma 2 (2) that R is $\mathbb{P}$-subnormal in PR. It is clear that G/P possesses an ordered Sylow tower of supersolvable type and all of its biprimary subgroups are supersolvable. Since $|\pi(G/P)| = |\pi(G)| - 1$, it follows by the inductive hypothesis, that G/P is w-supersolvable. Therefore, the Sylow r-subgroup PR/P is $\mathbb{P}$-subnormal in G/P. Lemma 3 (2) implies that the subgroup PR is $\mathbb{P}$-subnormal in G, and hence by Lemma 3 (3), the subgroup R is $\mathbb{P}$-subnormal in G. This is a contradiction.

2. By Lemma 13 (1), the class $\mathfrak{X}$ consists of finite groups which have an ordered Sylow tower of supersolvable type, so we can apply Lemma 4. Let $G \in \mathfrak{X}$ and suppose that H is an arbitrary subgroup of G. If A is a cyclic primary subgroup of H, then A is $\mathbb{P}$-subnormal in G. By Lemma 4 (1), the subgroup A is $\mathbb{P}$-subnormal in H, and hence $\mathfrak{X}$ is a hereditary class.

By Lemma 12, the class $\mathfrak{X}$ is closed under homomorphic image. By induction on the order of G, we verify that the class $\mathfrak{X}$ is closed under subdirect products. Let G be a group of least order with the following properties:

$$G/N_i \in \mathfrak{X},\ i = 1, 2,\ N_1 \cap N_2 = 1,\ G \notin \mathfrak{X}.$$

In this case, G has a primary cyclic subgroup A which is not $\mathbb{P}$-subnormal in G. Since $G/N_i \in \mathfrak{X}$, $i = 1, 2$, it follows that AN_i/N_i is $\mathbb{P}$-subnormal in G/N_i, and thus by Lemma 4 (2), $AN_1 \cap AN_2$ is $\mathbb{P}$-subnormal in G. If $K = AN_1N_2$ is a proper subgroup of G, then $K/N_i \in \mathfrak{X}$ because $\mathfrak{X}$ is a hereditary class. By the induction hypothesis, $K \in \mathfrak{X}$. It follows that A is $\mathbb{P}$-subnormal in $AN_1 \cap AN_2$, and by Lemma 3 (3), A is $\mathbb{P}$-subnormal in G, which is a contradiction. Therefore, $G = AN_1N_2$. Assume that $G = AN_1$. Then

$$N_2 \simeq N_1N_2/N_1 \subseteq G/N_1 \simeq A/A \cap N_1.$$

Thus N_2 is cyclic and AN_2 is supersolvable by Theorem VI.10.1 [3]. It follows that A is $\mathbb{P}$-subnormal in AN_2, AN_2 is $\mathbb{P}$-subnormral in G, and by Lemma 3 (3),

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A is $\mathbb{P}$-subnormal in G, which is a contradiction. Thus our assumption is false and $AN_1 \neq G \neq AN_2$.

The subgroup $D = N_1 \cap AN_2$ is normal in $AN_2 = H \neq G$. Hence the group H contains two normal subgroups D and N_2 such that

$$D \cap N_2 \subseteq N_1 \cap N_2 = 1,$$

$$H/N_2 \subset G/N_2 \in \mathfrak{X},\ H/N_2 \in \mathfrak{X},$$
$$G/N_1 = (AN_2)N_1/N_1 \simeq AN_2/N_1 \cap AN_2 = H/D \in \mathfrak{X}.$$

By the inductive assumption, $H \in \mathfrak{X}$. It follows that A is $\mathbb{P}$-subnormal in H, H is $\mathbb{P}$-subnormal in G, and hence A is $\mathbb{P}$-subnormal in G. This contradicts the fact that G has a primary cyclic subgroup which is not $\mathbb{P}$-subnormal in G. Thus $\mathfrak{X}$ is formation.

We prove that $\mathfrak{X}$ is a saturated formation by induction on $|G|$. Suppose that $\Phi(G) \neq 1$ and $G/\Phi(G) \in \mathfrak{X}$. Since by Lemma 13, the quotient group $G/\Phi(G)$ possesses an ordered Sylow tower of supersolvable type, it follows that G possesses an ordered Sylow tower of supersolvable type.

Let N be a minimal normal subgroup of G. It is clear that

$$\Phi(G)N/N \subseteq \Phi(G/N),\ G/\Phi(G)N \simeq \big(G/\Phi(G)\big)/\big(\Phi(G)N/\Phi(G)\big) \in \mathfrak{X},$$

$$(G/N)/(\Phi(G/N)) \simeq \big((G/N)/(\Phi(G)N/N)\big)/\big((\Phi(G/N)/(\Phi(G)N/N)\big) \in \mathfrak{X},$$

because

$$(G/N)/(\Phi(G)N/N) \simeq G/\Phi(G)N \in \mathfrak{X}.$$

By the inductive hypothesis, we have $G/N \in \mathfrak{X}$. Since $\mathfrak{X}$ is a formation, this implies that N is a unique minimal normal subgroup of G, $N \subseteq \Phi(G)$, N is a p-subgroup for the largest $p \in \pi(G)$, and $O_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G, P is normal in G.

Suppose that G has a primary cyclic subgroup A which is not $\mathbb{P}$-subnormal in G. Since $G/N \in \mathfrak{X}$, it folows that the quotient group AN/N is $\mathbb{P}$-subnormal in G/N, and by Lemma 3 (2), AN is $\mathbb{P}$-subnormal in G. By Lemma 3 (3), we see that A is not $\mathbb{P}$-subnormal in AN, and Lemma 5 implies that the orders of A and N are coprime. Therefore, AP/N is a biprimary subgroup in which the Sylow subgroups AN/N and P/N are both $\mathbb{P}$-subnormal. Theorem 2.13 (2) [1] implies that AP/N is supersolvable. By Lemma 15, $\Phi(P) = P \cap \Phi(G)$, thus $N \subseteq \Phi(P) \subseteq \Phi(AP)$, and by Theorem VI.8.6 [3], we deduce that AN is supersolvable. Lemma 2 (2) implies that A is $\mathbb{P}$-subnormal in AN, which is a contradiction.

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3. Let $G \in \mathfrak{X}$ and B is a biprimary group with cyclic Sylow subgroup R. By Lemma 16 (1), G possesses an ordered Sylow tower of supersolvable type. If R is normal in B, then B/R is primary, it follows that B is supersolvable. If R is not normal in B, then $B = PR$, where P is a normal Sylow subgroup of B. By hypothesis, we conclude that R is $\mathbb{P}$-subnormal in G, and by Lemma 4 (1), R is $\mathbb{P}$-subnormal in B. Hence, by Lemma 10 (2), B is supersolvable.

Conversely, suppose that G possesses an ordered Sylow tower of supersolvable type and every biprimary subgroup of G with cyclic Sylow subgroup is supersolvable. Assume that $G \notin \mathfrak{X}$. Let us choose among all such groups a group G with the smallest possible order. Then G contains a cyclic non-$\mathbb{P}$-subnormal r-subgroup R. Since G possesses an ordered Sylow tower of supersolvable type, we deduce that a Sylow p-subgroup P for the largest prime $p \in \pi(G)$ is normal. If $p = r$, then $R \subseteq P$, it follows that R is $\mathbb{P}$-subnormal in G, which is a contradiction. Thus $p \neq r$ and PR is biprimary with cyclic Sylow subgroup R. By hypothesis, PR is supersolvable, and by Lemma 2 (2), R is $\mathbb{P}$-subnormal in PR. The quotient group G/P possesses an ordered Sylow tower of supersolvable type and every its biprimary subgroup with cyclic Sylow subgroup is supersolvable. Thus $G/P \in \mathfrak{X}$. Since PR/P is a cyclic r-subgroup, we see that PR/P is $\mathbb{P}$-subnormal in G/P. It follows by Lemma 3 (2) that PR is $\mathbb{P}$-subnormal in G. Now by Lemma 3 (3), we obtain that R is $\mathbb{P}$-subnormal in G. This is a contradiction. The assertion is proved.

4. Let $G \in \mathcal{M}(\mathfrak{X})$, and let q be the smallest prime divisor of $|G|$. Consider an arbitrary proper subgroup H of G. Since $H \in \mathfrak{X}$, so by Lemma 13 (1), the subgroup H has an ordered Sylow tower of supersolvable type, in particular, H is q-nilpotent. By Theorem IV.5.4 [3], the group G is either q-nilpotent or a q-closed Schmidt group. If G is a q-closed Schmidt group, then G is a biprimary minimal non-supersolvavle group whose non-normal Sylow subgroup is cyclic. In this case, the statement is true.

Suppose that G is a q-nilpotent group. Then $G = [G_{q'}]G_q$. Since $G_{q'} \in \mathfrak{X}$, it follows by Lemma 13 (1), that $G_{q'}$ possesses an ordered Sylow tower of supersolvable type, and thus G possesses an ordered Sylow tower of supersolvable type. Let N be a minimal normal subgroup of G.

First, assume that $\Phi(G) = 1$. In this case, $G = [N]M$, where M is some maximal subgroup of G. Since $G \notin \mathfrak{X}$, then G contains a primary cyclic non-$\mathbb{P}$-subnormal subgroup. Let A be a subgroup of least order among these subgroups. Since

$$AN/N \subseteq G/N \simeq M \in \mathfrak{X},\ AN/N \simeq A/A \cap N,$$

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it follows that AN/N is $\mathbb{P}$-subnormal in G/N, and by Lemma 3 (2), the subgroup AN is $\mathbb{P}$-subnormal in G. If $AN \neq G$, then $AN \in \mathfrak{X}$, it follows that A is $\mathbb{P}$-subnormal in AN, and by Lemma 3 (3), the subgroup A is $\mathbb{P}$-subnormal in G, which is a contradiction. Therefore $AN = G$, in particular, G is biprimary. Let H be an arbitrary maximal subgroup of G. Then either $A^x \subseteq H$, $x \in G$ or $N \subseteq H$. If $A^x \subseteq H$, then $A^x = H$ because N is a minimal normal subgroup of $AN = G$ and H is cyclic. If $N \subseteq H$, then by the Dedekind identity, $H = (A \cap H)N$. By the choice of A we can conclude that $A \cap H$ is $\mathbb{P}$-subnormal in G. Now by Theorem 2.13 (2) [1], H is supersolvable. So in the case of $\Phi(G) = 1$, we proved that G is a biprimary minimal non-supersolvavle group.

Let $\Phi(G) \neq 1$. According to statement 2 of the theorem, $G/\Phi(G) \notin \mathfrak{X}$, and so by the inductive hypothesis, $G/\Phi(G)$ is a biprimary minimal non-supersolvable group. It follows from the structure of such groups that $G/\Phi(G)$ possesses an ordered Sylow tower. Since $\pi(G) = \pi(G/\Phi(G))$, we deduce that G is a biprimary group which possesses an ordered Sylow tower: $G = [P]Q$, where P and Q are Sylow subgroups of G. Since $G \notin \mathfrak{X}$, then there exists a primary cyclic non-$\mathbb{P}$-subnormal subgroup. Let A be a subgroup of least order among these subgroups. If $PA \neq G$, then $PA \in \mathfrak{X}$, and thus A is $\mathbb{P}$-subnormal in PA. Since PA is $\mathbb{P}$-subnormal in G, it follows by Lemma 3 (3) that A is $\mathbb{P}$-subnormal in G, this is a contradiction. Therefore, $PA = G$. Let H be an arbitrary maximal subgroup of G. Then either $P \subseteq H$ or $A^x \subseteq H$, $x \in G$. If $P \subseteq H$, then $H = [P](A \cap H)$ by the Dedekind identity. By the choice of A, we can conclude that $A \cap H$ is $\mathbb{P}$-subnormal in G. Now by Theorem 2.13 (2) [1], the subgroup H is supersolvable. If $A^x \subseteq H$, then $H = [P \cap H]A^x$. Since $H \in \mathfrak{X}$, we deduce that A^x is $\mathbb{P}$-subnormal in H, and thus H is supersolvable by Theorem 2.13 (2) [1]. Hence, in the case of $\Phi(G) \neq 1$, we proved that G is a biprimary minimal non-supersolvavle group.

Therefore, in any case every minimal non-$\mathfrak{X}$-group is a biprimary minimal non-supersolvable group. We conclude by Lemma 14, that every non-normal Sylow subgroup of G is cyclic.

The theorem is proved.

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