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On the $\mathfrak{X}\Phi$-hypercentre of finite groups ☆

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ABSTRACT

Let G be a finite group, $\mathfrak{X}$ a class of groups. A chief factor H/K of G is called $\mathfrak{X}$-central provided $[H/K](G/C_G(H/K)) \in \mathfrak{X}$. Let $Z_{\mathfrak{X}\Phi}(G)$ be the product of all normal subgroups H of G such that all non-Frattini G-chief factors of H are $\mathfrak{X}$-central. Then we say that $Z_{\mathfrak{X}\Phi}(G)$ is the *$\mathfrak{X}\Phi$-hypercentre* of G. Our main result here is the following (Theorem 1.4): *Let $X \leqslant E$ be normal subgroups of a group G. Suppose that every non-cyclic Sylow subgroup P of X has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) is weakly S-permutable in G. If X is either E or $F^*(E)$, then $E \leqslant Z_{\mathfrak{U}\Phi}(G)$.* Here $\mathfrak{U}$ is the class of all supersoluble finite groups.

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1. Introduction

Throughout this paper, all groups are finite. We use $\mathfrak{N}$ and $\mathfrak{U}$ to denote the class of all nilpotent groups and the class of all supersoluble groups, respectively. The symbol $[A]B$ denotes the semidirect product of the groups A and B where B is an operator group of A. A chief factor H/K of a group G is called Frattini provided $H/K \leqslant \Phi(G/K)$. A subgroup H of a group G is said to permute with a subgroup T if $HT = TH$.

Let $\mathfrak{X}$ be a class of groups. A chief factor H/K of a group G is called $\mathfrak{X}$-central provided $[H/K](G/C_G(H/K)) \in \mathfrak{X}$ (see [1]). Otherwise, it is called $\mathfrak{X}$-eccentric. The product of all normal subgroups of G whose G-chief factors are $\mathfrak{X}$-central in G is called the $\mathfrak{X}$-hypercentre of G and denoted by $Z_{\mathfrak{X}}(G)$ [3, p. 389]. Note that for any $\mathfrak{N}$-central chief factor H/K of G we have $C_G(H/K) = G$. Hence the $\mathfrak{N}$-hypercentre of G coincides with the hypercentre $Z_\infty(G)$ of G.

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The $\mathfrak{X}$-hypercentre essentially influences the structure of a group. For example, if all subgroups of G with prime order and order 4 are contained in $Z_{\mathfrak{N}}(G)$, then G is nilpotent (N. Ito). If all these subgroups are in $Z_{\mathfrak{U}}(G)$, then G is supersoluble (Huppert, Doerk). If all subgroups of G with prime order are in $Z_{\mathfrak{U}}(G)$, then G is soluble (Gaschütz). Note also that if G has a normal subgroup E such that $G/E \in \mathfrak{X}$ and $E \leqslant Z_{\mathfrak{X}}(G)$, then $G \in \mathfrak{X}$, for many concrete classes $\mathfrak{X}$.

In this paper we investigate the following subgroup, which inherits some key properties of the $\mathfrak{X}$-hypercentre.

Definition 1.1. Let $Z_{\mathfrak{X}\Phi}(G)$ be the product of all normal subgroups of G whose non-Frattini G-chief factors are $\mathfrak{X}$-central in G. Then we say that $Z_{\mathfrak{X}\Phi}(G)$ is the *$\mathfrak{X}\Phi$-hypercentre of G*.

The subgroup $Z_{\mathfrak{X}\Phi}(G)$ is characteristic in G and every non-Frattini G-chief factor of $Z_{\mathfrak{X}\Phi}(G)$ is $\mathfrak{X}$-central in G (Lemma 2.3).

We omit the letter $\mathfrak{X}$ when $\mathfrak{X} = \mathfrak{N}$. So, the $Z_{\Phi}(G)$-hypercentre of G is the product of all normal subgroups H of G such that all non-Frattini G-chief factors of H are central. The $\mathfrak{U}\Phi$-hypercentre of G is the product of all normal subgroups H of G such that all non-Frattini G-chief factors of H have prime order.

Example 1.2. Let V be a simple $\mathbb{F}_3 A_4$-module which is faithful for the alternating group A_4. Then we may consider V as a $\mathbb{F}_3 SL_2(3)$-module with $C_{SL_2(3)}(V) = Z$ where Z is a unique minimal normal subgroup of $SL_2(3)$. Let $E = [V]SL_2(3)$, and let $A = A_3(E)$ be the 3-Frattini module of E (see [4] or [3, p. 853]), and let G be a non-splitting extension of A by E. By Corollary 1 in [4], $VZ = O_{3',3}(E) = C_E(A/\mathit{Rad}\,A)$. Hence for some normal subgroup N of G we have $A/N \leqslant \Phi(G/N)$ and $G/C_G(A/N) \simeq A_4$. Thus $|A/N| > 3$, so $Z_{\mathfrak{U}\Phi}(G/N) = Z_{\Phi}(G/N) = (A/N)(D/N)$ where D/N is a unique normal subgroup of G/N with order 2. On the other hand, $Z_{\mathfrak{U}}(G/N) = Z_{\infty}(G/N) = D/N$.

Example 1.3. Let $r < p < q$ be primes and suppose r divides $p-1$. Let S be a non-abelian group of order pr, C_p and C_q groups of order p and q, respectively. Let $A = [Q]C_p$ where Q is a simple $\mathbb{F}_q C_p$-module which is faithful for C_p, and $B = [P]C_q$ where P is a simple $\mathbb{F}_p C_q$-module which is faithful for C_q. Let $H = A \times B \times S$, and let V be a projective envelope of a trivial $\mathbb{F}_p H$-module. Let $G = [V]H$. Let $C = C_H(V)$, and let C_0 be the intersection of the centralizers in H of all G-chief factors of V. Then $\Phi(G) = \mathit{Rad}(V)$ by Lemma B.3.14 in [3], and $C = O_{p'}(H) = Q$, $C_0 = O_{p',p}(H)$ by Theorem VII.14.6 in [5]. Suppose that all G-chief factors of V are cyclic. Then H/C_0 is an abelian group of exponent dividing $p-1$. Since q does not divide $p-1$, $C_q \leqslant C_0$. Hence $C_q \leqslant Q$. This contradiction shows that G has a Frattini chief factor K/L such that $|K/L| > p$ and for every G-chief factor M/N between K and V we have $|M/N| = p$. Hence $Z_{\mathfrak{U}\Phi}(G/L) = (V/L)(ZL/L) \neq Z_{\Phi}(G/L) = V/L$ where Z is a unique normal subgroup of H with order p.

Our main goal here is to prove the following

Theorem 1.4. *Let $X \leqslant E$ be normal subgroups of a group G. Suppose that every non-cyclic Sylow subgroup P of X has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) is weakly S-permutable in G. If X is either E or $F^*(E)$, then $E \leqslant Z_{\mathfrak{U}\Phi}(G)$.*

In this theorem $F^*(E)$ is the generalized Fitting subgroup of E, that is, the product of all normal quasinilpotent subgroups of E.

We shall prove Theorem 1.4 in Section 4. The proof of this theorem consists of many steps and the following useful fact is one of the important stages in the proof of Theorem 1.4.

Theorem 1.5. *Let E be a normal subgroup of a group G, and p be a prime dividing $|E|$. Suppose that a Sylow p-subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$*

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and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) not having a p-nilpotent supplement in G is weakly S-permutable in G.

(I) *If p is the smallest prime dividing $|G|$, then $E/O_{p'}(E) \leqslant Z_{\Phi}(G/O_{p'}(E))$.*
(II) *If p is the smallest prime dividing $|E|$, then $E/O_{p'}(E) \leqslant Z_{\mathfrak{U}\Phi}(G/O_{p'}(E))$.*

We shall prove Theorem 1.5 in Section 3.

Finally, note that the main results of the papers [2,7–19] and of some other papers (see Section 5 in [2]) are special cases of Theorem 1.4. In Section 5 we discuss some other applications of the generalized $\mathfrak{X}$-hypercentre concept.

All unexplained notations and terminologies are standard. The reader is referred to [3] or [20–22] if necessary.

2. Preliminaries

The following lemma is obvious.

Lemma 2.1. *Let H/K and E/T be chief factors of a group G. If H/K and E/T are G-isomorphic, then $[H/K](G/C_G(H/K)) \simeq [E/T](G/C_G(E/T))$.*

Lemma 2.2. *Let H be a normal subgroup of a group G. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be G-chief series of H. Then there exists a one-to-one correspondence between the chief factors of $\mathcal{H}_1$ and those of $\mathcal{H}_2$ such that corresponding factors are G-isomorphic and such that the Frattini (in G) chief factors of $\mathcal{H}_1$ correspond to the Frattini (in G) chief factors of $\mathcal{H}_2$.*

Proof. The assertion is a strengthened form of [3, A, Theorem 9.13] with the same proof. □

We say that a subgroup H of a group G is *$\mathfrak{X}\Phi$-hypercentral* in G provided $H \leqslant Z_{\mathfrak{X}\Phi}(G)$.

Lemma 2.3. *Let $Z = Z_{\mathfrak{X}\Phi}(G)$ and N and T be normal subgroups of the group G. Then*

(1) *Every non-Frattini G-chief factor of Z is $\mathfrak{X}$-central in G.*
(2) *$ZN/N \leqslant Z_{\mathfrak{X}\Phi}(G/N)$.*
(3) *If $TN/N \leqslant Z_{\mathfrak{X}\Phi}(G/N)$ and $(|T|, |N|) = 1$, then $T \leqslant Z$.*

Proof. (1) Suppose that A and B are normal subgroups of G such that all non-Frattini G-chief factors of A and B are $\mathfrak{X}$-central in G. We shall prove by induction on G that all non-Frattini G-chief factors of AB are $\mathfrak{X}$-central in G. Suppose that $A \cap B \neq 1$, and let N be a minimal normal subgroup of G contained in $A \cap B$. Then by induction $(A/N)(B/N) = AB/N$ is $\mathfrak{X}\Phi$-hypercentral in G/N. Since by hypothesis N is either a Frattini G-chief factor or $\mathfrak{X}$-central in G, the result follows from Lemma 2.2.

Finally, assume that $A \cap B = 1$. If H/K is a non-Frattini G-chief factor and $AB \geqslant H > K \geqslant B$, then H/K is G-isomorphic to a non-Frattini G-chief factor $H \cap A/K \cap A$. Therefore, by Lemma 2.1, H/K is $\mathfrak{X}$-central in G. Hence $AB \leqslant Z$.

(2) Let H/K be a non-Frattini G-chief factor of G and $N \leqslant K < H \leqslant NZ$. Then H/K is G-isomorphic to a non-Frattini G-chief factor $H \cap Z/K \cap Z$. Therefore, by Lemma 2.1, H/K is $\mathfrak{X}$-central in G. Hence we have (2).

(3) Let H/K be any non-Frattini G-chief factor of T, M a maximal subgroup of G such that $K \leqslant M$ and $HM = G$. Since $(|T|, |N|) = 1$, $N \leqslant M$. Hence $(NH/N)/(NK/N)$ is a non-Frattini G/N-chief factor of $TN/N \leqslant Z_{\mathfrak{X}\Phi}(G/N)$. Hence NH/NK both is $\mathfrak{X}$-central in G and G-isomorphic to H/K. Therefore $T \leqslant Z$. □

Lemma 2.4. *(See [23, Theorem 9.15].) Let $\mathfrak{X}$ be one of the classes $\mathfrak{N}$ or $\mathfrak{U}$. Then $G/C_G(Z_{\mathfrak{X}}(G)) \in \mathfrak{X}$.*

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We shall use the following special case of Theorem 1 in [24].

Lemma 2.5. *Let A be a p'-group of automorphisms of a p-group P of odd order. Assume that every subgroup of P with prime order is A-invariant. Then A is cyclic.*

The following lemma is, in fact, a fragment in the proof of Theorem 1 in [24].

Lemma 2.6. *Let V be an S-permutable subgroup of order 4 of a group G.*

(1) *If $V = A \times B$, where $|A| = |B| = 2$ and A is S-permutable in G, then B is S-permutable in G.*
(2) *If $V = \langle x \rangle$ is cyclic, then $\langle x^2 \rangle$ is S-permutable in G.*

Proof. Let Q be a Sylow q-subgroup of G such that $q \neq 2$. By hypothesis, $VQ = QV$ and $AQ = QA$. Since $|VQ : AQ| = 2 = |AQ : Q|$, Q is normal in VQ. Therefore $BQ = QB$.

(2) See the proof of (1). □

Before continuing, we shall need to know a few facts about S-permutable subgroups.

Lemma 2.7. *(See [6].) Let G be a group and $H \leqslant K \leqslant G$, $V \leqslant G$. Then*

(1) *If H and V are S-permutable in G, then $H \cap V$ is S-permutable in G.*
(2) *Suppose that H is normal in G. Then K/H is S-permutable in G/H if and only if K is S-permutable in G.*
(3) *If H is S-permutable in G, then H is subnormal in G.*

Lemma 2.8. *(See [25, Lemma A].) Suppose that H is a p-group for some prime p. Then H is S-permutable in G if and only if $O^p(G) \leqslant N_G(H)$.*

Lemma 2.9. *Let $G = [P]Q$ be a Schmidt group (i.e., a minimal non-nilpotent group) where P is a Sylow p-subgroup of G. If every cyclic subgroup of P with order p or order 4 (if P is a non-abelian 2-group) not having a p-nilpotent supplement in G is weakly S-permutable in G, then $|P| = p$.*

Proof. See the proof of Lemma 2.12 in [2]. □

We shall need the following modification of Lemma 2.2 in [2].

Lemma 2.10. *Let G be a group, p be the smallest prime divisor of $|G|$ and P a non-cyclic Sylow p-subgroup of G. Let $E \neq 1$ be a normal subgroup of G contained in P. If either every maximal subgroup of P has a p-nilpotent supplement in G or every maximal subgroup of E has a p-nilpotent supplement in G, then G is p-nilpotent.*

Proof. Suppose that every maximal subgroup of P has a p-nilpotent supplement in G. Let $M_1T_1 = G$ where T_1 is p-nilpotent and M_1 is maximal in P. We can assume that $T_1 = N_G(H_1)$ for some Hall p'-subgroup H_1 of G. Clearly, $P = M_1(P \cap T_1)$. Suppose that $P \cap T_1 \neq P$. Then we can choose a maximal subgroup M_2 in P containing $P \cap T_1$. By assumption, $G = M_2T_2$ where T_2 is p-nilpotent. Again, we can assume that $T_2 = N_G(H_2)$ for some Hall p'-subgroup H_2 of G. By [26] we have $H_1^x = H_2$ for some $x \in G$. Therefore, $G = M_1T_1 = M_2T_2 = M_2T_1^x = M_2T_1$ and $P = M_2(P \cap T_1) = M_2$, a contradiction. Hence $P \cap T_1 = P$, which implies the p-nilpotency of G.

Now suppose that every maximal subgroup of E has a p-nilpotent supplement in G. We shall prove by induction on $|G|$ that G is p-nilpotent. First note that for any non-identity normal subgroup N contained in E, G/N is p-nilpotent. Indeed, if $N = E$, it is clear. Let $N \neq E$. Then $E/N \neq 1$ and every maximal subgroup of E/N has a p-nilpotent supplement in G/N. Hence G/N is p-nilpotent by induction. Let $N = \Phi(G) \cap E \neq 1$. It is well known that the class of all p-nilpotent groups is a

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saturated formation (see [3]). Hence G is p-nilpotent. Finally, suppose that $\Phi(G) \cap E = 1$. Then by [23, Lemma 7.9], E is the direct product of some minimal normal subgroups of G. Without loss we may assume that E is a minimal normal subgroup of G and $|E| \neq p$. Let M be a maximal subgroup of E, T a p-nilpotent supplement of M in G. Then $E = M(E \cap T)$, which implies $E \cap T \neq 1$. But $ET = G$, and so $E \cap T$ is normal in G, which contradicts the minimality of E. □

In our proofs we shall use the following known properties of the generalized Fitting subgroup $F^*(G)$ (see Chapter X in [27]).

Lemma 2.11. *Let G be a group. Then*

(1) *If N is a normal subgroup of G, then $F^*(N) \leqslant F^*(G)$.*
(2) *If N is a normal subgroup of G and $N \leqslant F^*(G)$, then $F^*(G)/N \leqslant F^*(G/N)$.*
(3) *$F(G) \leqslant F^*(G) = F^*(F^*(G))$. If $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*
(4) *$F^*(G) = F(G)E(G)$ and $F(G) \cap E(G) = Z(E(G))$, where $E(G)$ is the layer of G (see [27, p. 128]).*
(5) *$C_G(F^*(G)) \leqslant F(G)$.*

The following lemma is a direct corollary of [27, Theorem X,13.6].

Lemma 2.12. *Let P be a normal p-subgroup of a group G contained in $Z(G)$. Then $F^*(G/P) = F^*(G)/P$.*

3. Proof of Theorem 1.5

Suppose that this theorem is false and consider a counterexample (G, E) for which $|G||E|$ is minimal.

(1) $O_{p'}(E) = 1$.

Suppose that $O_{p'}(E) \neq 1$. By [2, Lemma 2.10(4)] the hypothesis holds for $(G/O_{p'}(E), E/O_{p'}(E))$. Hence by the choice of (G, E) the theorem is true for $(G/O_{p'}(E), E/O_{p'}(E))$, and hence for (G, E), a contradiction.

(2) *Either $E = G$ or $E = P$.*

Suppose that $E \neq G$. By [2, Lemma 2.10(3)] the hypothesis is still true for (E, E), so E is p-supersoluble by the choice of (G, E). But since by hypothesis p is the smallest prime dividing E, E is p-nilpotent. But by (1), $O_{p'}(E) = 1$. Hence $E = P$.

(3) $O_{p'}(G) = 1$.

Suppose that $V = O_{p'}(G) \neq 1$. Then by (1) and (2), $E = P$. By [2, Lemma 2.10(4)] the hypothesis is still true for $(G/V, EV/V)$. Hence the theorem is true for $(G/V, EV/V)$ by the choice of (G, E). Now from (1) and Lemma 2.3(3) we deduce that the theorem is true for (G, E), a contradiction.

(4) $|D| > p$.

Suppose that $|D| = p$. First we show that G does not have a cyclic chief factor of the form E/V. Suppose G does. Then V is not cyclic, so $|V|$ is not prime. Let V_p be a Sylow p-subgroup of V. Suppose that $|V_p| = p$. Since p is the smallest prime dividing $|E|$, V is p-nilpotent. But since $O_{p'}(V)$ *char* V, $O_{p'}(V) \leqslant O_{p'}(E) = 1$. Hence $|V| = p$, a contradiction. Therefore $|V_p| > p$. Hence the hypothesis is still true for the pair (G, V), which implies $V \leqslant Z_{\mathfrak{U}\Phi}(G)$ by the choice of (G, E). Hence $E \leqslant Z_{\mathfrak{U}\Phi}(G)$. If p is the smallest prime dividing $|G|$, it follows that $E \leqslant Z_\Phi(G)$. This contradiction shows G does not have a cyclic chief factor of the form E/V.

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First assume that p is the smallest prime dividing $|G|$. Then G has no p-closed Schmidt subgroup of the form $H = [H_p]H_q$, where $H_p \leqslant E$. Indeed, by [2, Lemma 2.10(3)] every subgroup of H with order p and order 4 (if H_p is a non-abelian 2-group) not having a p-nilpotent supplement in G is weakly S-permutable in H. But then by Lemma 2.9, $|H_p/\Phi(H_p)| = p$, which contradicts the minimality of p. Since every non-p-nilpotent group has a p-closed Schmidt subgroup [22, Satz IV, 5.4], $G \neq E$. Hence by (2), $E = P$ and $x \in C_G(P)$ for any p'-element x of G. Hence G is p-nilpotent, a contradiction.

Now suppose that p is not the smallest prime dividing $|G|$. Then $p > 2$ and by (2), $E = P$. Suppose that some subgroup H of P with $|H| = p$ has a p-nilpotent supplement T in G. Then $T \cap P$ is maximal in P, so it is normal in G. But then from $|P : T \cap P| = |PT : T| = |G : T| = p$ it follows that $E = P$ has a cyclic G-chief factor $P/T \cap P$.

This contradiction shows that every subgroup H of P with $|H| = p$ is weakly S-permutable in G. Suppose that some subgroup H of P with $|H| = p$ is not S-permutable in G. Then by [2, Lemma 2.10(5)], G has a normal subgroup M such that $HM = G$ and $|G : M| = p$. Hence $P \cap M \neq P$ is normal in G and in view of the isomorphism $G/M \simeq P/P \cap M$, $E = P$ has a cyclic G-chief factor $P/P \cap M$, a contradiction. Hence every subgroup H of P with order $|H| = p$ is S-permutable in G. Suppose that $P \nsubseteq O^p(G)$. Then since every G-chief factor V/K above $O^p(G)$ is central (i.e. $C_G(V/K) = G$), from the G-isomorphism $O^p(G)P/O^p(G) \simeq P/P \cap O^p(G)$ it follows that G has a cyclic chief factor of the form E/K, a contradiction. Therefore $P \leqslant O^p(G)$. Now let G_p be a Sylow p-subgroup of G. Consider the series $1 \leqslant \Omega_1(P) \leqslant \Omega_2(P) \leqslant \cdots \leqslant \Omega_t(P) = P$. Since all members of this series are characteristic in P, the series may be refined to a G_p-chief series

$$1 = P_0 \leqslant P_1 \leqslant \cdots \leqslant P_r = P. \qquad (*)$$

By [2, Lemma 2.4] every factor $\Omega_i(P)/\Omega_{i-1}(P)$ is elementary. Hence every subgroup of $\Omega_i(P)/\Omega_{i-1}(P)$ is normalized by every p'-element of G. Hence the series $(*)$ is a chief series of G. Thus $E = P \leqslant Z_{\mathfrak{U}\Phi}(G)$, a contradiction. Hence $|D| > p$.

(5) *Suppose that $|P : D| > p$. Then G does not have a normal maximal subgroup M with $|G : M| = p$ and $MP = G$.*

Otherwise, the hypothesis holds for $(G, E \cap M)$. Hence the theorem is true for $(G, E \cap M)$ by the choice of (G, E). On the other hand, from the G-isomorphism $G/M \simeq E/M \cap E$ we deduce that $E/M \cap E$ is a central chief factor of G. Hence the theorem is true for (G, E), a contradiction.

(6) *Suppose that $|P : D| > p$. Then every subgroup H of P with order $|H| = |D|$ not having a p-nilpotent supplement in G is S-permutable in G.*

Assume that P has a subgroup H such that $|H| = |D|$ and H neither has a p-nilpotent supplement in G nor is S-permutable in G. Then by [2, Lemma 2.10(5)], G has a normal subgroup M such that $PM = G$ and $|G : M| = p$, which contradicts (5).

(7) *$|N| \leqslant |D|$ for any minimal normal subgroup N of G contained in P.*

Assume that $|D| < |N|$. If some subgroup H of N with order $|H| = |D|$ has a p-nilpotent supplement T in G, then $TN = G$ and $T \neq G$. Hence $N \cap T$ is a proper non-identity subgroup of N, because $N = N \cap HT = H(N \cap T)$. But evidently $N \cap T$ is normal in G, which contradicts the minimality of N. Hence every subgroup H of N with order $|H| = |D|$ is weakly S-permutable in G and so by [2, Lemma 2.11] some maximal subgroup of N is normal in G, a contradiction. Thus we have (7).

(8) *If $E = P$ and p is the smallest prime dividing $|G|$, then P is a Sylow subgroup of G.*

Let G_p be a Sylow p-subgroup of G. Suppose that $E = P \neq G_p$ and let Q be a Sylow q-subgroup of G, where $q \neq p$. Then $|PQ| < |G|$ and by [2, Lemma 2.10(3)] the hypothesis holds for (PQ, P).

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Hence $P \leqslant Z_{\Phi}(PQ)$, so PQ is nilpotent. Hence $Q \leqslant C_G(P)$. Now let $1 = P_0 \leqslant P_1 \leqslant \cdots \leqslant P_t = P$ where P_{i+1}/P_i is a chief factor of G_p, $i = 0, 1, \ldots, t-1$. Then P_{i+1}/P_i is a chief factor of G, so $P \leqslant Z(G)$, which contradicts the choice of (G, E). Hence $P = G_p$.

(9) *Suppose that $p = 2$, $|P : D| > 2$ and some subgroup H of P with order 4 has a 2-nilpotent supplement T in G. Then H is not cyclic, $G/T_G \simeq A_4$, every subgroup of H with order 2 is not S-permutable in G, and T_G is a 2-group.*

In view of (5), $|G : T| = 4$. By considering the permutation representation of G/T_G on the right cosets of T/T_G one can see that G/T_G is isomorphic to some subgroup of the symmetric group S_4 of degree 4. But since by (5), G does not have a subgroup M with $|G : M| = 2$, then $G/T_G \simeq A_4$. It follows that $H \simeq HT_G/T_G$ is not cyclic. Since by (3), $O_{2'}(G) = 1$, we deduce that $O_{2'}(T_G) = 1$. Hence T_G is a 2-group. Suppose that some subgroup V of H with order 2 is S-permutable in G and let Q be a Sylow 3-subgroup of T. Then $V \leqslant N_G(Q)$. On the other hand, since T is 2-nilpotent and $|T| = 2^n 3$, $T \leqslant N_G(Q)$. Hence $|G : N_G(Q)| = 2$, a contradiction. Thus we have (9).

(10) *If P is non-abelian 2-group and $|P : D| > 2$, then $|D| > 4$.*

We use here some arguments in the proof of Theorem 1 in [24].

Since P is a non-abelian 2-group, it has a cyclic subgroup $H = \langle x \rangle$ with order 4. Suppose that $|D| = 4$. Then by (6) and $|P : D| > 2$ we know that every subgroup of P with order 4 not having a 2-nilpotent supplement in G is S-permutable in G. Hence in view of (9), H is S-permutable in G. Then by Lemma 2.6(2), $\langle x^2 \rangle$ is S-permutable in G. Now note that if G has a subgroup $V = A \times B$ with order 4, where $|A| = 2$ and A is S-permutable in G, then V and B are S-permutable in G by (9) and Lemma 2.6(1). Therefore some subgroup Z of $Z(P)$ with $|Z| = 2$ is S-permutable in G. Hence every subgroup of P with order 2 is S-permutable in G, which contradicts (4).

(11) *If N is an abelian minimal normal subgroup of G contained in E, then the hypothesis is still true for $(G/N, E/N)$.*

If either $p > 2$ and $|N| < |D|$ or $p = 2$ and $2|N| < |D|$ or $|P : D| = p$, it is clear. So let $|P : D| > p$ and either $p > 2$ and $|N| = |D|$ or $p = 2$ and $|N| \in \{|D|, |D| : 2\}$. By (6) every subgroup H of P with order $|D|$ not having a p-nilpotent supplement in G is S-permutable in G. Besides, in view of (4), $|D| > p$. Suppose that $|N| = |D|$. Then N is non-cyclic and hence every subgroup of G containing N is non-cyclic. Let $N \leqslant K \leqslant P$, where $|K : N| = p$. Since K is non-cyclic, it has a maximal subgroup $L \neq N$. If at least one of the subgroups L or N has a p-nilpotent supplement in G, then K does. Otherwise, $K = LN$ is S-permutable in G, as it is the product of two S-permutable in G subgroups. Thus if either $p > 2$ or P/N is abelian, the hypothesis is true for $(G/N, E/N)$ by [2, Lemma 2.10(2)(4)]. Next suppose that P/N is a non-abelian 2-group. Then P is non-abelian, so $|D| > 4$ by (10). Let $N \leqslant K \leqslant V$ where $|V : N| = 4$ and $|V : K| = 2$. Let K_1 be a maximal subgroup of V such that $V = K_1 K$. Suppose that K_1 is cyclic. Then $N \nsubseteq K_1$, so $V = K_1 N$, which implies $|N| = 4$. But then $|D| = 4$, which contradicts (10). Hence K_1 is non-cyclic and hence as above one can show that K_1 either is S-permutable in G or has a 2-nilpotent supplement in G. Therefore every subgroup of P/N with order 2 and order 4 not having a p-nilpotent supplement in G/N is weakly S-permutable in G/N.

Finally, suppose that $|D| = 2|N|$. If $|N| > 2$, then as above one can show that every subgroup of P/N with order 2 and order 4 (if P/N is non-abelian) not having a 2-nilpotent supplement in G/N is weakly S-permutable in G/N. Now, suppose that $|N| = 2$ and P/N is non-abelian. Then P is non-abelian and $|D| = 4$, which contradicts (10). Hence we have (11).

(12) $E = G$.

Suppose that $E = P$ and let N be any minimal normal subgroup of G contained in P. Then by (11) the hypothesis holds for $(G/N, E/N)$. Hence $E/N \leqslant Z_{\mathfrak{U}\Phi}(G/N)$, $N \nsubseteq \Phi(G)$ and $|N| > p$. Therefore $\Phi(G) \cap E = 1$. Hence by [23, Lemma 7.9], P is the direct product of some minimal normal subgroups

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of G. In view of [2, Lemma 2.11], $P \neq N$. Hence for some minimal normal subgroup R of G contained in P we have $R \neq N$. Then by [3, Lemma A,9.11], $NR/N \nsubseteq \Phi(G/N)$. Therefore $|R| = |NR/N| = p$, which implies that the theorem is true for (G, E), a contradiction. Therefore we have (12).

(13) *Some maximal subgroup of P does not have a p-nilpotent supplement in G* (this follows from Lemma 2.10).

(14) *E is p-soluble.*

By (11) we need only to show that $P_G \neq 1$. Suppose that this false. Then by Lemma 2.7(3) and [23, Theorem 7.7] every non-identity subgroup of P is not S-permutable in G.

First suppose that $|P : D| = p$. By (13) at least one of the maximal subgroups of P, M say, does not have a p-nilpotent supplement in G. Since $M_{sG} = 1$, by hypothesis M has a subnormal complement T in G. By (12) the order of a Sylow p-subgroup of T is equal to p, so T is p-nilpotent, which contradicts the choice of M. Hence we may assume that $|P : D| > p$. In this case, by (6), every subgroup H of P with order $|H| = |D|$ not having a p-nilpotent supplement in G is S-permutable in G. Hence we have to conclude that every subgroup H of P with order $|H| = |D|$ has a p-nilpotent supplement in G and so every maximal subgroup of P has a p-nilpotent supplement in G, which contradicts (13). Thus we have (14).

Final contradiction. Let N be a minimal normal subgroup of G. Then in view of (1), (12) and (14), $N \leqslant P$. Hence by (11) and the choice of G for every minimal normal subgroup N of G the quotient G/N is p-nilpotent. Thus $|N| > p$, $N \nsubseteq \Phi(G)$ and $N = O_p(G) = F(G)$ is the only minimal normal subgroup of G. Hence $G = [N]M$ for some maximal subgroup of G. Assume that $|P : D| = p$. For every maximal subgroup A of P containing N we have $AM = G$, so $M \simeq G/N$ is a p-nilpotent supplement of A in G. Thus by (13) some maximal subgroup V of P neither contains N nor has a p-nilpotent supplement in G. Hence by hypothesis V is weakly S-permutable in G. Let $L = V_{sG}$ and let T be a subnormal subgroup of G such that $VT = G$ and $T \cap V \leqslant L$. Suppose that $L = V$. Clearly $N \cap V$ is normal in P. Hence by Lemmas 2.7(1) and 2.8, $N \cap V$ is normal in G, which implies $|N| = p$. This contradiction shows that $L \neq V$, so $T \neq G$.

Suppose that $L = 1$. By (12) the order of a Sylow p-subgroup of T is equal to p, so T is p-nilpotent, which contradicts the choice of V. Thus $L \neq 1$. By Lemma 2.7(3), $L \leqslant O_p(G) = N$ and hence $L \leqslant N \cap V$. Suppose that $N \leqslant T$. Then $T \cap V \leqslant N \cap V \leqslant T \cap V$ and so $T \cap V = N \cap V$. Therefore from $T \cap V \leqslant L \leqslant N \cap V$ we have $T \cap V = L$. Therefore $L = N \cap V$ is S-permutable in G and thus for every Sylow q-subgroup Q of G we have $Q \leqslant N_G(L)$. On the other hand, $N \cap V$ is normal in P. Hence L is a non-identity subgroup of P which is normal in G. Hence $N \leqslant L \leqslant V$, a contradiction. Therefore we have to conclude that $N \nsubseteq T$. Since T is subnormal in G, it contains all Sylow q-subgroups of G for all primes $q \neq p$. Hence G/T_G is a p-group. Thus $G \simeq G/N \cap T_G$ is p-nilpotent, a contradiction.

Therefore we may assume that $|P : D| > p$. Then by (6) every subgroup H of P satisfying $|H| = |D|$ and not having a p-nilpotent supplement in G is S-permutable in G. Since every S-permutable subgroup of G contained in P is contained in $O_p(G) = N$, it follows that every subgroup H of P different from N and satisfying $|H| = |D|$ has a p-nilpotent supplement in G. Therefore every maximal subgroup of P has a p-nilpotent supplement in G, which contradicts (13). This contradiction completes the proof of this theorem.

4. Proof of Theorem 1.4

First assume that $X = E$. Suppose that in this case the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. If V is a normal Hall subgroup of E, then V is normal in G and the hypothesis also holds for (G, V) and $(G/V, E/V)$ (see (1) in the proof of Theorem 1.4 in [2]). If $1 \neq V \neq E$, then $V \leqslant Z_{\mathfrak{U}\Phi}(G)$ and $E/V \leqslant Z_{\mathfrak{U}\Phi}(G/V)$ by the choice of (G, E). Hence $E \leqslant Z_{\mathfrak{U}\Phi}(G)$, a contradiction. Therefore for all normal Hall subgroups $V \neq 1$ of E we have $E = V$. By Theorem 1.5(I), $E/O_{p'}(E) \leqslant Z_{\Phi}(E/O_{p'}(E))$ where p is the smallest prime dividing $|E|$. Since the class of all nilpotent groups is a saturated formation (see 1. in Section 5), it follows that $E/O_{p'}(E)$ is nilpotent, so E is p-nilpotent. Hence E is p-group, which in view of Theorem 1.5(II) implies that $E \leqslant Z_{\mathfrak{U}\Phi}(G)$, a contradiction.

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Now suppose that $X = F^*(E)$. Suppose that in this case the theorem is false and let (G, E) be a counterexample with $|G||E|$ minimal. Let $F = F(E)$ and $F^* = F^*(E)$. If E is soluble we use p to denote the smallest prime divisor of $|F|$. And let p be the largest prime divisor of $|F|$ if E is non-soluble. Let P be a Sylow p-subgroup of F.

(1) *$F^* = F \neq E$ and $C_G(F) = C_G(F^*) \leqslant F$* (see (1) in [2, Theorem 1.3]).
(2) *$P \leqslant Z_{\mathfrak{U}\Phi}(G)$ and $E/P \nleqslant Z_{\mathfrak{U}\Phi}(G/P)$.*

Since P is characteristic in $F = F^*$ and F is characteristic in E, P is normal in G. Hence by the case $X = E$, $P \leqslant Z_{\mathfrak{U}\Phi}(G)$. Therefore $E/P \nleqslant Z_{\mathfrak{U}\Phi}(G/P)$, otherwise, $E \leqslant Z_{\mathfrak{U}\Phi}(G)$, which contradicts the choice of (G, E).

(3) *If Y is a proper normal subgroup of G and $E \leqslant Y$, then $E \leqslant Z_{\mathfrak{U}\Phi}(Y)$.*

By Lemma 2.11(1), $F^*(Y) \leqslant F^* = F \leqslant Y$ and so $F^*(Y) = F^*$. Thus the hypothesis is still true for (Y, E) and hence $E \leqslant Z_{\mathfrak{U}\Phi}(Y)$, by the choice of (G, E).

(4) *If $E \neq G$, E is supersoluble* (this follows directly from (3)).
(5) *Assume that E is soluble, $V/P = F(E/P)$ and Q is a Sylow q-subgroup of V where q divides $|V/P|$. Then $q \neq p$ and either $Q \leqslant F$ or $p > q$ and $C_Q(P) = 1$* (see (4) in [2, Theorem 1.3]).
(6) *$p > 2$.*

Assume that $p = 2$. Suppose that E is soluble. In this case by (5) we have $F/P = F(E/P)$. Besides, by (1) and Lemma 2.11(3), $F^*(E/P) = F(E/P) = F^*/P$. Thus by [2, Lemma 2.10(4)] the hypothesis is still true for $(G/P, E/P)$. Hence $E/P \leqslant Z_{\mathfrak{U}\Phi}(G/P)$, which contradicts (2). Therefore E is not soluble. Hence $P = F(E)$, since in this case p is the largest prime dividing $|F|$. Since by (1), $E \neq F$, E contains the subgroup $V = PQ$ where Q is a q-group for some odd prime q. By the case $X = E$, V is nilpotent. Thus $Q \leqslant C_E(F)$. But by (1), $C_E(F) = C_E(F^*) \leqslant F$, a contradiction. Hence we have (6).

(7) *Some minimal subgroup of P is not S-permutable in G.*

Suppose that every minimal subgroup of P is S-permutable in G. Let $P_0 = \Omega_1(P)$ and $C = C_G(P_0)$. It is clear that C is normal in G.

First suppose that E is soluble. Let $V/P = F(E/P)$ and Q be a Sylow q-subgroup of V where q divides $|V/P|$. Then by (5) either $Q \leqslant F$ or $C_Q(P) = 1$. In the second case Q is cyclic by (5) and Lemma 2.5. Thus by [2, Lemma 2.10(4)] the hypothesis holds for G/P, so $E/P \leqslant Z_{\mathfrak{U}\Phi}(G/P)$, which contradicts (2). Hence E is not soluble. Note that in this case $E = G$ by (4). We show that every minimal subgroup L of P is normal in G. But first we prove that $O^p(G) = G$. Indeed, assume that $O^p(G) \neq G$. By Lemma 2.11(1), $F^*(O^p(G)) \leqslant F^*$. Hence $F^*(O^p(G)) = F^* \cap O^p(G) = F \cap O^p(G)$. Therefore by (6) and [2, Lemma 2.10(3)] the hypothesis is still true for $(O^p(G), O^p(G))$. Thus $O^p(G)$ is supersoluble by the choice of G. But then G is soluble, which implies the solubility of E, a contradiction. Therefore we have to conclude that $O^p(G) = G$, so by Lemma 2.8, $G = O^p(G) \leqslant N_G(L)$, since L is S-permutable in G. Therefore every minimal subgroup of P is normal in G and hence $P_0 \leqslant Z(F)$. Next we show that the hypothesis is still true for $(G/P_0, C \cap E/P_0)$. Indeed, $F^* = F \leqslant F^*(C \cap E)$ and by Lemma 2.11(1), $F^*(C \cap E) \leqslant F^*$. Hence $F^*(C \cap E) = F^*$ and so by Lemma 2.12, $F^*(C \cap E/P_0) = F^*/P_0 = F/P_0$, since $P_0 \leqslant Z(C)$. Now by (6) and [2, Lemmas 2.4 and 2.10(4)], we know that the hypothesis is still true for $(G/P_0, C \cap E/P_0)$. Hence $C \cap E/P_0 \leqslant Z_{\mathfrak{U}\Phi}(G/P_0)$, by the choice of (G, E). On the other hand, since by Lemma 2.4, G/C is supersoluble, every G-chief factor between E and $E \cap C$ has prime order. Hence $E \leqslant Z_{\mathfrak{U}\Phi}(G)$, a contradiction.

(8) *P is non-cyclic* (this follows directly from (7)).

By (8), P is non-cyclic and so by hypothesis P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is weakly S-permutable in G.

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(9) $|D| > p$.

Suppose that $|D| = p$. By (7), P has a subgroup H such that $|H| = p$ and H is not S-permutable in G. By [2, Lemma 2.10(5)] the subgroup H has a normal complement T in G. Then the hypothesis is true for (G, V) where $V = T \cap E$. Indeed, evidently $F^*(V) \leqslant F^*(E)$. On the other hand, since $|G : T| = p$, every Sylow q-subgroup of $F = F^*$, where $q \neq p$, is contained in T. Thus the hypothesis is still true for (G, V). But since T is a proper subgroup of G and $ET = G$, $|V| < |E|$. Hence $V \leqslant Z_{\mathfrak{U}\Phi}(G)$, by the choice of (G, E). But $G/T = ET/T \simeq E/E \cap T$ is a cyclic group, so $E \leqslant Z_{\mathfrak{U}\Phi}(G)$. This contradiction completes the proof of (9).

(10) *If L is a minimal normal subgroup of G and $L \leqslant P$, then $|L| > p$.*

Assume that $|L| = p$. Let $C_0 = C_E(L)$. Then the hypothesis is true for $(G/L, C_0/L)$. Indeed, since $F = F^* \leqslant C_0$ and $L \leqslant Z(F)$, we have $F^*(C_0/L) = F^*/L$. On the other hand, if H/L is a subgroup of G/L such that $|H| = |D|$, we have $1 < |H/L| < |P/L|$, by (9). Besides, H/L is weakly S-permutable in G/L, by [2, Lemma 2.10(2)]. Now, by [2, Lemma 2.10(2)] and by (6) we see that the hypothesis still holds for $(G/L, C_0/L)$. Hence $C_0/L \leqslant Z_{\mathfrak{U}\Phi}(G/L)$, which implies $E \leqslant Z_{\mathfrak{U}\Phi}(G)$, a contradiction.

(11) *$E = G$ is not soluble and $F = P$* (see (11) and (14) in [2, Theorem 1.3]).

Final contradiction. Since by (6), $p \neq 2$, and by (1) and (11), $F^* = F = P = F^*(G)$, G is supersoluble by Theorem 1.3 in [2]. This contradiction completes the proof of the theorem.

5. Final remarks

1. Recall that a formation is a homomorph $\mathfrak{X}$ of groups such that each group G has a smallest normal subgroup whose quotient is still in $\mathfrak{X}$. A formation $\mathfrak{X}$ is said to be saturated (solubly saturated) if it contains each group G with $G/\Phi(G) \in \mathfrak{X}$ (with $G/\Phi(R) \in \mathfrak{X}$, for some soluble normal subgroup R of G, respectively).

2. We say that a chief factor H/K of a group G is a *solubly-Frattini chief factor* of G if $H/K \leqslant \Phi(R/K)$ for some soluble normal subgroup R/K of G/K. By analogy with Definition 1.1 we introduce the following

Definition 5.1. Let $Z_{\mathfrak{X}\Phi(\mathfrak{S})}(G)$ be the product of all normal subgroups of G whose non-solubly-Frattini G-chief factors are $\mathfrak{X}$-central in G. Then we say that $Z_{\mathfrak{X}\Phi(\mathfrak{S})}(G)$ is the *$\mathfrak{X}\Phi(\mathfrak{S})$-hypercentre of G*.

The importance of this concept and the concept of $\mathfrak{X}\Phi$-hypercentre in the theory of (solubly) saturated formations is connected with the following observation.

Proposition 5.2. *Let $\mathfrak{X}$ be a class of groups and E a normal subgroup of G with $G/E \in \mathfrak{X}$. Suppose that at least one of the following hold*:

(1) *$\mathfrak{X}$ is a saturated formation and $E \leqslant Z_{\mathfrak{X}\Phi}(G)$.*
(2) *$\mathfrak{X}$ is a solubly saturated formation and $E \leqslant Z_{\mathfrak{X}\Phi(\mathfrak{S})}(G)$.*

Then $G \in \mathfrak{X}$.

3. It is clear that $Z_{\mathfrak{X}}(G) \leqslant Z_{\mathfrak{X}\Phi(\mathfrak{S})}(G) \leqslant Z_{\mathfrak{X}\Phi}(G)$. The following example shows that in general, $Z_{\mathfrak{X}}(G) < Z_{\mathfrak{X}\Phi(\mathfrak{S})}(G) < Z_{\mathfrak{X}\Phi}(G)$.

Example 5.3. It is well known [28, Chapter 4] that the Schur multiplier of the Mathieu group M_{22} is cyclic of order 12. Hence there is a group A with a cyclic normal subgroup $R = \langle a \rangle$ such that $R = \Phi(A) \cap Z(A)$, $A/R \simeq M_{22}$ and $|R| = 4$. Let p be an odd prime and C_p a group with $|C_p| = p$.

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Let $G = A \wr C_p = [K]C_p$, where K is the base group of the regular wreath product G. Then $Z_{\mathfrak{N}}(G) < Z_{\mathfrak{N}\Phi(\mathfrak{S})}(G) = \langle a^2 \rangle^{\natural} < Z_{\mathfrak{N}\Phi}(G) = R^{\natural}$ (we use here the terminology in [3, Chapter A]).

4. Researches of many authors are connected with analysis of the following general question: *Let $\mathfrak{X}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{X}$. Under what conditions on E then, does G belong to $\mathfrak{X}$*? (see Section 5 in [2] or the survey [29]). Almost all results in this direction may be improved by proving that under certain conditions the assumptions imply that $E \leqslant Z_{\mathfrak{U}\Phi}(G)$ is true. As a partial illustration for this we obtain from Theorem 1.4 the following stronger versions of Theorems 1.3 and 1.4 in [2].

Corollary 5.4. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) is weakly S-permutable in G. Then $E \leqslant Z_{\mathfrak{U}\Phi}(G)$. In particular, $G \in \mathfrak{F}$.*

Corollary 5.5. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) not having a supersoluble supplement in G are weakly S-permutable in G. Then $E \leqslant Z_{\mathfrak{U}\Phi}(G)$. In particular, $G \in \mathfrak{F}$.*

5. A subgroup H of a group G is called nearly normal in G if G has a normal subgroup T such that $T \cap H \leqslant H_G$ and $HT = H^G$ [30]. By using the same arguments as in the proof of Theorem 1.4 the following result may be proved.

Theorem 5.6. *Let $X \leqslant E$ be normal subgroups of a group G. Suppose that every maximal subgroup of every Sylow subgroup of X is nearly normal in G. If X is either E or $F^*(E)$, then $E \leqslant Z_{\mathfrak{U}\Phi(\mathfrak{S})}(G)$.*

In view of Proposition 5.2, Theorem 3.8 in [30] is a corollary of Theorem 5.6.

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