

Finite groups with semi-subnormal Schmidt subgroups

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ABSTRACT. A Schmidt group is a non-nilpotent group in which every proper subgroup is nilpotent. A subgroup A of a group G is *semi-normal* in G if there exists a subgroup B of G such that $G = AB$ and AB_1 is a proper subgroup of G for every proper subgroup B_1 of B . If A is either subnormal in G or is semi-normal in G , then A is called a *semi-subnormal* subgroup of G . In this paper, we establish that a group G with semi-subnormal Schmidt $\{2, 3\}$ -subgroups is 3-soluble. Moreover, if all 5-closed Schmidt $\{2, 5\}$ -subgroups are semi-subnormal in G , then G is soluble. We prove that a group with semi-subnormal Schmidt subgroups is metanilpotent.

1. Introduction

All groups in this paper are finite. A Schmidt group is a non-nilpotent group in which every proper subgroup is nilpotent. These groups were first considered by O. Y. Schmidt [1]. He proved that a Schmidt group is biprimary, one of its Sylow subgroups is normal and another one is cyclic. Reviews about the structure of Schmidt groups and their applications in the theory of finite groups are available in [2], [3].

Since every non-nilpotent group contains a Schmidt subgroup, Schmidt groups are universal subgroups of finite groups. So the properties of

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Schmidt subgroups contained in a group have a significant influence on the group structure. Groups with some restrictions on Schmidt subgroups were investigated in many papers. For example, groups with subnormal Schmidt subgroups were studied in [4]–[5], and groups with Hall Schmidt subgroups were described in [6].

A subgroup A of a group G is *semi-normal* in G if there exists a subgroup B of G such that $G = AB$ and AB_1 is a proper subgroup of G for every proper subgroup B_1 of B . Obviously, every subgroup of prime index is semi-normal. Also a quasi-normal subgroup (i. e. a subgroup of G , that permutes with all subgroups of G) is semi-normal. In the simple group $PSL(2, 5)$ a subgroup A which is isomorphic to the alternating group A_4 is a semi-normal Schmidt subgroup, but A is not quasi-normal and not subnormal.

Some properties of semi-normal subgroups were obtained in [7]–[9]. The criteria for solubility of a group with some semi-normal Schmidt subgroups were established in [10].

We introduce the following concept, which combines subnormality and semi-normality.

Definition. A subgroup A of a group G is called *semi-subnormal* in G if A is either subnormal in G or semi-normal in G .

In this paper, we establish that a group G with semi-subnormal Schmidt $\{2, 3\}$ -subgroups is 3-soluble. Moreover, if all 5-closed Schmidt $\{2, 5\}$ -subgroups are semi-subnormal in G , then G is soluble. We prove that a group with semi-subnormal Schmidt subgroups is metanilpotent.

2. Preliminary results

The terminology in the article as in [11]–[12]. We write $Y \leq X$ if Y is a subgroup of a group X . Recall that $A^G = \langle A^g \mid g \in G \rangle$ is the subgroup generated by all subgroups of G that are conjugate to A . A group with a normal Sylow p -subgroup is called p -closed. Let π be a set of primes. We say G is a π -group if every prime divisor of $|G|$ lies in π . Let us agree to call the $S_{(p,q)}$ -group a Schmidt group with a normal Sylow p -subgroup and a cyclic Sylow q -subgroup.

Lemma 1 ([11, 2.41; 2.43; 5.31]). *Let H be a subnormal subgroup of a group G .*

(1) *If $U \leq G$, then $U \cap H$ is subnormal in U . In particular, if $H \leq V \leq G$, then H is subnormal in V .*

(2) *If N is a normal subgroup of G , then HN/N is a subnormal subgroup of G/N .*

(3) If K is a subnormal subgroup of G , then $H \cap K$ and $\langle H \cup K \rangle$ are subnormal subgroups of G .

(4) $\pi(H) = \pi(H^G)$.

(5) If \mathfrak{X} is a Fitting class and $H \in \mathfrak{X}$, then $H^G \in \mathfrak{X}$.

Lemma 2. *If X is a subnormal $S_{\langle p, q \rangle}$ -subgroup of a group G , then X^G is a p -closed $\{p, q\}$ -subgroup.*

Proof. Lemma 1 (4) implies that X^G is a $\{p, q\}$ -subgroup. The class of all p -closed subgroups is a Fitting class [11, 5.3]. By Lemma 1 (5), X^G is p -closed. \square

Lemma 3 ([4, Lemma 2]). *If K and D are subgroups of a group G , D is normal in K and K/D is an $S_{\langle p, q \rangle}$ -subgroup, then each minimal supplement L to the subgroup D in K has the following properties:*

(1) L is a p -closed $\{p, q\}$ -subgroup;

(2) all proper normal subgroups of L are nilpotent;

(3) L contains an $S_{\langle p, q \rangle}$ -subgroup $[P]Q$ such that Q is not contained in D , and $L = ([P]Q)^L = Q^L$.

Lemma 4 ([13, Lemma 1.6]). *If H is a subgroup of a soluble group G and $|G : H|$ is a prime number, then G/H_G is supersoluble.*

Here $H_G = \bigcap_{x \in G} H^x$ is the core of H in G .

The Frattini and Fitting subgroups of a group G are denoted by $\Phi(G)$ and $F(G)$ respectively. The intersection of all normal subgroups of G , whose quotient groups belong to \mathfrak{F} is called \mathfrak{F} -residual of G and denoted by $G^{\mathfrak{F}}$.

Lemma 5 ([12, 24.2; 24.3]). *If \mathfrak{F} is a saturated formation, G is a soluble minimal non- \mathfrak{F} -group, then the following statements hold:*

(1) $G^{\mathfrak{F}}$ is a p -group for some $p \in \pi(G)$;

(2) $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G ;

(3) $G^{\mathfrak{F}}\Phi(G) = F(G)$.

Lemma 6. *If G is a soluble minimal non-metanilpotent group, then $G/F(G)$ is a Schmidt group.*

Proof. The class of all metanilpotent groups is a saturated hereditary formation [12, p. 36] and coincides with the product $\mathfrak{N}\mathfrak{N} = \mathfrak{N}^2$. First, let $\Phi(G) = 1$. According to Lemma 5(3), $G^{\mathfrak{N}^2} = F(G)$. Since $G^{\mathfrak{N}^2}$ is a p -group for some $p \in \pi(G)$ by Lemma 5(1), we have $F(G) = O_p(G)$. Since $G \notin \mathfrak{N}^2$, it follows that $G/F(G) \notin \mathfrak{N}$.

Let $U/F(G)$ be a proper subgroup of $G/F(G)$. Then U is metanilpotent and $U/F(U)$ is nilpotent. Since G is soluble, we obtain $O_p(G) = F(G) \leq U$, it implies

$$C_G(F(G)) = Z(F(G)), \quad O_{p'}(U) = 1, \quad O_p(G) = F(G) \leq F(U) = O_p(U).$$

Let $H/F(G) = F(G/F(G))$. Then $H/F(G)$ is a p' -subgroup and

$$H \cap O_p(U) = O_p(G) = F(G).$$

Since G/H is nilpotent, $UH/H \simeq U/U \cap H$ and $U/O_p(U)$ are nilpotent. So

$$U/U \cap H \cap O_p(U) = U/F(G)$$

is nilpotent. Therefore, all proper subgroups of $G/F(G)$ are nilpotent and $G/F(G)$ is a Schmidt group.

Let $\Phi(G) \neq 1$. According to Lemma 5 (3), the subgroup $F(G) = G^{\mathfrak{N}^2} \Phi(G)$. Since \mathfrak{N}^2 is a hereditary formation, we have $G/\Phi(G) \notin \mathfrak{N}^2$ and $G/\Phi(G)$ is a minimal non- \mathfrak{N}^2 -group. By the hypothesis, G is soluble, so $F(G/\Phi(G)) = F(G)/\Phi(G)$ by [11, 4.21], and

$$G/F(G) \simeq (G/\Phi(G))/(F(G)/\Phi(G)) = (G/\Phi(G))/(F(G/\Phi(G))).$$

Since $\Phi(G/\Phi(G)) = 1$, by the above $(G/\Phi(G))/F(G/\Phi(G))$ is a Schmidt group, therefore $G/F(G)$ is a Schmidt group. \square

Example 1. In the simple group $PSL(2, 5)$, all its proper subgroups are metanilpotent, they are even metabelian. Therefore, the condition of solubility of a group in Lemma 6 is not superfluous.

3. Properties of semi-subnormal subgroups

Lemma 7. (1) If H is a semi-subnormal subgroup of a group G and $H \leq X \leq G$, then H is semi-subnormal in X .

(2) If H is a semi-subnormal subgroup of a group G , N is a normal subgroup of G , then HN is semi-subnormal in G , and HN/N is semi-subnormal in G/N .

(3) If H is a semi-subnormal subgroup of a group G and Y is nonempty set of elements in G , then the subgroup

$$H^Y = \langle H^y \mid y \in Y \rangle$$

is semi-subnormal in G . In particular, H^g is semi-subnormal in G for any $g \in G$.

Proof. Suppose that H is a subnormal subgroup of a group G . Then it follows from Lemma 1 that statements (1)-(3) are true. If H is semi-normal in G , then assertions (1)-(3) are proved in [10, Lemma 2.5]. \square

Proposition 1. *Let A be a semi-subnormal subgroup of a group G . Then the following statements holds.*

- (1) *If A is a 2-nilpotent subgroup, then A^G is soluble.*
- (2) *If A is soluble and 3 does not divide the order of A , then A^G is soluble.*
- (3) *Let p be the smallest prime divisor of the order of G . If p does not divide the order of A , then p does not divide the order of A^G . In particular, if A is a subgroup of odd order, then A^G is of odd order.*

Proof. If A is a semi-normal subgroup of G , then all three statements are known [10, lemmas 10 and 11]. If A is a subnormal subgroup of G , then all three statements follow from Lemma 1 (4-5). \square

Proposition 2. *If A is a semi-subnormal subgroup of a soluble group G and r is the largest prime from $\pi(G)$, then $O_r(A) \leq O_r(G)$.*

Proof. It suffices to prove that $O_r(A)$ is subnormal in G . If A is a subnormal subgroup of G , then $O_r(A)$ is subnormal in G and $O_r(A) \leq O_r(G)$ by Lemma 1 (5). Let A be a semi-normal subgroup of G and Y be a minimal supplement to A in G such that A is permutable with all subgroups of Y . We apply induction on the order of the group. Since Y is soluble, there is a subgroup X of prime index i.e $|Y : X| = t$ is a prime number. By the hypothesis, A is permutable with X . Besides A is a semi-subnormal subgroup of AX by Lemma 7 (1). By induction, $O_r(A)$ is subnormal in AX . If AX is normal in G , then $O_r(A)$ is subnormal in G . Suppose that AX is not normal in G , i. e. $AX \neq (AX)_G$. Since $G = AY$ and $AX < G$, we have

$$1 \neq |G : AX| = \frac{|A||Y|}{|A \cap Y|} : \frac{|A||X|}{|A \cap X|} = \frac{|Y : X|}{|A \cap Y : A \cap X|} = t.$$

By Lemma 4, $G/(AX)_G$ is supersoluble and is isomorphic to a subgroup of the symmetric group S_t of degree t . Since

$$t = |G : AX| = |G/(AX)_G : A/(AX)_G|,$$

then t does not divide $|AX/(AX)_G|$. If $t = r$, then $AX/(AX)_G$ is a r' -group. Hence

$$O_r(A) \leq O_r(AX) \leq (AX)_r \leq (AX)_G$$

and $O_r(A)$ is subnormal in G . If $t \neq r$, then $t < r$ and $G/(AX)_G$ is a r' -group. Therefore, $A_r \leq (AX)_G$ and again $O_r(A)$ subnormal in G . Consequently, $O_r(A) \leq O_r(G)$. \square

4. Semi-subnormal Schmidt subgroups

Lemma 8. *Suppose that all $S_{\langle p,q \rangle}$ -subgroups of a group G are semi-subnormal in G .*

(1) *If H is a subgroup of G , then all $S_{\langle p,q \rangle}$ -subgroups of H are semi-subnormal in H .*

(2) *If N is a normal subgroup of G , then all $S_{\langle p,q \rangle}$ -subgroups of G/N are semi-subnormal in G/N .*

Proof. 1. The statement follows from Lemma 7 (1).

2. Let S/N be an $S_{\langle p,q \rangle}$ -subgroup of G/N and L be a minimal subgroup of S such that $S = LN$. By Lemma 3, L contains an $S_{\langle p,q \rangle}$ -subgroup A such that $L = A^L$. By the hypothesis, A is semi-subnormal in G , and by Lemma 7 (3), L is semi-subnormal in G . Now by Lemma 7 (2), the subgroup $LN/N = S/N$ is semi-subnormal in G/N . \square

Theorem 1. *Let A be a semi-subnormal Schmidt subgroup of a group G .*

(1) *If A^G is insoluble, then $A/Z(A) \simeq A_4$.*

(2) *If A^G is a simple group, then $A^G \simeq PSL(2, 5)$.*

Proof. If A is a subnormal subgroup of G , then $\pi(A^G) = \pi(A)$ by Lemma 2, so A^G is soluble. Further, we assume that A is a semi-normal subgroup of G . If A^G is insoluble, then A is a 2-closed $\{2, 3\}$ -subgroup according to [10, Theorem 1]. We deduce from the properties of Schmidt groups [3, Theorem 1.2] that $A/Z(A) \simeq A_4$. Let A^G be a simple group. By Lemma 7, the subgroup A is semi-normal in A^G and $|A^G : A|$ is a prime number by [10, Lemma 7]. Now $Z(A) = 1$ by [14, V.7.2] and $A \simeq A_4$. In view of [[15], Theorem 1], we obtain that $A^G \simeq PSL(2, 5)$. \square

Theorem 2. *If all $\{2, 3\}$ -subgroups of a group G are semi-subnormal in G , then G is 3-soluble.*

Proof. We use induction on the order of G . Let N be a normal subgroup of G . By Lemma 8, all Schmidt $\{2, 3\}$ -subgroups are semi-subnormal in N and in G/N . If $G \neq N \neq 1$, then by induction, N and G/N are 3-soluble. This implies that G likewise is 3-soluble. We will therefore assume that G is simple, in particular there are no subnormal Schmidt

subgroups. Consequently, all Schmidt $\{2, 3\}$ -subgroups are semi-normal in G . According to [10, Theorem 2], G is 3-soluble. \square

Corollary 1. *If all Schmidt $\{2, 3\}$ -subgroups and all 5-closed Schmidt $\{2, 5\}$ -subgroups of a group G are semi-subnormal, then G is soluble.*

In Corollary 1, three types of Schmidt subgroups: $S_{\langle 2,3 \rangle}$ -subgroups, $S_{\langle 3,2 \rangle}$ -subgroups, and $S_{\langle 5,2 \rangle}$ -subgroups should be semi-subnormal. The following examples of simple groups show that the condition of semi-subnormality of all these types of Schmidt subgroups is not superfluous.

Example 2. In $PSL(2, 3^3)$, there are no $S_{\langle 5,2 \rangle}$ -subgroups and $S_{\langle 3,2 \rangle}$ -subgroups [16]; so the condition of being semi-subnormal for $S_{\langle 2,3 \rangle}$ -subgroups is not superfluous.

Example 3. In $SL(2, 8)$, there are no $S_{\langle 5,2 \rangle}$ -subgroups and $S_{\langle 2,3 \rangle}$ -subgroups [16]; so the groups with semi-subnormal Schmidt $S_{\langle 5,2 \rangle}$ -subgroups and $S_{\langle 2,3 \rangle}$ -subgroups may be insoluble and the condition of being semi-subnormal for $S_{\langle 3,2 \rangle}$ -subgroups is not superfluous.

Example 4. In $Sz(8)$, there are no $\{2, 3\}$ -subgroups [16]; so the groups with semi-subnormal Schmidt $\{2, 3\}$ -subgroups may be insoluble and the condition of being semi-subnormal for $S_{\langle 5,2 \rangle}$ -subgroups is not superfluous.

Theorem 3. *If all Schmidt subgroups of a group G are semi-subnormal in G , then G is metanilpotent.*

Proof. By Corollary 1, G is soluble. We use induction on the order of G . According to Lemma 8, in each proper subgroup and in each quotient group of G different from G , all Schmidt subgroups are semi-subnormal. By induction, they are metanilpotent. The class of all metanilpotent groups is a hereditary saturated formation. By [17, Lemma 8], G is primitive, and:

$$G = [N]M, \quad M < \cdot G, \quad N = F(G) = O_p(G) = C_G(N),$$

for some $p \in \pi(G)$. According to Lemma 6, the quotient group $G/N \simeq M$ is a Schmidt group. By the hypothesis, M is either subnormal in G or semi-normal in G . If M is subnormal in G , then M is normal in G and $M \leq C_G(N) = N$, a contradiction. If M is semi-normal in G , then $|G : M| = |N| = p$ is a prime number by [10, Lemma 7]. Hence G is supersoluble and therefore metanilpotent. \square

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