On the residual of a factorized group with widely supersoluble factors

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Abstract. Let $\mathbb{P}$ be the set of all primes. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G, if either $H = G$, or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \ldots \leq H_n = G$, $|H_i : H_{i-1}| \in \mathbb{P}$, $\forall i$. A group G is called *widely supersoluble*, w-supersoluble for short, if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G. A group $G = AB$ with $\mathbb{P}$-subnormal w-supersoluble subgroups A and B is studied. The structure of its w-supersoluble residual is obtained. In particular, it coincides with the nilpotent residual of the $\mathcal{A}$-residual of G. Here $\mathcal{A}$ is the formation of all groups with abelian Sylow subgroups. Besides, we obtain new sufficient conditions for the w-supersolubility of such group G.

Keywords. widely supersoluble groups, mutually sn-permutable subgroups, $\mathbb{P}$-subnormal subgroup, the $\mathfrak{X}$-residual.

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Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [6]. The formations of all nilpotent, supersoluble groups and groups with abelian Sylow subgroups are denoted by $\mathfrak{N}$, $\mathfrak{U}$ and $\mathcal{A}$, respectively. The notation $Y \leq X$ means that Y is a subgroup of a group X and $\mathbb{P}$ be the set of all primes. Let $\mathfrak{X}$ be a formation. Then $G^{\mathfrak{X}}$ denotes the $\mathfrak{X}$-residual of G.

By Huppert's Theorem [6, VI.9.5], a group G is supersoluble if and only if for every proper subgroup H of G there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G, \ |H_i : H_{i-1}| \in \mathbb{P}, \ \forall i. \tag{1}$$

So naturally the following definition.

A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G, if either $H = G$, or there is a chain subgroups (1). We use the notation $H \ Psn \ G$. This definition was proposed in [13] and besides, in this paper w-*supersoluble* (widely supersoluble) groups, i.e. groups with $\mathbb{P}$-subnormal Sylow subgroups, were investigated. Denote by $\mathrm{w}\mathfrak{U}$ the class of all w-supersoluble groups.

The factorizable groups $G = AB$ with w-supersoluble factors A and B were investigated in [8], [10], [11], [14]. There are many other papers devoted to study factorizable groups, and the reader is referred to the book [1] and the bibliography therein. A criteria for w-supersolvability was obtained by A. F. Vasil'ev, T. I. Vasil'eva and V. N. Tyutyanov [14].

Theorem A. [14, Theorem 4.7] *Let $G = AB$ be a group which is the product of two* w-*supersoluble subgroups A and B. If A and B are $\mathbb{P}$-subnormal in G and $G^{\mathcal{A}}$ is nilpotent, then G is* w-*supersoluble.*

We recall that two subgroups A and B of a group G are said to be *mutually sn-permutable* if A permutes with all subnormal subgroups of B and B permutes with all subnormal subgroups of A. If A and B are mutually *sn*-permutable subgroups of a group $G = AB$, then we say that G is a *mutually sn-permutable product* of A and B, see [4]. In soluble groups, mutually *sn*-permutable factors are $\mathbb{P}$-subnormal [14, Lemma 4.5]. The converse is not true, see the example 3.1 below.

A. Ballester-Bolinches, W. M. Fakieh and M. C. Pedraza-Aguilera [3] obtained the following results for the *sn*-permutable product of the w-supersoluble subgroups.

Theorem B. *Let $G = AB$ be the mutually sn-permutable product of subgroups A and B. Then the following hold:*

(1) *if A and B are* w-*supersoluble and N is a minimal normal subgroup of G, then both AN and BN are* w-*supersoluble,* [3, Theorem 3]*;*

(2) *if A and B are* w-*supersoluble and $(|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$, then G is* w-*supersoluble,* [3, Theorem 5].

Present paper extends the Theorems A and B. We prove the following result.

Theorem 1. *Let A and B be* w-*supersoluble $\mathbb{P}$-subnormal subgroups of G and $G = AB$. Then the following hold:*

(1) $G^{\mathrm{w}\mathfrak{U}} = (G^{\mathcal{A}})^{\mathfrak{N}}$*;*

(2) *if N is a nilpotent normal subgroup of G, then both AN and BN are* w-*supersoluble;*

(3) *if* $(|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$*, then* G *is* w*-supersoluble.*

Theorem A follows from assertion (1) of Theorem 1. Theorem B follows from assertions (2) and (3) of Theorem 1 since the group G in Theorem B is soluble.

1 Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called *supersoluble*. Recall that a *p-closed* group is a group with a normal Sylow p-subgroup and a *p-nilpotent* group is a group with a normal Hall p'-subgroup.

Denote by G', $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, centre, Fitting and Frattini subgroups of G respectively. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$.

Let $\mathfrak{F}$ be a formation. Recall that the $\mathfrak{F}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. We define $\mathfrak{X}\mathfrak{Y} = \{G \in \mathfrak{E} \mid G^{\mathfrak{Y}} \in \mathfrak{X}\}$ and call $\mathfrak{X}\mathfrak{Y}$ the *formation product* of $\mathfrak{X}$ and $\mathfrak{H}$. Here $\mathfrak{E}$ is the class of all finite groups.

If H is a subgroup of G, then $H_G = \bigcap_{x \in G} H^x$ is called *the core* of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*.

A simple check proves the following lemma.

Lemma 1.1. *Let* $\mathfrak{F}$ *be a saturated formation and* G *be a group. Assume that* $G \notin \mathfrak{F}$*, but* $G/N \in \mathfrak{F}$ *for all non-trivial normal subgroups* N *of* G*. Then* G *is a primitive group.*

Lemma 1.2. ([6, Theorem II.3.2]) *Let* G *be a soluble primitive group and* M *is a primitivator of* G*. Then the following statements hold:*

(1) $\Phi(G) = 1$*;*

(2) $F(G) = C_G(F(G)) = O_p(G)$ *and* $F(G)$ *is an elementary abelian subgroup of order* p^n *for some prime* p *and some positive integer* n*;*

(3) G *contains a unique minimal normal subgroup* N *and moreover,* $N = F(G)$*;*

(4) $G = F(G) \rtimes M$ *and* $O_p(M) = 1$.

Lemma 1.3. ([2, Proposition 2.2.8, Proposition 2.2.11]) *Let* $\mathfrak{F}$ *and* $\mathfrak{H}$ *be formations,* K *be normal in* G*. Then the following hold:*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$*;*

(2) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$;
(3) *if* $\mathfrak{H} \subseteq \mathfrak{F}$, *then* $G^{\mathfrak{F}} \le G^{\mathfrak{H}}$;
(4) *if* $G = HK$, *then* $H^{\mathfrak{F}}K = G^{\mathfrak{F}}K$.

Recall that a group G is said to be *siding* if every subgroup of the derived subgroup G' is normal in G, see [12, Definition 2.1]. Metacyclic groups, t-groups (groups in which every subnormal subgroup is normal) are siding. The group $G = (Z_6 \times Z_2) \rtimes Z_2$ (IdGroup(G)=[24,8]) [5] is siding, but not metacyclic and a t-group.

Lemma 1.4. *Let G be siding. Then the following hold:*
(1) *if N is normal in G, then G/N is siding;*
(2) *if H is a subgroup of G, then H is siding;*
(3) *G is supersoluble*

Proof. 1. By [6, Lemma I.8.3], $(G/N)' = G'N/N$. Let A/N be an arbitrary subgroup of $(G/N)'$. Then

$$A \le G'N, \ A = A \cap G'N = (A \cap G')N.$$

Since $A \cap G' \le G'$, we have $A \cap G'$ is normal in G. Hence $(A \cap G')N/N$ is normal in G/N.

2. Since $H \le G$, it follows that $H' \le G'$. Let A be an arbitrary subgroup of H'. Then $A \le G'$ and A is normal in G. Therefore A is normal in H.

3. We proceed by induction on the order of G. Let $N \le G'$ and $|N| = p$, where p is prime. By the hypothesis, N is normal in G. By induction, G/N is supersoluble and G is supersoluble. □

Lemma 1.5. ([9, Lemma 3]) *Let H be a subgroup of G, and N be a normal subgroup of G. Then the following hold:*
(1) *if $N \le H$ and H/N $\mathbb{P}sn$ G/N, then H $\mathbb{P}sn$ G;*
(2) *if H $\mathbb{P}sn$ G, then $(H \cap N)$ $\mathbb{P}sn$ N, HN/N $\mathbb{P}sn$ G/N and HN $\mathbb{P}sn$ G;*
(3) *if $H \le K \le G$, H $\mathbb{P}sn$ K and K $\mathbb{P}sn$ G, then H $\mathbb{P}sn$ G;*
(4) *if H $\mathbb{P}sn$ G, then H^g $\mathbb{P}sn$ G for any $g \in G$.*

Lemma 1.6. ([9, Lemma 4]) *Let G be a soluble group, and H be a subgroup of G. Then the following hold:*
(1) *if H $\mathbb{P}sn$ G and $K \le G$, then $(H \cap K)$ $\mathbb{P}sn$ K;*
(2) *if H_i $\mathbb{P}sn$ G, $i = 1, 2$, then $(H_1 \cap H_2)$ $\mathbb{P}sn$ G.*

Lemma 1.7. ([9, Lemma 5]) *If H is a subnormal subgroup of a soluble group G, then H is $\mathbb{P}$-subnormal in G.*

Lemma 1.8. ([13, Theorem 2.7]) *The class* w$\mathfrak{U}$ *is a hereditary saturated formation.*

Lemma 1.9. (1) *If* $G \in \mathrm{w}\mathfrak{U}$*, then* $G^{\mathcal{A}}$ *is nilpotent,* [13, Theorem 2.13].

(2) $G \in \mathrm{w}\mathfrak{U}$ *if and only if every metanilpotent subgroup of* G *is supersoluble,* [7, Theorem 2.6].

(3) $G \in \mathrm{w}\mathfrak{U}$ *if and only if* G *has a Sylow tower of supersoluble type and every biprimary subgroup of* G *is supersoluble,* [9, Theorem B].

2 Factorizable groups with $\mathbb{P}$-subnormal w-supersoluble subgroups

Lemma 2.1. ([14, Theorem 4.4]) *Let* A *and* B *be* $\mathbb{P}$*-subnormal subgroups of* G*, and* $G = AB$*. If* A *and* B *have an ordered Sylow tower of supersoluble type, then* G *has an ordered Sylow tower of supersoluble type.*

Proof of Theorem 1 (1). If G is w-supersoluble, then $G^{\mathrm{w}\mathfrak{U}} = 1$ and $G^{\mathcal{A}}$ is nilpotent by Lemma 1.9 (1). Consequently $G^{\mathrm{w}\mathfrak{U}} = 1 = (G^{\mathcal{A}})^{\mathfrak{N}}$ and the statement is true. Further, we assume that G is non-w-supersoluble. Since $\mathrm{w}\mathfrak{U} \subseteq \mathfrak{N}\mathcal{A}$, it follows that

$$G^{(\mathfrak{N}\mathcal{A})} = (G^{\mathcal{A}})^{\mathfrak{N}} \leq G^{\mathrm{w}\mathfrak{U}}$$

by Lemma 1.3 (2-3). Next we check the converse inclusion. For this we prove that $G/(G^{\mathcal{A}})^{\mathfrak{N}}$ is w-supersoluble. By Lemma 1.3 (1),

$$(G/(G^{\mathcal{A}})^{\mathfrak{N}})^{\mathcal{A}} = G^{\mathcal{A}}(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}} = G^{\mathcal{A}}/(G^{\mathcal{A}})^{\mathfrak{N}}$$

and $(G/(G^{\mathcal{A}})^{\mathfrak{N}})^{\mathcal{A}}$ is nilpotent. The quotients

$$G/(G^{\mathcal{A}})^{\mathfrak{N}} = (A(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}})(B(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}},$$

$$A(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}} \simeq A/A \cap (G^{\mathcal{A}})^{\mathfrak{N}},$$

$$B(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}} \simeq B/B \cap (G^{\mathcal{A}})^{\mathfrak{N}},$$

hence the subgroups $A(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}}$ and $B(G^{\mathcal{A}})^{\mathfrak{N}}/(G^{\mathcal{A}})^{\mathfrak{N}}$ are w-supersoluble by Lemma 1.8 and by Lemma 1.5 (2), they are $\mathbb{P}$-subnormal in $G/(G^{\mathcal{A}})^{\mathfrak{N}}$. By Theorem A, $G/(G^{\mathcal{A}})^{\mathfrak{N}}$ is w-supersoluble. □

Lemma 2.2. *Let* G *be a group, and* A *be a subgroup of* G *such that* $|G : A| = p^{\alpha}$*, where* $p \in \pi(G)$ *and* $\alpha \in \mathbb{N}$*. Suppose that* A *is* w*-supersoluble and* $\mathbb{P}$*-subnormal in* G*. If* G *is* p*-closed, then* G *is* w*-supersoluble.*

Proof. Let P be a Sylow p-subgroup of G. Since P is normal in G and $G = AP$, we have $G/P \simeq A/A \cap P \in \mathrm{w}\mathfrak{U}$, in particular, G is soluble. Because G is soluble, it follows that P is $\mathbb{P}$-subnormal in G by Lemma 1.7. Let Q be a Sylow q-subgroup of G, $q \neq p$. Then $Q \leq A^x$ for some $x \in G$. By Lemma 1.5 (4), A^x is $\mathbb{P}$-subnormal in G. Since $A^x \in \mathrm{w}\mathfrak{U}$, it follows that Q is $\mathbb{P}$-subnormal in A^x and Q is $\mathbb{P}$-subnormal in G by Lemma 1.5 (3). So, G is w-supersoluble. □

Lemma 2.3. *Let A and B be* w*-supersoluble $\mathbb{P}$-subnormal subgroups of G, and $G = AB$. Suppose that $|G : A| = p^\alpha$, where $p \in \pi(G)$. If p is the greatest in $\pi(G)$, then G is* w*-supersoluble.*

Proof. Since every w-supersoluble group has an ordered Sylow tower of supersoluble type, then by Lemma 2.1, G has an ordered Sylow tower of supersoluble type. Hence G is p-closed. By Lemma 2.2, we have that G is w-supersoluble. □

Theorem 2.1. *Let A be a* w*-supersoluble $\mathbb{P}$-subnormal subgroup of G, and $G = AB$. Then G is* w*-supersoluble in each of the following cases:*

(1) *B is nilpotent and normal in G;*
(2) *B is nilpotent and $|G : B|$ is prime;*
(3) *B is normal in G and is a siding group.*

Proof. We prove all three statements at the same time using induction on the order of G. Note that G is soluble in any case. By Lemma 1.7, B is $\mathbb{P}$-subnormal in G and G has an ordered Sylow tower of supersoluble type by Lemma 2.1. If N is a non-trivial normal subgroup of G, then AN/N is $\mathbb{P}$-subnormal in G/N by Lemma 1.5 (2) and $AN/N \simeq A/A \cap N$ is w-supersoluble by Lemma 1.8. The subgroup $BN/N \simeq B/B \cap N$ is nilpotent or a siding group by Lemma 1.4 (1). Hence $G/N = (AN/N)(BN/N)$ is w-supersoluble by induction. Since the formation of all w-supersoluble groups is saturated by Lemma 1.8, we have G is a primitive group by Lemma 1.1. By Lemma 1.2, $F(G) = N = G_p$ is a unique minimal normal subgroup of G and $N = C_G(N)$, where p is the greatest in $\pi(G)$.

Since A is $\mathbb{P}$-subnormal in G, it follows that G has a subgroup M such that $A \leq M$ and $|G : M|$ is prime. By Dedekind's identity, $M = A(M \cap B)$. The subgroup A is $\mathbb{P}$-subnormal in M. The subgroup $M \cap B$ satisfies the requirements (1)–(3). By induction, M is w-supersoluble.

1. If B is nilpotent and normal in G, then $B = N$. Hence $G = AN$ and A is a maximal subgroup of G. Since A is $\mathbb{P}$-subnormal in G, we have $|G : A| = p = |N|$ and G is supersoluble. Therefore G is w-supersoluble. So, in (1), the theorem is proved.

2. Let B be nilpotent and $|G : B| = q$, where q is prime. Besides, let $|G : M| = r$, where r is prime. If $q \neq r$, then $(|G : M|, |G : B|) = 1$. Since $G = MB$, M and B are $\mathbb{P}$-subnormal in G and w-supersoluble, it follows obviously that G is w-supersoluble. Hence $q = r$. If $q = p$, then N is not contained in M. Thus $G = N \rtimes M$ and $|N|$ is prime. Consequently G is supersoluble and therefore G is w-supersoluble. So, $q \neq p$. Then $G_p = N \leq M \cap B$. Since B is nilpotent, $G_p = B \leq M$. Because $G = MB$, we have $G = M$, a contradiction. So, in (2), the theorem is proved.

3. Let B is normal in G and is a siding group. If B is nilpotent, then G is w-supersoluble by (1). Hence $B' \neq 1$. Because B' is normal in G and nilpotent, we have $N = B'$. If N is not contained in M, then $G = N \rtimes M$ and $|N|$ is prime. Consequently G is supersoluble and therefore G is w-supersoluble. Let N be contained in M and N_1 be a subgroup of prime order of N such that N_1 is normal in M. Then N_1 is normal in B by definition of siding group. Hence N_1 is normal in G. Consequently G is w-supersoluble. So, in (3), the theorem is proved. □

Proof of Theorem 1 (2).

Note that by the Lemma 2.1, G is soluble. By Theorem 2.1 (1), Theorem 1 (2) is true. □

Proof of Theorem 1 (3). Assume that the claim is false and let G be a minimal counterexample. By Lemma 2.1, G has an ordered Sylow tower of supersoluble type. If N is a non-trivial normal subgroup of G, then AN/N and BN/N are $\mathbb{P}$-subnormal in G/N by Lemma 1.5 (2). Besides, $AN/N \simeq A/A \cap N$ and $BN/N \simeq B/B \cap N$ are w-supersoluble by Lemma 1.8. By Lemma 1.3, we have

$$(|(AN/N)/(AN/N)^{\mathcal{A}}|, |(BN/N)/(BN/N)^{\mathcal{A}}|) =$$

$$= (|AN/(AN)^{\mathcal{A}}N|, |BN/(BN)^{\mathcal{A}}N|) =$$

$$= (|AN/A^{\mathcal{A}}N|, |BN/B^{\mathcal{A}}N|) = \big(\frac{|A/A^{\mathcal{A}}|}{|S_1|}, \frac{|B/B^{\mathcal{A}}|}{|S_2|}\big),$$

$$S_1 = (A \cap N)/(A^{\mathcal{A}} \cap N),\ S_2 = (B \cap N)/(B^{\mathcal{A}} \cap N).$$

Since $(|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$, it follows that

$$(|(AN/N)/(AN/N)^{\mathcal{A}}|, |(BN/N)/(BN/N)^{\mathcal{A}}|) = 1.$$

The quotient $G/N = (AN/N)(BN/N)$ is w-supersoluble by induction.

Since the formation of all w-supersoluble groups is saturated by Lemma 1.8, we have G is a primitive group by Lemma 1.1. By Lemma 1.2, $F(G) = N =$

G_p is a unique minimal normal subgroup of G and $N = C_G(N)$, where p is the greatest in $\pi(G)$.

By Lemma 2.2, AN is w-supersoluble. If $AN = G$, then G is w-supersoluble, a contradiction. Hence in the future we consider that AN and BN are proper subgroups of G.

By Lemma 1.9 (1), $(AN)^{\mathcal{A}}$ is nilpotent. Since $N = C_G(N)$, we have $(AN)^{\mathcal{A}}$ is a p-group. Because $AN/(AN)^{\mathcal{A}} \in \mathcal{A}$, it follows that all Sylow r-subgroups of A are abelian, $r \neq p$. Since $A_p \leq G_p$, where A_p is a Sylow p-subgroup of A, we have $A \in \mathcal{A}$. Similarly, $B \in \mathcal{A}$. Hence $A^{\mathcal{A}} = 1 = B^{\mathcal{A}}$ and $(|A|, |B|) = (|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$. It is clear that G is w-supersoluble, a contradiction. □

3 Examples

The following example shows that for a soluble group $G = AB$ the mutually sn-permutability of subgroups A and B doesn't follow from $\mathbb{P}$-subnormality of these factors.

Example 3.1. The group $G = S_3 \rtimes Z_3$ (IdGroup=[18,3]) has $\mathbb{P}$-subnormal subgroups $A \simeq E_{3^2}$ and $B \simeq Z_2$. However A and B are not mutually sn-permutable.

The following example shows that we cannot omit the condition «G is p-closed» in Lemma 2.2.

Example 3.2. The group $G = (S_3 \times S_3) \rtimes Z_2$ (IdGroup=[72,40]) has a $\mathbb{P}$-subnormal supersoluble subgroups $A \simeq Z_3 \times S_3$. Besides $|G : A| = 2^2$ and Sylow 2-subgroup is maximal in G. Hence G is non-w-supersoluble.

The following example shows that in Theorem 2.1 (1) the normality of subgroup B cannot be weakened to $\mathbb{P}$-subnormality.

Example 3.3. The group $G = (Z_2 \times (E_{3^2} \rtimes Z_4)) \rtimes Z_2$ (IdGroup=[144,115]) is non-w-supersoluble and factorized by subgroups $A = D_{12}$ and $B = Z_{12}$. The subgroup A has the chain of subgroups $A < S_3 \times S_3 < Z_2 \times S_3 \times S_3 < G$ and B has the chain of subgroups $B < Z_3 \times (Z_3 \rtimes Z_4) < (Z_3 \times (Z_3 \rtimes Z_4)) \rtimes Z_2 < G$. Therefore A and B are $\mathbb{P}$-subnormal in G.

The following example shows that in Theorem 2.1 (2) it is impossible to weak the restrictions on the index of subgroup B.

Example 3.4. The alternating group $G = A_4$ is non-w-supersoluble and factorized by subgroups $A = E_{2^2}$ and $B = Z_3$. It is clear that A is supersoluble and $\mathbb{P}$-subnormal in G, and B is nilpotent and $|G : B| = 2^2$. The group $G = E_{5^2} \rtimes Z_3$ is non-w-supersoluble and has a nilpotent subgroup Z_3 of index 5^2. Therefore even for the greatest p of $\pi(G)$, the index of B cannot be equal p^α, $\alpha \geq 2$.

The following example shows that in Theorem 2.1 (3) the normality of subgroup B cannot be weakened to subnormality.

Example 3.5. The group $G = Z_3 \times ((S_3 \times S_3) \rtimes Z_2)$ (IdGroup=[216,157]) is non-w-supersoluble and factorized by $\mathbb{P}$-subnormal supersoluble subgroup $A \simeq S_3 \times S_3$ and subnormal siding subgroup $B \simeq Z_3 \times Z_3 \times S_3$.

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