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Colorings of groups without big monochrome symmetric subsets

I.V.Protasov

Let G be a group, $g \in G$. A mapping $s_g : G \to G$, $s_g(x) = gx^{-1}g$ is called *a symmetry*. A subset $A \subseteq G$ is called *symmetric* if $s_g(A) = A$ for some element $g \in G$.

Let G be an Abelian group of cardinality $\gamma > \aleph_0$. Suppose that the subgroup $\{g \in G : 2g = 0\}$ is of cardinality $< \gamma$. By [1, Theorem 3], there exists a coloring $\xi : G \to \{0, 1\}$ such that every ξ-monochrome symmetric subset of G is of cardinality $< \gamma$.

We prove the following non-abelian version of the above statement.

Theorem. *Let G be a group of cardinality $\gamma > \aleph_0$. Suppose that there exists a cardinal $\delta < \gamma$ such that*

$$card\{g \in G : g^2 = a\} \le \delta$$

for every element $a \in G$. Then there exists a coloring $\xi : G \to \{0, 1\}$ without ξ-monochrome symmetric subsets of cardinality γ.

We shall extract the proof from three Lemmas.

Lemma 1. *Let G be a group and let $\{G_\alpha : \alpha < \gamma\}$ be a family of its subgroups with the following properties*

(i) $G = \cup\{G_\alpha : \alpha < \gamma\}$,

(ii) $G_\alpha \subset G_{\alpha+1}$ for every ordinal $\alpha < \gamma$,

(iii) $G_\beta = \cup\{G_\alpha : \alpha < \beta\}$ for every limit ordinal $\beta < \gamma$,

(iv) $g^2 \notin G_\alpha$ for every ordinal $\alpha < \gamma$ and for every element $g \notin G_\alpha$.

Then there exists a coloring $\xi : G \to \{0, 1\}$ such that

$$\xi(g) \ne \xi(ag^{-1}b)$$

whenever $a, b \in G_\alpha$, $g \notin G_\alpha$, $\alpha < \gamma$.

Proof. Fix any ordinal $\alpha < \gamma$ and take any double coset $K = G_\alpha \times G_\alpha$ of $G_{\alpha+1}$ by G_α, $K \ne G_\alpha$. By (iv), $K^{-1} \ne K$. Hence, we can partition $G_{\alpha+1} \setminus G_\alpha$ into the pairs K^+, K^- of double cosets such that

$$(K^+)^{-1} = K^-.$$

Denote by G_α^+ and G_α^- the unions of the positive and negative double cosets respectively. Put

$$G^+ = \cup\{G_\alpha^+ : \alpha < \gamma\},\ G^- = \cup\{G_\alpha^- : \alpha < \gamma\}.$$

By (i), (ii), (iii), for every element $x \in G \setminus G_0$, there exists an ordinal $\alpha < \gamma$ such that $x \in G_{\alpha+1} \setminus G_\alpha$. Thus $G = G_0 \cup G^+ \cup G^-$. Define the coloring ξ as follows

$$\xi(x) = \begin{cases} 1, & if\ x \in G_0 \cup G^+, \\ 0, & if\ x \in G^- \end{cases}$$

Lemma 2. *Let G be a group, $X \subseteq G$ and let δ be a cardinal such that*

$$card\{g \in G : g^2 = a\} \le \delta$$

for every element $a \in G$. Then there exists a subgroup S of G such that

$$cardS \leq \max\{\xi_0, \delta, cardX\}$$

and $g^2 \notin S$ for every element $g \notin S$.

Proof. Denote by S_0 the subgroup generated by X. Suppose that we have chosen the subgroups $S_0, \ldots, S_n$. Denote by S_{n+1} the subgroup generated by the subset $S_n \cup \{g \in G : g^2 \in S_n\}$. Put $S = \cup\{S_n : n < \omega\}$.

Lemma 3. *Let G be a group of cardinality $\gamma > \aleph_0$. Suppose that there exists a cardinal $\delta < \gamma$ such that*

$$cardg \in G : g^2 = a\} \leq \delta$$

for every element $a \in G$. Then there exists a family $\{G_\alpha : \alpha < \gamma\}$ of subgroups satisfying Lemma 1 and having the additional property

(v) $cardG_\alpha < \gamma$ for every ordinal $\alpha < \gamma$.

Proof. Fix a minimal well-ordering $\{g_\alpha : \alpha < \gamma\}$ of G. Put $X = \{g_0\}$ and use Lemma 2 to find a subgroup G_0 such that card$G_0 = \max\{\aleph_0, \delta\}$ and $g^2 \in G_0$ implies $g \in G_0$. Suppose that we have constructed the family $\{G_\alpha : \alpha < \beta\}$ for some ordinal $\beta < \gamma$. If β is a limit ordinal put $G_\beta = \cup\{G_\alpha : \alpha < \beta\}$. If $\beta = \alpha + 1$ take a minimal element $g_\mu \notin G_\alpha$ and put $X = G_\alpha \cup \{g_\mu\}$. By Lemma 2, there exists a subgroup $G_{\alpha+1}$ such that $X \subseteq G_{\alpha+1}$ and cardG_α=card$G_{\alpha+1}$.

Proof of Theorem. By Lemma 3, there exists a family $\{G_\alpha : \alpha < \gamma\}$ of subgroups of G with properties (i)-(v). Apply Lemma 1 to point out the desired coloring $\xi : G \to \{0, 1\}$.

Remark 1. The above arguments work also in some countable cases. Let G be a countable locally finite group without an element of order 2. Choose an increasing sequence of subgroups $\{G_n : n < \omega\}$ with $G = \cup\{G_n : n < \omega\}$. By Lemma 1, there exists a coloring $\xi : G \to \{0, 1\}$ with only finite monochrome symmetric subsets.

Remark 2. Suppose that a group G satisfies the assumptions of Theorem. Let $\{G_\alpha : \alpha < \gamma\}$ be a family of subgroups of G with properties (i)-(v). Consider the coloring $\xi : G \to \{0, 1\}$ given by Lemma 1. Put $D_0 = \xi^{-1}(0)$, $D_1 = \xi^{-1}(1)$. Then the subsets D_0, D_1 are dense in every topology on G such that the mappings $x \mapsto x^{-1}$, $x \mapsto gx$, $g \in G$ are continuous and every nonempty open subset of G has cardinality γ.

Abstract. A subset A of group G is called symmetric if $A = gA^{-1}g$ for some element $g \in G$. For a wide class of groups, we construct the colorings $\xi : G \to \{0, 1\}$ such that every ξ-monochrome symmetric subset of G is of cardinality $\leq cardG$

References

[1] I.V.Protasov, *Monochromatic symmetric subsets in the colorings of Abelian groups,* Dokl. NAN Ukr., 1999, №11, 54–57.

Киевский Национальный университет
им. Тараса Шевченко

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