[illegible]

On non-trivial factorizations of one-generated ω-saturated formations

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All groups considered in this paper are finite.

Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and such that each group G has the smallest normal subgroup (denoted by $G^{\mathfrak{F}}$) with quotient in $\mathfrak{F}$. The product $\mathfrak{M}\mathfrak{H}$ of the formations $\mathfrak{M}$ and $\mathfrak{H}$ is the class $(G|G^{\mathfrak{H}} \in \mathfrak{M})$.

Let $p \in \omega \subseteq \mathbb{P}$. A formation $\mathfrak{F}$ is called ω-saturated if $\mathfrak{F}$ contains every group G with $G/(O_{\omega}(G) \cap \Phi(G)) \in \mathfrak{F}$. A formation $\mathfrak{F}$ is called $\mathfrak{N}_p$-saturated [2] if $\mathfrak{F}$ contains every group G with $G/\phi(O_p(G)) \in \mathfrak{F}$

The intersection of all ω-saturated formations which contain some fixed group G is called a one-generated ω-saturated formation.

A non-trivial factorization of a formation $\mathfrak{F}$ [1] is a product $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \ldots \mathfrak{F}_t$, $t \geq 2$, where $\mathfrak{F}_i \neq (1)$ for all $i = 1, 2, \ldots, t$. In this note answering Question 19 from [1] we give the description of non-trivial factorizations of one-generated ω-saturated formations.

Lemma 1. *Let $\mathfrak{M}, \mathfrak{H}$ be non-empty formations such that $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ for some one-generated saturated formation $\mathfrak{F}$. Assume that $\mathfrak{M} \neq (1)$. Then every simple group $A \in \mathfrak{M}$ is abelian.*

Lemma 2. *Suppose that $\mathfrak{M}\mathfrak{H}$ is a $\mathfrak{N}_q$-saturated formation where $q \in \omega$ such that $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ for some one-generated ω-saturated formation $\mathfrak{F}$. Suppose that for some prime p we have $\mathfrak{N}_p \subseteq \mathrm{S}(\mathfrak{H})$. And let $\mathfrak{M} \neq (1)$. Then $|A| = p$ for each simple group A in $\mathfrak{M}$.*

Lemma 3. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be a product of formations $\mathfrak{M}$ and $\mathfrak{H}$. Suppose that each simple group in $\mathfrak{M}$ is abelian. Suppose that there are a group $A \in \mathfrak{M}$ and a natural number m such that for all groups $B \in \mathfrak{H}$ with $|B| \geq m$ the $\mathfrak{H}$-residual of the wreath product $T = A \wr B$ is not contained subdirectly in the base group of T. Then there is a group Z_p of prime order p such that $Z_p \in \mathfrak{M} \cap \mathfrak{H}$ and $\mathfrak{N}_p \subseteq \mathrm{S}(\mathfrak{H})$.*

Lemma 4. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ where every simple group in $\mathfrak{M}$ has a prime order p. Then $D = A^{\mathfrak{H}} \wr (A/A^{\mathfrak{H}}) \in \mathfrak{F}$ for all groups $A \in \mathfrak{F}$.*

Lemma 5. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$. And let $\mathfrak{N}_p\mathfrak{H} = \mathfrak{H}$ for some prime p. If for every simple group $A \in \mathfrak{M}$ we have $|A| = p$, then $\mathfrak{F} = \mathfrak{H}$.*

Lemma 6. *Let $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ where $\mathfrak{F}$ is a one-generated ω-saturated formation, $\mathfrak{M}$ is a ω-saturated formation. Let m be the minimal ω-local satellite of $\mathfrak{M}$. Suppose that there are a prime $p \in \omega$ and a simple group A such that $A \in m(p)$ and $|A| \neq p$. Then $A \notin \mathfrak{H}$ and the formation $\mathfrak{H}$ is abelian.*

Lemma 7. *Let $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ where $\mathfrak{F}$ is a one-generated ω-saturated formation. Suppose that there is a simple group $A \in \mathfrak{M}$ with prime order such that $|A| \notin \omega$. Then $\mathfrak{H}$ is abelian.*

Lemma 8. *Suppose that $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ where for every simple group $A \in \mathfrak{M}$ we have $|A| = p$. Suppose that there is a group $B \in \mathfrak{M}$ such that for all groups $A \in \mathfrak{H}$ the $\mathfrak{H}$-residual of the group $T = B \wr A$ is not contained subdirectly in the base group of the wreath product T. Then $\mathfrak{H} = \mathfrak{N}_p\mathfrak{H}$.*

Lemma 9. *Let $\mathfrak{M}\mathfrak{H} \subseteq \mathfrak{F}$ where $\mathfrak{F}$ is a one-generated ω-saturated formation, $\mathfrak{M}$ and $\mathfrak{H}$ are non-identity formations. Suppose that $\mathfrak{M}\mathfrak{H}$ is either a $\mathfrak{N}_q$-saturated formation where $q \in \omega$. If $\mathfrak{F} \neq \mathfrak{M}\mathfrak{H}$, then $\mathfrak{M}$ is an q-saturated formation such that $\mathfrak{M} \subseteq \mathfrak{N}_q\mathfrak{N}_{q'}$.*

Lemma 10. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be the product of non-identity formations $\mathfrak{M}$ and $\mathfrak{H} \neq \mathfrak{F}$. Suppose that $\mathfrak{M} \not\subseteq \mathfrak{N}_p$. Then $\mathfrak{F}$ is a one-generated ω-saturated formation if and only if the following conditions are true:*

(a) $\mathfrak{M}$ is a one-generated ω-saturated subformation in $\mathfrak{N}_p\mathfrak{N}_{p'}$;

(b) $\mathfrak{H}$ is an abelian one-generated formation and $\pi(\mathfrak{H}) \cap \omega \subseteq \pi(\mathfrak{M})$; (c) for all groups $A \in \mathfrak{M}$ and $B \in \mathfrak{H}$ we have $(|A/F_\omega(A)|, |B|) = 1$, $(|A/O_\omega(A)|, |B|) = 1$.

Lemma 11. *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be the product of non-identity formations $\mathfrak{M}$ and $\mathfrak{H} \neq \mathfrak{F}$. Suppose $\mathfrak{M} \subseteq \mathfrak{N}_\omega$. Then $\mathfrak{F}$ is a one-generated ω-saturated formation if and only if $|\pi(\mathfrak{M})| < \infty$, $\pi(\mathfrak{H}) \cap \omega \subseteq \pi(\mathfrak{M})$, $\mathfrak{H}(\omega')$ is a one-generated formation and either $|\pi(\mathfrak{M})| > 1$ and $\mathfrak{H}$ is a one-generated formation or $\pi(\mathfrak{M}) = \{p\}$ for some prime p and $\mathfrak{H}(p)$ is a one-generated formation.*

We use $F_\omega(G)$ to denote the intersection $\cap_{p\in\omega} O_{p',p}(G)$

Theorem 1. *The product*

$$\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_t$$

is a non-trivial factorization of some one-generated ω-saturated formation $\mathfrak{F}$ if and only if $\mathfrak{F}_i \neq (1)$ for all $i = 1, 2, \dots, t$ and one of the following statements is true:

(1) there exist an index i, a prime $p \in \omega$ and a one-generated formation $\mathfrak{H}$ such that

$$\mathfrak{F} = \mathfrak{F}_i \dots \mathfrak{F}_t = \mathfrak{N}_p\mathfrak{H}$$

$\pi(\mathfrak{F}) \cap \omega = \{p\}$ and if $i > 1$, there $|A| = p$ for all groups A in $\mathfrak{F}_1 \dots \mathfrak{F}_{i-1}$;

(2) there exist an index $i < t$ and a prime $p \in \omega$ such that $\mathfrak{F}_1 \dots \mathfrak{F}_i = \mathfrak{N}_p$ and if $\mathfrak{H} = \mathfrak{F}_{i+1} \dots \mathfrak{F}_t$, then $\pi(\mathfrak{H}) \cap \omega \subseteq \{p\}$ and the formations $\mathfrak{H}(p)$ and $\mathfrak{H}(p')$ are one-generated;

(3) $t = 2, \mathfrak{F}_1 \subseteq \mathfrak{N}_\omega, \mathfrak{F}_2$ is a one-generated formation and $1 < |\pi(\mathfrak{F}_1)| < \infty$;

(4) $t = 2$, $\mathfrak{F}_1$ is a one-generated ω-local formation in $\mathfrak{N}_\omega\mathfrak{N} \setminus \mathfrak{N}_o mega$; $\mathfrak{F}_2$ is an abelian one-generated formation such that $\pi(\mathfrak{H}) \cap \omega \subseteq \pi(\mathfrak{F}_1)$ and for all groups $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$ it is true that $(|A/F_\omega(A)|, |B| = 1, |(A/O_\omega(A)|, |B| = 1$;

(5) $t = 3, \mathfrak{F}_1 \subseteq \mathfrak{N}_\omega$, $1 < |\pi(\mathfrak{F}_1)| < \infty$, $\mathfrak{F}_3$ is a one-generated abelian formation and for every $p \in \pi(\mathfrak{F}_1)$ the formation $\mathfrak{F}_2(p)$ is a one-generated nipotent formation and for all groups $A \in \mathfrak{F}_2$ and $B \in \mathfrak{F}_3$ it is true that $\pi(A/O_p(A)) \cap \pi(B) = \varnothing$.

Резюме. Нетривиальная факторизация формации $\mathfrak{F}$ [1]— это произведение $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2 \dots \mathfrak{F}_t$, $t \geq 2$, где $\mathfrak{F}_i \neq (1)$ для всех $i = 1, 2, \dots, t$. В этой заметке, отвечая на вопрос 19 из [1], мы даём описание нетривиальных факторизаций однопорожденных ω-насыщенных формаций.

References

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