Известия Гомельского государственного университета имени
Ф.Скорины, 2001, Вопросы алгебры — 17, 2001
2000 Mathematics Subject Classification: 20D10

On s-normal subgroups of finite groups

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All groups considered are finite. The maximal subgroups play an important role in the study of finite groups. And first of all it is connected with the fact that the most interesting classes of finite groups may be characterized in the terms of maximal subgroups. In particular, we know that a group G is nilpotent if and only if every its maximal subgroup is normal in G; a group G is supersoluble if and only if every maximal subgroup of G has prime index in G. A number of interesting characterizations of soluble groups are obtained by using the concept of normal index of maximal subgroups. We remind that a normal index $\eta|G : M|$ of a maximal subgroup M of a group G is the order of a chief factor H/K of G such that $HM = G$ and $K \subseteq M$. It was proved by Deskins in [1] a group G is soluble if and only if $|G : M| = \eta|G : M|$ for all maximal subgroups M. In paper [2] Yanming introduced the following concept: A subgroup H of a group G is called c-normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \subseteq H_G$. Using this concept Wang obtained [2] a number of new interesting characterizations of soluble and π-soluble groups. He also proved [2] that a maximal subgroup M of a group G is c-normal in G if and only if $|G : M| = \eta|G : M|$.

We note that the concept of a c-normal subgroup generalizes the concept of normal subgroup. Professor Skiba suggested us the following concept generalizing the concept of a subnormal subgroup.

Definition. *Let H be a subgroup of a group G. We say that H is s-normal in G if there exists a subnormal subgroup T of G such that $TH = G$ and $T \cap H \subseteq H_{sG}$ where H_{sG} is the largest subnormal subgroup of G contained in H.*

It is clear that every c-normal subgroup of G is a s-normal subgroup of G, but in general the converse is not true. For example, let $G = Z_p \wr Z_q$ where Z_p and Z_q are groups of prime orders p and q respectively. Then, evidently, every subgroup T of G such that $T \cong Z_p$ is s-normal, but not c-normal in G.

In this note using the concept of s-normal subgroup we give some new characterizations of soluble groups.

Lemma 1. *Let H be a subgroup of a group G. Then the following statements are true:*
(1) If H is s-normal in G and $H \leq K \leq G$, then H is s-normal in K;
(2) If H is s-normal in G and $K \lhd G$, then HK/K is s-normal in G/K;
(3) Let $K \leq H$ where $K \lhd G$. If H/K is s-normal in G/K, then H is s-normal in G.

Theorem 1. *Let N be a normal subgroup of a group G. Then N is p-soluble if every maximal subgroup M of G such that $N \not\subseteq M$ and $p \nmid |G : M|$ is s-normal in G.*

Theorem 2. *Let N be a normal subgroup of a group G. Then N is soluble if and only if every maximal subgroup of G not containing N is s-normal in G.*

In the case when $N = G$ we obtain from this theorem

Theorem 3. *A group G is soluble if and only if every maximal subgroup of G is s-normal in G.*

Since every c-normal subgroup of G is s-normal in G we have also from Theorem 3.2.

Theorem 4. *Let N be a normal subgroup of a group G. Then N is soluble if and only if every maximal subgroup of G not containing N is c-normal in G.*

Corollary 1. [2]. *A group G is soluble if and if every maximal subgroup of G is c-normal in G.*

Corollary 2. *Let N be a normal subgroup of a group G. Then N is soluble if and only if $|G:M| = \eta|G:M|$ every maximal subgroup of G such that $N \not\leq M$.*

Corollary 3. [1]. *A group G is soluble if and only if $|G:M| = \eta|G:M|$ for all maximal subgroups M of G.*

Theorem 5. *A group G is π-soluble if and only if G has a maximal s-normal π-soluble subgroup.*

Corollary 4. *A group G is soluble if and only if G has a maximal s-normal soluble subgroup.*

Corollary 5 [2]. *A group is soluble if and only if G has a maximal c-normal soluble subgroup.*

Corollary 6 [4]. *Let π be a set of primes. If a group G has a π-soluble maximal subgroup M such that $\eta(G:M) = |G:M|$, then G is π-soluble.*

Резюме. Пусть H — подгруппа конечной группы G. Мы используем символ $H_{..G}$, обозначая наибольшую субнормальную в G подгруппу, содержащуюся в H. Подгруппа H называется s-нормальной в G, если существует субнормальная в G подгруппа T такая, что $HT = G$ и $H \cap T \subseteq H_{..G}$. Используя этот понятие, мы получаем новые характеризации для разрешимых и π-разрешимых конечных групп.

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Received 26.05.2001

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