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On minimal τ-closed ω-local non-nilpotent formations

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

All groups considered are finite. Let Θ be some set of formations. Formations belonging to Θ are called Θ-formations. Recall [1] that a Θ-formation $\mathfrak{F}$ is called a minimal non-nilpotent Θ-formation if the formation $\mathfrak{F}$ is non-nilpotent but all proper Θ-subformations of $\mathfrak{F}$ are nilpotent. In paper [2] the minimal non-nilpotent local formations were described. Formations of that type turned out to be precisely local formations generated by a group G, where G is either a simple non-abelian group or a Schmidt group. In [3, 4] analogs of that result in the classes of hereditary and normally hereditary formations have been obtained. These results have been applied for the research of local formations with the given subformation systems (see for more details Chapter 4 in [5] or [6]).

During the study of ω-local formations more useful were minimal ω-local non-nilpotent formations, i.e. ω-local non-nilpotent formations in which all of their proper ω-local subformations are nilpotent, their description was found in [7, 8].

In this paper we describe the minimal τ-closed ω-local non-nilpotent formations, corollaries of which are all the results mentioned above.

We use standard terminology [9, 10]. In addition we shall need some definitions and notations from the work of L.A.Shemetkov and A.N. Skiba [11] and the concept of τ-closed functor given by A.N.Skiba [6].

Recall that a Skiba subgroup functor τ associates with every group G a system of its subgroups $\tau(G)$, such that the following conditions are satisfied:

1) $G \in \tau(G)$ for any group G;

2) for any epimorphism $\varphi : A \to B$ and for any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$. Everywhere further τ is a subgroup functor in the given meaning. A formation $\mathfrak{F}$ is called τ-closed [6] if $\tau(G) \subseteq \mathfrak{F}$ for any group $G \in \mathfrak{F}$.

It is convenient to denote the class of all τ-closed formations by the symbol τ.

Let ω be an arbitrary non-empty set of primes. Every function of the kind

$$f : \omega \cup \{\omega'\} \to \{\text{group formations}\}$$

is called a ω-local satellite. A satellite f is called τ-valued if all values of f belong to τ. Let $G_{\omega d}$ denote the largest normal subgroup N in G such that $\omega \cap \pi(H/K) \neq \varnothing$ for every composition factor H/K from $N(G_{\omega d} = 1$, if $\omega \cap \pi(Soc(G)) = \varnothing)$. For an arbitrary satellite f the symbol $LF_\omega(f)$ denotes the class $\{G \mid G/G_{\omega d} \in f(\omega')$ and $G/F_p(G) \in f(p)$ for all $p \in \omega \cap \pi(G)\}$. If the formation $\mathfrak{F}$ is such that $\mathfrak{F} = LF_\omega(f)$, then it is said that it is ω-local, and f is a ω-local satellite of that formation.

Let $\mathfrak{X}$ be an arbitrary set of groups, p a prime, then

$$\mathfrak{X}(F_p) = \begin{cases} \mathrm{form}(G/F_p(G) \mid G \in \mathfrak{X}), & \text{if } p \in \pi(\mathfrak{X}), \\ \varnothing, & \text{if } p \notin \pi(\mathfrak{X}). \end{cases}$$

The symbol $\tau\,\mathrm{form}\,\mathfrak{X}$ denotes the intersection of all τ-closed formations containing the set of groups $\mathfrak{X}$. Following [11] we denote by $\tau^\omega\,\mathrm{form}\,\mathfrak{X}$ the intersection of all τ-closed ω-local formations containing the set of groups $\mathfrak{X}$.

The minimal τ-valued ω-local satellite of the formation $\mathfrak{F}$ is a satellite f with the following values:

$$f(\omega') = \tau\,\mathrm{form}(G/G_{\omega d} \mid G \in \mathfrak{F})$$

and

$$f(p) = \tau\,\mathrm{form}(\mathfrak{F}(F_p))$$

for all $p \in \omega$. A satellite H of the formation $\mathfrak{H}$ is called canonical if $H(\omega') = \mathfrak{H}$ and $H(p) = \mathfrak{N}_p\mathfrak{H}(F_p)$ for all $p \in \omega$.

Theorem. *A formation $\mathfrak{F}$ is a minimal τ-closed ω-local non-nilpotent formation if and only if $\mathfrak{F} = \tau^\omega\,\mathrm{form}\,G$, where G is a monolithic group with a monolith $P = G^{\mathfrak{N}}$ and either $\pi = \pi(P) \cap \omega = \varnothing$ and all proper τ-subgroups of G are nilpotent or $\pi \neq \varnothing$ and one of the following conditions is fulfilled:*

1) $|\pi| \geq 2$, $G = P$ and the group G does not have non-trivial τ-subgroups,

2) $\pi = \{p\}$ is a singleton, and either G is a Schmidt group or $P = G^{\mathfrak{N}_p}$ is a non-abelian group and all proper τ-subgroups of G are p-groups.

Proof. Necessity. The formation $\mathfrak{N}$ of nilpotent groups has a ω-local satellite H, such that

$$H(a) = \begin{cases} \mathfrak{N}_p, & \text{if } a = p \in \omega \\ \mathfrak{N}, & \text{if } a = \omega'. \end{cases}$$

Hence, since $\mathfrak{F}$ is a minimal τ-closed ω-local non-nilpotent formation, it follows by the theorem of work [12] that $\mathfrak{F} = \tau^\omega\,\mathrm{form}\,G$ where G is a monolithic group with a monolith $P = G^{\mathfrak{N}}$ such that either $\pi = \pi(P) \cap \omega \neq \varnothing$, $\Phi(G) = 1$ and $f(p)$ is a minimal τ-closed non-nilpotent-formotion for all $p \in \pi$, or $\pi = \varnothing$ and $f(\omega')$ is a minimal τ-closed non-$\mathfrak{N}$-formation, where f is the minimal τ-valued ω-local satellite of the formation $\mathfrak{F}$. Since $P = G^{\mathfrak{N}}$, we have $\Phi(G) = 1$.

Let us consider the case $\pi \neq \varnothing$. Let P be a non-abelian group. Then according to Lemma 5 [11] the formation $\mathfrak{F}$ has a minimal ω-local τ-valued satellite f such that $f(p) = \tau\,\mathrm{form}(G/F_p(G))$ for every $p \in \pi(G) \cap \omega$. Since $F_p(G) = 1$ for every prime $p \in \pi$, for all mentioned p we have $f(p) = \tau\,\mathrm{form}\,G$. Hence the formation $\tau\,\mathrm{form}\,G$ is a minimal τ-closed non-$\mathfrak{N}_p$-formation for any prime $p \in \pi$. Since $\Phi(G) = 1$, according to Lemma 2.1.5 [6] $\tau\,\mathrm{form}((G/P) \cup \mathfrak{X})$ is the maximal τ-closed subformation of the formation $\tau\,\mathrm{form}\,G$, where $\mathfrak{X}$ is the set of all proper τ-subgroups of the group G.

Since P is a non-abelian group, $|\,\pi(P)\,| \geq 2$. Let $\{p, q\} \in \pi$ and $p \neq q$. Then $(\{G/P\} \cup \mathfrak{X}) \in \mathfrak{N}_p \cap \mathfrak{N}_q = (1)$. Thus $G = P$ and it does not have non-trivial τ-subgroups.

Now we suppose that $\pi = \{p\}$. Then $((G/P) \cup \mathfrak{X}) \in \mathfrak{N}_p$. It is obvious that $G \notin \mathfrak{N}_p$. Hence the group G satisfies Condition 2).

Now let P be an abelian p-group. Then

$$f(p) = \tau\,\mathrm{form}(G/F_p(G)) = \tau\,\mathrm{form}(G/P) = \tau\,\mathrm{form}\,H,$$

where H is a subgroup of the group G such that $G = [P]H$. It means that $\tau\,\mathrm{form}\,H$ is a minimal τ-closed non-$\mathfrak{N}_p$-formation. Since $G/P \simeq H \in \mathfrak{N}$ and the formation $\mathfrak{N}$ is hereditary, $s\,\mathrm{form}\,H \subseteq \mathfrak{N}$. Thus, according to Theorem 2.4. [9],

$$\tau\,\mathrm{form}\,H = \mathrm{form}\,H = s\,\mathrm{form}\,H.$$

Let M be a group of minimal order from $s\,\mathrm{form}\,H \setminus \mathfrak{N}_p$. If $s\,\mathrm{form}\,M \subset s\,\mathrm{form}\,H$, then $s\,\mathrm{form}\,M \subseteq \mathfrak{N}_p$. A contradiction. Therefore $s\,\mathrm{form}\,M = s\,\mathrm{form}\,H$. Let us consider that group M. According to the choice of the group M, it is a minimal non-$\mathfrak{N}_p$-group. Thus all its Sylow subgroups are p-groups. It means that M is a p-group. A contradiction. Therefore M is a group of prime order q, where $q \neq p$. Thus the formation $s\,\mathrm{form}\, M = s\,\mathrm{form}\, Z_p$ is a hereditaty formation generated by the group of prime order q. Since $H \in s\,\mathrm{form}\, Z_q$, H is a group of exponent q. Since $G = [P]H$ and $P = C_G(P)$, H is a irreducible abelian group of automorphisms for P. Therefore H is a cyclic group. But the order and the exponent of the cyclic group H coincide. Thus we have $|H| = q$. So, the group G satisfies Condition 2).

Let now P be a ω'-group. Then $G_{\omega d} = 1$ and

$$f(\omega') = \tau\,\mathrm{form}(G/G_{\omega d}) = \tau\,\mathrm{form}\, G.$$

According to Lemma 2.1.5 [6], $\tau\,\mathrm{form}(\mathfrak{X} \cup (G/P))$ is a maximal τ-closed subformation of the formation $\tau\,\mathrm{form}\, G$. Therefore $\tau\,\mathrm{form}(\mathfrak{X} \cup (G/P)) \subseteq \mathfrak{N}$. Thus all proper τ-subgroups of the group G are nilpotent and $G^{\mathfrak{N}} = P$.

Sufficiency. Since $P = G^{\mathfrak{N}}$, we have $\Phi(G) = 1$. Let condition 1) be satisfied. According to Lemma 5 [11], the minimal τ-valued ω-local satellite f of the formation $\mathfrak{F}$ is such that

$$f(p) = \tau\,\mathrm{form}(G/F_p(G)) = \tau\,\mathrm{form}(G),$$

for every $p \in \pi$. According to Lemma 2.1.5 [6], $\tau\,\mathrm{form}(\mathfrak{X} \cup (G/P)) \subseteq \mathfrak{N}_p$ for erery prime $p \in \pi$. Therefore according to Theorem in [12] $\mathfrak{F} = \tau^{\omega}\,\mathrm{form}\, G$ is a minimal τ-closed ω-local non-nilpotent formation.

Let condition 2) be satisfied and P be an abelian p-group. Then G is a Schmidt group. From a well-known description of Schmidt groups it follows that $G = [P]H$, where $P = G_G(P)$ is a minimal normal p-subgroup and $\mid H \mid = q$, where q is a prime. It means that H is a minimal non-$\mathfrak{N}_p$-group with a monolith $Q = H$. In this case $\Phi(H) = 1$. Then $f(p) = \tau\,\mathrm{form}\, H$. Thus, according to Lemma 2.1.5 [6], it is easy to see that $f(p) = \tau\,\mathrm{form}\, H$ is a minimal τ-closed non-$(H(p))$-formation. Therefore, according to theorem in [12] $\mathfrak{F} = \tau^{\omega}\,\mathrm{form}\, G$ is a minimal τ-closed ω-local non-nilpotent formation. The rest of the cases are studied similarly. The theorem is proved.

Резюме. Установлено строение минимальных τ-замкнутых ω-локальных ненильпотентных формаций конечных групп.

References

[1] L.A.Shemetkov, *Screens of gratduated formations,* Proc. VI All-Union Symposium on the theory of groups. Kiev: Nauk. Dumka, (1980), 37–50 (Russian)

[2] A.N.Skiba, *On critical formations. In the book: "Infinite groups and related algebraic structures"*, Kiev, (1993), 258–268 (Russian)

[3] A.N.Skiba, *On minimal s-closed local non-π-supersoluble formations. In the book: "Research of normal and subgroup structure of finite groups"*, Minsk: Nauka i tehnika, (1984), 53–58 (Russian)

[4] V.M.Sel'kin, *On minimal local normally hereditary non-supersoluble formations,* Voprosy Algebry (Problems in Algebra), V. 13 (1998), 172–176 (Russian)

[5] L.A.Shemetkov, A.N.Skiba, *Formations of Algebraic Systems,* Nauka, Moscow, 1989 (Russian)

[6] A.N.Skiba, *Algebra of Formations,* Belaruskaya Nauka, Minsk, 1997 (Russian)

[7] Jaraden Jehad J., *On formations with systems of hereditary subformations,* Izvestiya Vuzow. Matematika, V. 10 (1996), 39–44 (Russian).

[8] I.N.Safonova, *On minimal ω-local non-$\mathfrak{L}$-formations,* Vesti NAN Belarusi, V. 2 (1999), 23–27 (Russian).

[9] L.A.Shemetkov, *Formations of Finite Groups,* Nauka, Moscow, 1978 (Russian).

[10] K.Doerk, D.Hawkes, *Finite Soluble Groups,* Walter de Gruyter, Berlin–New York, 1992.

[11] L.A.Shemetkov, A.N.Skiba, *Multiply ω-local formations and Fitting classes of finite groups,* Siberian Advances in Mathematics, 10:2 (2000), 1–30.

[12] V.M.Sel'kin, A.N.Skiba, *On $\mathfrak{H}_{\Theta^\omega}$-critical formations,* Voprosy Algebry (Problems in Algebra), V. 14 (1999), 127–131, (Russion).

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