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A description of finite groups with $\mathfrak{F}$-abnormal or $\mathfrak{F}$-subnormal subgroups

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

In the paper we consider only finite groups. Let $\mathfrak{F}$ be a non-empty hereditary formation. A subgroup H of a group G is said to be:

1) $\mathfrak{F}$-subnormal if there exists a maximal chain of subgroups

$$G = H_0 \supset H_1 \supset \ldots \supset H_n = H$$

such that for each $i \geq 1$, H_i is $\mathfrak{F}$-normal in H_{i-1};

2) $\mathfrak{F}$-abnormal if in any maximal chain of subgroups

$$G = H_0 \supset H_1 \supset \ldots \supset H_n = H$$

H_i is $\mathfrak{F}$-abnormal in H_{i-1} for each $i \geq 1$.

In the paper we obtain the description of groups in which any proper subgroup is either $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal where $\mathfrak{F}$ is an arbitrary soluble superradical formation. As a consequence of the obtained theorem, in the case when $\mathfrak{F}$ is the formation of all nilpotent groups, all soluble p-nilpotent groups, every soluble $\check{S}$-formation, we have the results of [1–3].

We use standard notations [5, 7]. $\pi(\mathfrak{F})$ is the set of all prime divisors of groups in $\mathfrak{F}$. If $\mathfrak{F} = LF(F)$ then $\sigma(\mathfrak{F}) = \{p \in \pi(\mathfrak{F}) : F(p) \neq \mathfrak{F}\}$. $\pi'(\mathfrak{F}) = \mathbb{P} \setminus \pi(\mathfrak{F})$.

Lemma 1. *Let $\mathfrak{F}$ be a nonempty hereditary formation. Then the following assertions hold:*

1) if H is a subgroup of G and $G^{\mathfrak{F}} \subseteq H$, then H is $\mathfrak{F}$-subnormal in G;

2) if $H$is an $\mathfrak{F}$-subnormal subgroup of G and K is a subgroup of G, then $H \cap K$ is $\mathfrak{F}$-subnormal in K;

3) if H_1 and H_2 are $\mathfrak{F}$-subnormal subgroups of a group G, then $H_1 \cap H_2$ is $\mathfrak{F}$-subnormal in G;

4) if H is $\mathfrak{F}$-subnormal in K and K is $\mathfrak{F}$-subnormal in G, then H is an $\mathfrak{F}$-subnormal subgroup of the group G.

A formation $\mathfrak{F}$ is said to be superradical if it satisfies the following requirements:

1) $\mathfrak{F}$ is a normally hereditary formation;

2) $G = AB$, where A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of G, always implies $G \in \mathfrak{F}$

Lemma 2. *Let $\mathfrak{F} \neq \varnothing$ be a soluble hereditary superradical formation. Then $\mathfrak{F}$ is a local formation.*

Proof. Let $G = AB$, where A and B are normal $\mathfrak{F}$-subgroups of G. Since

$$G/A = AB/A \simeq A/A \cap B \in \mathfrak{F},$$

it follows that $G^{\mathfrak{F}} \subseteq A$ and $G^{\mathfrak{F}} \subseteq B$. Then, by lemma 1, A and B are $\mathfrak{F}$-subnormal subgroups of G. Since $\mathfrak{F}$ is a superradical formation, it follows that $G \in \mathfrak{F}$. Thus, $\mathfrak{F}$ is a radical formation. By theorem 1 of [4] $\mathfrak{F}$ is a local formation. Lemma is proved.

Theorem. *Let $\mathfrak{F} \neq \varnothing$ be a soluble hereditary superradical formation. Then any proper subgroup of a group G is either $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal if and only if G is a $\pi(\mathfrak{F})$-soluble group of one of the following types:*

1) $G \in \mathfrak{F}$;

2) $G = [G_{q'}]G_q$, *where* $G_{q'} \in \mathfrak{F}$, G_q *is an* $\mathfrak{F}$*-projector of* G, G_q *is a cyclic group,* $\pi(G^{\mathfrak{F}}) \subseteq \pi(\mathfrak{F})$, $G_{q'} \times G_q^*$ *is the unique normal maximal subgroup of the group* G, G_q^* *is a maximal subgroup in* G_q;

3) G *is a* $\pi'(\mathfrak{F})$*-group;*

4) $G = G_{p'}G_p$, *where* $p \in \pi(\mathfrak{F})$, $\pi(G_{p'}) \subseteq \pi'(\mathfrak{F})$, $|G_p| = p$, G_p *is an* $\mathfrak{F}$*-projector of the group* G, $N_G(K)$ *is a* p'*-group, where* K *is any* p'*-subgroup from* G.

Proof. Necessity. According to Lemma 2, $\mathfrak{F}$ is a local formation. We consider the following two cases.

1) $G^{\mathfrak{F}} \subset G$. Since $G^{\mathfrak{F}}$ is an $\mathfrak{F}$-subnormal subgroup, every subgroup from $G^{\mathfrak{F}}$ is $\mathfrak{F}$-subnormal in $G^{\mathfrak{F}}$. By Lemma 1, any subgroup from $G^{\mathfrak{F}}$ is $\mathfrak{F}$-subnormal in G. Since $\mathfrak{F}$ is a local formation, we have $G^{\mathfrak{F}} \in \mathfrak{F}$. In view of the fact that $\mathfrak{F}$ is a soluble formation and $G/G^{\mathfrak{F}} \in \mathfrak{F}$, we obtain that G is a soluble group.

Let H be an $\mathfrak{F}$-projector of the group G. Let H_1 and H_2 be non-conjugate maximal subgroups of H. Then

$$G = G^{\mathfrak{F}}H_1G^{\mathfrak{F}}H_2.$$

Obviously, $G^{\mathfrak{F}}H_1$ and $G^{\mathfrak{F}}H_2$ are $\mathfrak{F}$-subnormal subgroups of the group G. Since $\mathfrak{F}$ is a local formation and any subgroup in $G^{\mathfrak{F}}H_i$ $(i = 1, 2)$ is a $\mathfrak{F}$-subnormal subgroup of the group G, by Lemma 1 in $G^{\mathfrak{F}}H$, it follows that $G^{\mathfrak{F}}H_i \in \mathfrak{F}$, $i = 1, 2$. Since $\mathfrak{F}$ is a superradical formation, we have $G \in \mathfrak{F}$, a contradiction. Thus, H is a cyclic q-group. Obviously, G_q is an $\mathfrak{F}$-abnormal subgroup of the group G and $G_q \in \mathfrak{F}$. By Theorem 15.1 of [5], G_q is an $\mathfrak{F}$-projector of the group G. According to Theorem 15.3 of [5], we have $H = G_q$.

By induction on the order of the group one can show that $|G^{\mathfrak{F}}|$ is not divisible by q.

Let N be a minimal normal subgroup of the group G and $N \subseteq G^{\mathfrak{F}}$. Let N be a p-group $(p \neq q)$. If $G/N \in \mathfrak{F}$, then $N = G^{\mathfrak{F}}$. Assume now that $G/N \notin \mathfrak{F}$. By induction, $|G^{\mathfrak{F}}/N|$ is not divisible by q. But then $|G^{\mathfrak{F}}|$ is not divisible by q. Assume now that N is a q-group. If $\Phi(G) \neq 1$, then, by induction, $|G^{\mathfrak{F}}/G^{\mathfrak{F}} \cap \Phi(G)|$ is not divisible by q. If $G^{\mathfrak{F}}$ is divisible by q, then, by Lemma 4.4 of [5], $G^{\mathfrak{F}}Q \times K$, where Q is a Sylow q-subgroup from $G^{\mathfrak{F}}$, contained in $\Phi(G)$. Then

$$(G/K)^{\mathfrak{F}} = QK/K \subseteq \Phi(G/K).$$

Since $\mathfrak{F}$ is saturated we have $G/K \in \mathfrak{F}$. But then $G^{\mathfrak{F}} \subseteq K$, a contradiction. Thus, $\Phi(G) = 1$. But then

$$G = N \times M = NM_qM_{q'}.$$

Since $N \subseteq \Phi(G_q)$, we have $G = M$, a contradiction. Thus, $G^{\mathfrak{F}} = G_{q'}$.

We show that the maximal subgroup G_q^* from G_q is normal in G. As above, it is easy to show that $G_{q'}G_q^* \in \mathfrak{F}$. Obviously, $G_{q'}G_q^*$ is a normal subgroup in G. We consider the subgroup $G_pG_q^*$, where $p \neq q$. Obviously, G_p is normal in $G_pG_q^*$. we assume that $N_G(G_q^*) \neq G$. Since $G_pG_q^* \in \mathfrak{F}$, we have

$$G_pG_q^*/F_p(G_pG_q^*) \in f(p),$$

where f is the maximal integrated local screen of the formation $\mathfrak{F}$. From this it follows that $q \in \pi(f(p))$. Since $\mathfrak{F}$ is an superradical formation, by Theorem 1 of [6] we have

$$\mathfrak{F} = \bigcap_{p \in \pi(\mathfrak{F})} \mathfrak{S}_{p'}\mathfrak{S}_{\pi(f(p))} \bigcap \mathfrak{S}_{\pi(\mathfrak{F})}.$$

$\mathfrak{F}$ has a local screen h such that $h(p) = \mathfrak{S}_{\pi(f(p))}$. From this it is easy to show that $G_pG_q^* \in \mathfrak{F}$. Since G_q is $\mathfrak{F}$-abnormal in G, it follows that G_qG_p is $\mathfrak{F}$-abnormal in G. By Theorem 15.1

of [5], G_pG_q is a $\mathfrak{F}$-projector of the group G, a contradiction. Thus, G_q^* is normal in $G_pG_q^*$, where p is any prime number in $\pi(G)$. From this it follows that G_q^* is a normal subgroup of G.

We show that $\pi(G^{\mathfrak{F}}) \subseteq \sigma(\mathfrak{F})$. We assume the opposite. Then there exists a prime number $p \in \pi(G^{\mathfrak{F}})$ such that $f(p) = \mathfrak{F}$. We consider the subgroup G_pG_q. Obviously, G_p is normal in G_pG_q. Since $G_q \in \mathfrak{F}$ and

$$G_pG_q/F_p(G_pG_q) \in f(p), \qquad G_pG_q/F_q(G_pG_q) \in f(q),$$

it follows that $G_pG_q \in \mathfrak{F}$. As above, it is easy to show that this is impossible. We have a contradaction, i.e., $\pi(G^{\mathfrak{F}}) \subseteq \sigma(\mathfrak{F})$.

Assume now that $G^{\mathfrak{F}} = G$. We show that G is a $\pi(\mathfrak{F})$-soluble group. Let p be any prime number from $\pi(G)$ such that $p \in \pi(\mathfrak{F})$. Obviously, any proper subgroup of G is $\mathfrak{F}$-abnormal in G. In view of the fact that $G_p \in \mathfrak{F}$, we obtain $|G_p| = p$. From this we have the $\pi(\mathfrak{F})$-solubility of G.

We consider a subgroup G_p, where p is any prime number from $\pi(G) \cap \pi(\mathfrak{F})$. We have proved above that $|G_p| = p$. By Theorem 15.1 of [5] G_p is an $\mathfrak{F}$-projector of the group G. By Theorem 15.5 of [5] $\mathfrak{F}$-projectors of the group G are conjugate. This means that $G = G_{p'}G_p$, where $p \in \pi(\mathfrak{F})$, $\pi(G_{p'}) \subseteq \pi'(\mathfrak{F})$. Since all proper subgroups of the group G are $\mathfrak{F}$-abnormal in G, it follows that $N_G(K)$ is a p'-group, where K is a p'-subgroup.

Sufficiency. Let G be a group from 1). Since $\mathfrak{F}$ is a hereditary formation, it follows that every proper subgroup in G is $\mathfrak{F}$-subnormal in G.

Let G be a group from 2). Let K be a proper subgroup of the group G. If $|K|$ is not divisible by q, then $K \subseteq GF^{\mathfrak{F}}$. Since $G^{\mathfrak{F}}$ is $\mathfrak{F}$-subnormal in G, $G^{\mathfrak{F}} \in \mathfrak{F}$ and $\mathfrak{F}$ is a local formation, it follows that K is an $\mathfrak{F}$-subnormal subgroup of the group G. Assume that $|K|$ is divisible by q. If $G_q \subseteq K$, then from the fact G_q is an $\mathfrak{F}$-projector of G it follows that K is $\mathfrak{F}$-abnormal in G. Let $G_q \not\subseteq K$. Obviously, $K \subseteq G^{\mathfrak{F}} \times H^*$. Since $G^{\mathfrak{F}} \times H^*$ is $\mathfrak{F}$-subnormal in G and $G^{\mathfrak{F}} \times H^* \in \mathfrak{F}$, it follows that K is $\mathfrak{F}$-subnormal in G.

Let G be a group from 3). Obviously, every subgroup in G is $\mathfrak{F}$-abnormal in G.

Let G be a group from 4). Let K be a proper subgroup of the group G. If $|K|$ is divisible by q, then $G_q \subseteq K$. Since G_p is $\mathfrak{F}$-abnormal in G, it follows that K is $\mathfrak{F}$-abnormal in G. Assume that $|K|$ is not divisible by q. Then K is a $\pi'(\mathfrak{F})$-group. Since $N_G(K)$ is a $\pi'(\mathfrak{F})$-group, it is easy to prove that K is an $\mathfrak{F}$-abnormal subgroup of the group G. The theorem is proved.

Резюме. Пусть $\mathfrak{F}$ — разрешимая наследственная сверхрадикальная формация. Получено описание конечной группы G, у которой каждая собственная подгруппа либо $\mathfrak{F}$-субнормальна, либо $\mathfrak{F}$-абнормальна.

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