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A note on supersoluble Fitting classes

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

1. Introduction

It can be considered standard knowledge that the class of all supersoluble groups is not a Fitting class; not so clear was for some time whether there are Fitting classes of (not only nilpotent but also) supersoluble groups. Examples of Fitting classes with this property were given by Menth [3], Traustason [4] and the author [1]. The purpose of this note is to show that the classes exhibited by Menth can be used for a modification to obtain countably many classes of $\{3,p\}$-groups. The method of modification was mentioned earlier (see [2]); here we will describe this method in some more detail and prove the Fitting class property. We will first describe the supersoluble group of minimal order contained in this Fitting class, this leads to the key section corresponding to the class, and we will then show in several steps that in fact a Fitting class is defined.

2. The minimal supersoluble group contained in the class

Let F be a field of order p^q where p is a prime with $p = 3k+1$ and q is relatively prime to $3p$. The set of upper unitriangular 5×5 matrices $M = (m_{ij})$ described by
$m_{12} = m_{23} = 2a$; $m_{34} = m_{45} = 2b$,
$m_{13} = 2a^2$; $m_{24} = c$; $m_{35} = 2b^2$,
$m_{14} = d$; $m_{25} = e$; $m_{15} = f$,
with $a, b, c, d, e, f \in F$ consitutes a group of nilpotency class 4. The central elements are obtained in the case that all parameters except f vanish, we will consider the quotient group modulo the center and call it T, suppressing the dependance on F for the moment. Now T is nilpotent of class 3, further $T/T' \cong F^+ \times F^+$, $T'/T_3 \cong F^+$, and $T_3 \cong F^+ \times F^+$. The modification to Menth's example is the substitution of the prime field by F, since T constructed like this for the prime field is isomorphic to the free nilpotent class 3 exponent p group of degree 2.
The original group of unitriangular matrices can be extended by adding the diagonal matrix $diag(n, n^2, 1, n, n^2)$, where $1 + n + n^2 = 0$ and n is an element of the prime field. Taking the appropriate quotient group as before, we find a quotient group U of order $3|F|^5$. This supersoluble group U is the group for which we want to construct the smallest Fitting class containing it.

3. Forming the key section

We consider here Fitting classes of extensions of p-groups by 3-groups. For this case we reduce to such members of the Fitting class which are not nontrivial normal products with one of the factors nilpotent. This is obtained by assigning $O^p(L)/O_3(L)$ to every member L of the Fitting class; the groups found in this way constute the set $\mathfrak{S}$ of key sections of the

Fitting class $\mathfrak{F}$. In turn, the set $\mathfrak{S}$ is a key section to some Fitting class of extensions of p-groups by 3-groups, if the following is true:
(a) If $R \in \mathfrak{S}$ and N is a normal subgroup of R, then $O^p(N) \in \mathfrak{S}$,
(b) If R, S are normal subgroups of the product RS and $R, S \in \mathfrak{S}$, then $RS/O_3(RS) \in \mathfrak{S}$.

We define the set $\mathfrak{S}$ in the following way:$L \in \mathfrak{S}$ if
(i) the maximal normal p-subgroup P of L is a central product of groups isomorphic to T as defined in the previous section, and every of these central factors is normal in L,
(ii) for every of these central factors R isomorphic to T we have $L/C(R) \cong U$ with U as in the previous section,
(iii) there are no nontrivial normal 3-subgroups and no normal subgroups of index p in L.
It is obvious from the construction that the set $\mathfrak{S}$ satisfies condition (a). We will need more information for the proof that also (b) is satisfied.

4. The central products

We consider here P and T in more detail. First, we mention a fact that can be verified without too much trouble:

$$\text{If } x \in T \setminus T', \text{ then } |[x,T] \cap T_3| = |F|,$$

$$\text{If } x \in T' \setminus T_3, \text{ then } [x,T] = T_3.$$

We noted earlier that $|T_3| = |F|^2$.- Now P is a central product of isomorphic copies of T, and so $P/Z(P)$ is a direct product of isomorphic copies of $T/Z(T)$. By the famous Theorem of Krull, Remak and Schmidt (see for instance [5] p. 80), a direct product of directly irreducible factors is unique except for forming central automorphisms. Assume for instance that $R/Z(R) \cong T/Z(T)$ is not left invariant, but the generators $v_jZ(P)$ of $R/Z(R)$ are mapped onto $v_jz_jrZ(P)$, where $z_jZ(S) \neq 1$ belongs to $Z_2(S)/Z(S)$ of some other direct factor $S/Z(S)$ and r is contained in the product of the remaining factors. We know that $|[v_j,P]| = |[v_j,R]| = |F|$ but $|[z_j,P]| = |[z_j,S]| = |F|^2$ and so $|[v_jz_jr,P]| \geq |F|^2$. This shows that such a factor z_j can not occur and we have that any two decriptions of P as central products of groups isomorphic to T coincide modulo $Z(P)$. The construction has the further property that $Z(T) = Z(U)$ and so also $Z(P) = Z(L)$.

5. The normal products

We begin with two groups $L, M \in \mathfrak{S}$ with maximal normal p-subgroups P, Q and consider their normal product LM and its maximal normal p-subgroup PQ. Since P and Q are nilpotent of class 3, their product is nilpotent of class at most 6, and $p > 6$. Therefore an element $s \in Q$ can not permute p central factors of P by conjugation, since the subgroup generated by s and these factors would have nilpotency class p at least. We deduce that the conjugations by elements of Q induce automorphisms on $P/Z(P)$ that leave all direct factors invariant. By symmetry the same is true for elements $t \in P$ and $Q/Z(Q)$. Consider now one of the central factors $R \cong T$ of P. We have seen that $RZ(P)/Z(P)$ is a direct factor in every description of $P/Z(P)$ as direct product of irreducible factors. Now $RZ(P) = RZ(L)$ and therefore $[RZ(P), L] = [R, L] = R$, and P is a unique central product of L-invariant factors isomorphic to T. The analogous statement is true for M and Q.

We consider the action of Q on the quotient L/L' of some factor L, here the condition that q and $3p$ are relatively prime yields that this action is decribed by an F-linear mapping since field automorphisms have order dividing q. The same happens with automorphisms of order 3: they are also F-linear mappings.

We are now able to follow all the steps described by Menth in [3], section 3 one by one and come to the same conclusion, namely that the normal product will have the same form modulo its maximal normal 3- subgroup. This means that the class of groups described by this key section is in fact a Fitting class.

6. Subclasses

In the given case we have just one minimal non-nilpotent Fitting subclass (see [2]), since whenever we begin with some member of a Fitting subclass of one of our classes which itself has commutator subgroup of index 3, by the structure we may use Lemma 1 of [2] to obtain a group of the following form: If T and V is as in section 2 and v is an element of order 3 in V, then $V = \langle v, T\rangle$. Let K, L be two isomorphic copies of V and denote the isomorphism mapping K onto L by σ. First form the central product $M = (K \times L)/Z$ where $Z = \langle z^{-1}z^{\sigma} | z \in Z(K)\rangle$. If $k \in K$ is the image of v under the isomorphism from V to K, we reduce to the subgroup $N = \langle v^{-1}v^{\sigma}Z, M'\rangle$. This is a group which is contained in all non-nilpotent subclasses of our Fitting class, and therefore it generates the smallest non-nilpotent Fitting class connected with T.
Unlike the situation of Menth [3] we have here more cases of central products with complete covering of the two centers, substituting Z in the quotient group by some $W = \langle z^{-1}z^{\tau} | z \in Z(K)\rangle$ where τ is some isomorphism of $Z(K)$ onto $Z(L)$. We recall that there are now two isomorphisms from $Z(K)$ onto $Z(L)$: one is the mapping σ^*, the restriction of σ to $Z(K)$, the other is τ. The extension of KL as described by an element of order 3 is characterized by the quotient $\sigma^*\tau^{-1}$, it is easily seen that this quotient is independant of automorphisms of L; on the other hand, change of the reference system (by an automorphism) in K leads to conjugation by the automorphism of K restricted to $Z(K)$.

Let $\mathfrak{F}$ be a Fitting class containing a group as described before. If $\mathfrak{P}$ is the set of groups of this form in $\mathfrak{F}$, we denote by Ξ the set of quotients $\sigma^*\tau^{-1}$. We know that Ξ can be described as a set of linear mappings belonging to $GL(2q,p)$, that a certain subgroup $H \cong \Gamma L(F)$ (the set of all invertible semilinear mappings of a vectorspace of dimension 2 over F) is fixed, and that Ξ is invariant with respect to conjugation by some elements of H, namely those which are preimages of elements $\det(\rho)\rho \in \Gamma L(F)$. Also, by exchanging K and L, we see that Ξ is invariant with respect to inversion. Finally, taking two groups of the described form and "identitfying" the second factor of one with the first factor of the other and taking the element of order 3 that fixes this factor elementwise, we find out that the set Ξ is closed with respect to multiplication. This shows that Ξ is in fact a subgroup of $GL(2q,p)$, it is normalized by $\Sigma L(2,F)$, the set of semilinear mappings coming from linear mappings with determinant 1. Assume now that $rs = q$ with $r + s < q$. Then there is a subgroup $S \cong \Gamma L(2r, p^s)$ of $GL(2q,p)$ which contains H. We may choose $\Xi = S$ to obtain a Fitting class, and different divisors r of q lead to different Fitting classes. The smallest Fitting class, contained in all the other ones, is the one with $\Xi = \langle 1 \rangle$.

The (only) minimal characteristic subgroup of K is $Z(K)$. This is the reason for the following fact: If a Fitting class contains a non-nilpotent extension D of a central product of two groups isomorphic to K by a group of order 3 such that $|Z(D)| > |Z(K)|$, then the same Fitting class also contains a similar extension of a direct product of two copies of K. The same is true if there is a group A belonging to the class $\mathfrak{F}$ with sylow 3-subgroup of order 3, without normal subgroups of index p, and with a central factor B of the form described before in the maximal normal p-subgroup such that the intersection of B with the product of all the other central factors is not equal to $Z(B)$.

Резюме. Работа посвящена конструированию классов Фиттинга, составленных из конечных сверхразрешимых групп.

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