Известия Гомельского государственного университета имени
Ф.Скорины, 3(16) Вопросы алгебры (2000), 88–92
2000 Mathematics Subject Classification: 20D10

On factorizations of one-generated p-local formations

T.R.Vishnevskaya

Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

Introduction. All groups under consideration are finite. By Gaschütz [1] a formation is a class of groups which is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ of groups is said to be saturated (p-saturated) if it contains an arbitrary group G that has a normal subgroup N such that $G/N \in \mathfrak{F}$ and $N \subseteq \Phi(G)$ ($N \subseteq O_p(G) \bigcap \Phi(G)$ respectively).

Remind that a one-generated saturated (p-saturated) formation [2] is such a saturated (a p-saturated) formation $\mathfrak{F}$ such that $\mathfrak{F} = \bigcap \mathfrak{H}$ where $\mathfrak{H}$ ranges over all saturated (all p-saturated) formations containing some fixed group $G \in \mathfrak{F}$. The formations of that kind were first introduced by Gaschütz in [3] and they play a very important role in the classifications of formations (see Chapter 4 in [2] and Chapters 2–5 in [4]).

The product $\mathfrak{M}\mathfrak{H}$ of non-empty formations $\mathfrak{M}$ and $\mathfrak{H}$ is the class $(G \mid G^{\mathfrak{H}} \in \mathfrak{M})$ where $G^{\mathfrak{H}}$ is the intersection of kernels of epimorphisms of G onto groups in $\mathfrak{H}$.

It is well-known ([1], [5]), the product $\mathfrak{M}\mathfrak{H}$ of any two saturated formations $\mathfrak{M}$ and $\mathfrak{H}$ is a saturated formation. In the papers of Vedernikov [6] and Vorob'ev [7], the first examples of saturated formations of the form $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ were constructed, where both factors $\mathfrak{M}$ and $\mathfrak{H}$ are non-saturated formations. In this connection the following result of Skiba is very interesting:

Theorem 21 ([8]). *Let $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be a one-generated saturated formation where $\mathfrak{M}$ and $\mathfrak{H}$ are formations. If $\mathfrak{F} \neq \mathfrak{H}$, then $\mathfrak{M}$ is a saturated formation as well.*

In this paper we shall prove the analogoues result for p-saturated formations.

We use standard terminology [2, 9, 10] and some definitions and notations from [11].

Every function of the form

$$f : \{p, p'\} \to \{\text{group formations}\}$$

is called a p-local satellite. Following [11] we denote by G_{pd} the largest normal in G subgroup all composition factors H/K of which are pd-group (i.e. $p \mid |H/K|$). We put $G_{pd} = 1$ if G has no factors with that property.

Let f be a p-local sattelite. Then

$$LF_p(f) = (G \mid G/G_{pd} \in f(p') \text{ and } G/F_p(G) \in f(p) \text{ } if \text{ } p \mid |G|).$$

If a formation $\mathfrak{F}$ is such that $\mathfrak{F} = LF_p(f)$ for some p-local satellite f then following [11] we say that $\mathfrak{F}$ is a p-local formation and f is a p-local satellite of that formation.

Let

$$\mathfrak{X}(F_p) = \begin{cases} \text{form}(G/F_p(G) \mid G \in \mathfrak{X}) & \text{if } p \in \pi(\mathfrak{X}) \\ \varnothing & \text{if } p \notin \pi(\mathfrak{X}). \end{cases}$$

A p-local satellite f of the formation $\mathfrak{F}$ is called the minimal p-local satellite of $\mathfrak{F}$ if $f(p) = \mathfrak{F}(F_p)$ and $f(p') = \text{form}(G/G_{pd} \mid G \in \mathfrak{X})$.

The symbol $l_p \text{form}\, \mathfrak{X}$ denotes the intersection of all p-local formations containing $\mathfrak{X}$.

Lemma 1 ([12]). *A non-empty formation $\mathfrak{F}$ is p-local if and only if it is p-saturated.*

Lemma 2 ([2]). *Let $A \in \operatorname{form} G$, $m = |G|$. If H/K is a chief factor of the group A, then $|H/K| < m$.*

We use $A \wr B$ to denote the regular wreath product of groups A and B.

Lemma 3 ([8]). *Let $G = A \wr B = [K]B$ where $K = \prod_{b\in B} A_1^b$ is the base group of G and A_1 is the first copy of A in K. Let L_1 be a minimal normal subgroup in A_1. If $L_1 \not\subseteq Z(A_1)$, then $L = \prod_{b\in B} L_1^b$ is a minimal normal subgroup in G.*

Remind that a p-local formation $\mathfrak{F}$ is called a minimal p-local non-$\mathfrak{H}$-formation [11] if $\mathfrak{F} \not\subseteq \mathfrak{H}$, but $\mathfrak{F}_1 \subseteq \mathfrak{H}$ for each proper p-local subformation $\mathfrak{F}_1$ in $\mathfrak{F}$.

Lemma 4 ([14]). *If a p-local formation $\mathfrak{F} \not\subseteq \mathfrak{N}_p\mathfrak{N}$, then $\mathfrak{F}$ has a minimal p-local non-$\mathfrak{N}_p\mathfrak{N}$-subformation.*

Lemma 5 ([14]). *A p-local formation $\mathfrak{F}$ is a minimal p-local non-$\mathfrak{N}_p\mathfrak{N}$-formation if and only if $\mathfrak{F} = l_p \operatorname{form} G$ where G is a monolithic group with a monolith $R = G^{\mathfrak{N}_p\mathfrak{N}}$, and either R is a p'-group or R is a non-abelian pd-group.*

Theorem 22. *Let $\mathfrak{M}$, $\mathfrak{H}$ be formations and $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ be a one-generated p-local formation. If $\mathfrak{H} \neq \mathfrak{F}$, then $\mathfrak{M}$ is a p-local formation such that all p-local subformations of it are hereditary.*

Proof. Assume that the formation $\mathfrak{M}$ is not p-local. Then, by Lemma 1 there is a group A such that for some normal subgroup L in A with $L \subseteq O_p(A) \bigcap \Phi(A)$ and $A/L \in \mathfrak{M}$ we have $A \notin \mathfrak{M}$.

First suppose that for each simple group $M \in \mathfrak{M}$ we have $|M| = p$. We shall show that in this case the equality $\mathfrak{F} = \mathfrak{H}$ is true. Clearly $\mathfrak{H} \subseteq \mathfrak{F}$. Assume that $\mathfrak{F} \not\subseteq \mathfrak{H}$ and let D be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{H}$. Let $P = D^{\mathfrak{H}}$ be the $\mathfrak{H}$-residual of the group D. Since $D \in \mathfrak{F}$, we have $P \in \mathfrak{M}$. It is clear that P is a minimal normal subgroup of the group D. Hence $P = A_1 \times A_2 \times \ldots \times A_t$ where $A_1 \simeq A_2 \simeq \ldots \simeq A_t$ is a simple group. But $A_1, A_2, \ldots, A_t \in \mathfrak{M}$. Hence $|A_1| = |A_2| = \ldots = |A_t| = p$.

Let $B = A/L$, $E = B \wr D$ and $B_1 = A \wr D$. Let K be the base group of E and K_1 be the base group of B_1. Evidently $E^{\mathfrak{H}} \subseteq K$. We use B_1 to denote the first copy of B in K. And let F be the projection of $E^{\mathfrak{H}}$ in B_1. Suppose that $F = B_1$. Then $E^{\mathfrak{H}}$ is contained subdirectly in

$$K = \prod_{d\in D} B_1^d \in \mathfrak{M}.$$

Let A_1 be the first copy of the group A in K_1. And let L_1 be the subgroup of A_1 such that $L_1 = L^{\pi}$ where π is an isomorphism from A to A_1. Let $R = \prod_{d\in D} L_1^d$. Then there is an epimorphism $\varphi : E_1 \to E$ such that $\operatorname{Ker}\varphi = R$. It is clear that $R \subseteq O_p(E_1) \bigcap \Phi(E_1)$. But

$$E_1/R \simeq E \in \mathfrak{F}.$$

Since $E_1 \in \mathfrak{F}$, we have $E_1^{\mathfrak{H}} \in \mathfrak{M}$. Note that $E^{\mathfrak{H}} = ((E_1)^{\mathfrak{H}})^{\varphi}$, $K_1^{\varphi} = K$ and $E^{\mathfrak{H}}$ is contained subdirectly in K. Therefore E_1^{φ} is contained subdirectly in K_1. Thus there is an epimorphism from $E_1^{\mathfrak{H}}$ onto the group A, and so $A \in \mathfrak{M}$. This contradiction shows that $F \subset B_1$. Since $E^{\mathfrak{H}}$ is normal in E, F is normal in B_1. By Lemma 3.1.9 [4] the group $(B_1/F) \wr D$ belongs to the formation $\mathfrak{H}$. Let M be a normal subgroup in B_1 such that B_1/M is a simple group. It is clear that $B_1/M \in \mathfrak{M}$, and so B_1/M is a group of order p. It is also clear that the group

$T = (B_1/M) \wr D$ belongs to $\mathfrak{H}$. By Theorem 18.9 [10], $D \simeq E_0 \subseteq E = P \wr (D/P)$ where E_0 is a subgroup of $P \wr (D/P)$ such that $E_0 F(E) = E$. By Lemma 1.4.4 [4], $E_0 \in \operatorname{form} E$. Hence $D \in \operatorname{form} E$. It is clear that

$$E \in \mathrm{R}_0(Z_p \wr (D/P)) \subseteq \operatorname{form}(Z_p \wr (D/P))$$

where Z_p is a group of order p. Therefore

$$D \simeq E_0 \in \operatorname{form} E \subseteq \operatorname{form}((B_1/M) \wr D) \subseteq \mathfrak{H}.$$

This contradiction shows that $\mathfrak{F} \subseteq \mathfrak{H}$. So $\mathfrak{F} = \mathfrak{H}$. But, by hypothesis, $\mathfrak{F} \neq \mathfrak{H}$. Thus there is a simple group $X \in \mathfrak{M}$ such that $|X| \neq p$. We shall show that $\mathfrak{H}$ is an abelian formation.

Let $\mathfrak{F} = l_p \operatorname{form} G$ and $|G| = m$. Assume that X is an abelian q-group. Let $\mathbb{F}_q$ be the field with q elements and let $\overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$. And let M be a non-abelian group in $\mathfrak{H}$. Then there is a simple $\overline{\mathbb{F}_q}M$-module T with $\dim_{\overline{\mathbb{F}_q}} T \geq 2$. Let D be the external tensor product (see § 43 [13]) of m copy of the module T. Then the $\overline{\mathbb{F}_q}$ M-module D is simple (see § 27 [14]) and $\dim_{\overline{\mathbb{F}_q}}(D) \geq 2^m$. Hence there is a simple $\mathbb{F}_q M^m$-module L such that D is a direct composed of $L^{\overline{F}_q}$ (see § 29 [13]). Hence $\dim_{\mathbb{F}_q}(L) \geq 2^m$. Since $M \in \mathfrak{H}$ and L is an elementary abelian q-group, we have $R = [L]M^m \in \mathfrak{F}$. It is clear that L is a minimal normal subgroup in R and $|L| \geq q^{2^m}$. But then the group R/R_{pd} has a minimal normal subgroup LR_{pd}/R_{pd} of order

$$|LR_{pd}/R_{pd}| = |L/R_{pd} \bigcap L| = |L| \geq q^{2^m} > m.$$

Since $R \in \mathfrak{F}$, we have

$$R/R_{pd} \in f(p') = \operatorname{form}(G/G_{pd})$$

where f is the minimal p-local satellite of $\mathfrak{F}$. This contradicts to Lemma 2. Thus X is a non-abelian simple group.

Consider the group $D = A \wr M^m$ where M is a non-identity group in $\mathfrak{H}$. Then by Lemma 3, the group D is monolithic and its monolith L coincides with the base group of D. Hence $|L| = |A|^{|M|^m} > m$. It is clear that $D \in \mathfrak{F}$, and so

$$D \simeq D/D_{pd} \in f(p') = \operatorname{form}(G/G_{pd}).$$

A contradiction. Thus $\mathfrak{H}$ is an abelian formation.

Now let $E = A \wr D$ for some non-identity group $D \in \mathfrak{H}$. Let K be the base group of E and A_1 be the first copy of A in K. Let F be the projection of $E^{\mathfrak{H}}$ in A_1. Assume that $F \neq A_1$. Then the group $(A_1/F) \wr D \in \mathfrak{H}$, and so

$$D \subseteq Z(E) \subseteq C_E(K).$$

This contradiction shows that $F = A_1$. Now as above we can show that the group A belongs to the formation $\mathfrak{M}$. This contradiction shows that $\mathfrak{M}$ is a p-saturated formation, and so $\mathfrak{M}$ is p-local, by Lemma 1.

Now we shall show that $\mathfrak{M} \subseteq \mathfrak{N}_p\mathfrak{N}$. Assume that it is false. Then, by Lemma 4, $\mathfrak{F}$ has a minimal p-local non-$\mathfrak{N}_p\mathfrak{N}$-subformation $\mathfrak{F}_1$. By Lemma 5, $\mathfrak{F}_1 = l_p \operatorname{form} A$ where A is a monolithic group with the monolith $R = A^{\mathfrak{N}_p\mathfrak{N}}$ such that either R is a p'-group or R is a non-abelian pd-group. Let A be a soluble monolithic group with the monolith $R = A^{\mathfrak{N}_p\mathfrak{N}}$ such that R is a p'-group. As a product of two local formations $\mathfrak{N}_p$ and $\mathfrak{N}$, the formation

$\mathfrak{N}_p\mathfrak{N}$ is local as well. So $R \not\subseteq \Phi(A)$. Let M be a maximal subgroup in A such that $RM = A$. Then $R\bigcap M = 1$ and

$$C_A(R) = C_A(R)\bigcap RM = R(C_A(R)\bigcap M) = R_1 = R.$$

Consider the group $T = A \wr (D^m) = [K](D^m)$ where D is a non-identity group in $\mathfrak{H}$ and H is the base group of T. Let A_1 be the first copy of A in K and R_1 be the monolith of A_1. If $\mathfrak{H}$ is an abelian formation then, like proved above we can show that $T \in \mathfrak{F}$. So

$$T \simeq T/T_{pd} \in f(p') = \mathrm{form}(G/G_{pd}).$$

By lemma 3, the group T is monolithic and its monolith $L = \prod_{d\in D^m} R_1^d$. Hence $|L| \geq |R_1|^m > m$. This contradicts Lemma 2. Hence me may suppose that the formation $\mathfrak{H}$ is not abelian. From stated above we obtain that in this case for every simple group X in $\mathfrak{M}$ we have $|X| = p$. We shall show that in this case $\mathfrak{F} = \mathfrak{H}$.

It is not difficult to show that $\mathfrak{N}_p\mathfrak{H} = \mathfrak{H}$. Clearly $\mathfrak{H} \subseteq \mathfrak{F}$. Let $\mathfrak{F} \not\subseteq \mathfrak{H}$ and let B be a group of minimal order in $\mathfrak{F}\backslash\mathfrak{H}$. Let L be the monolith of the group B. Then from $B \in \mathfrak{F} = \mathfrak{M}\mathfrak{H}$ we have $L \in \mathfrak{M}$, and so L is a p-group. But $L = B^{\mathfrak{H}}$. Hence $B \in \mathfrak{N}_p\mathfrak{H} = \mathfrak{H}$. This contradiction shows that $\mathfrak{F} = \mathfrak{H}$. But, by hypothesis $\mathfrak{H} \neq \mathfrak{F}$. Hence $\mathfrak{M} \subseteq \mathfrak{N}_p\mathfrak{N}$. Using the result from [15] we see that in the formation $\mathfrak{M}$ all p-local subformations are hereditary. The Theorem is proved. □

Резюме. Доказано, что если однопорожденная формация $\mathfrak{F}$ конечных групп p-локальна, $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ и $\mathfrak{F} \neq \mathfrak{H}$, то формация $\mathfrak{M}$ p-локальна.

References

[1] W.Gaschütz, *Zur Theorie der endlichen auflösbaren Gruppen,* Math. Z. 80 (1963), 300–305.

[2] L.A.Shemetkov, A.N.Skiba, *Formations of algebraic systems,* Nauka, Moskow, 1989 (Russian).

[3] W.Gaschutz, *Selected topics in the theory of soluble groups,* Lectures given at the 9th Summer Research Institute of the Austral. Math. Soc. Canberra (1969), Notes by J. Locker.

[4] A.N.Skiba, *Algebra of formations,* Belaruskaja Navuka, Minsk, 1997 (Russian).

[5] L.A.Shemetkov, *On the product of formations,* Dokl. Akad. Nauk BSSR, 28:2 (1984), 101–103.

[6] V.A.Vedernikov, *Local formations of finite groups,* Mat. Zametki, 46:3 (1989), 32–37.

[7] N.T.Vorob'ev, *Factorizations of non-local formations of finite groups,* Voprosy Algebry (Problems in Algebra), 5 (1990), 21–24 (Russian).

[8] A.N.Skiba, *On non-trivial factorizations of one-generated local formations of finite groups,* Proc. Intern. Conf. Algebra Dedecated to the memory of A.I.Mal'cev (Novosibirsk, August 21–26, 1989), Amer Math. Soc., Providence (B.I.) (1992), 363–374.

[9] L.A.Shemetkov, *Formations of finite groups,* Nauka, Moskow, 1978 (Russian).

[10] K.Doerk, T.Hawkes, *Finite soluble groups,* Walter de Gruyter, Berlin–New York, 1992.

[11] L.A.Shemetkov, A.N.Skiba, *Multiply ω-local formations and Fitting classes of finite groups,* Sibirian Advances in Mathematics, 10:2 (2000), 1–30 (Russian).

[12] A.N.Skiba, L.A.Shemetkov, *On partially local formations,* Dokl. Akad. Nauk Belarusi, 39:3 (1995), 9–11 (Russian).

[13] C.W.Curtis, I.Reiner, *Representation theory of finite groups and associative algebras,* Pure and Appl. Math. V. 11. Interscience, New York, 1962; 2nd ed., 1966.

[14] V.N.Ryzhik, *On critical p-local formations,* Preprint № 58 (1997), Gomel University Preprints (Russian).

[15] Jaraden Jehad J., A.N.Skiba, *Partially local formations with systems hereditary formations,* Vesty Akad. Navuk Belarus. Ser. fiz.-mat. navuk, № 3 (1996), 13–16 (Russian).

Received May 30, 2000

Francisk Scorina Gomel State University
246019 Gomel, Belarus