Известия Гомельского государственного университета имени
Ф.Скорины, 3(16) Вопросы алгебры (2000), 93–98
2000 Mathematics Subject Classification: 20D10

On ω-saturated formations with systems of hereditary ω-saturated subformations

Jaraden Jehad J.

Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

1. Introduction

All groups considered are finite.

A formation is a class of groups which is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is called *hereditary* (*normally hereditary*) if $\mathfrak{F}$ contains all subgroups (all normal subgroups) of all its groups. The formation $\mathfrak{F}$ is called ω*-saturated* ($\varnothing \neq \omega \subseteq \mathbb{P}$) if it contains each group G with $G/O_\omega(G) \bigcap \Phi(G) \in \mathfrak{F}$.

We use l_ω form $\mathfrak{X}$ to denote the intersection of all ω-saturated formations containing the set $\mathfrak{X}$ of groups.

Let $\{\mathfrak{F}_i \mid i \in I\}$ be the set of all proper ω-saturated subformations of a formation $\mathfrak{F}$. And let $\mathfrak{M} = l_\omega \operatorname{form}(\bigcup_{i\in I} \mathfrak{F}_i)$. Then that formation $\mathfrak{F}$ is called [1]: (a) l_ω*-irreducible* if $\mathfrak{M} \neq \mathfrak{F}$; (b) l_ω*-reducible* if $\mathfrak{M} = \mathfrak{F}$.

In this note extending some results from [2] we prove the following.

Theorem 1.1. *Let $\mathfrak{F}$ be a ω-saturated formation. Then $\mathfrak{F} \subseteq \mathfrak{N}_\omega\mathfrak{N}$ (i.e. every $\mathfrak{F}$-group is an extension of a nilpotent ω-group by a nilpotent group) if at least one of the following two conditions is true:*

(1) $\mathfrak{F}$ is soluble and each l_ω-irreducible ω-saturated subformation of $\mathfrak{F}$ is normally hereditary;

(2) each l_ω-irreducible ω-saturated subformation of $\mathfrak{F}$ is hereditary.

Remark 1.2 ([2]). *If $\mathfrak{F} \subseteq \mathfrak{N}_\omega\mathfrak{N}$, then every ω-saturated subformation of $\mathfrak{F}$ is hereditary.*

2. Preliminaries

In this section we collect some definitions and notations as well as some known results which we will need later on.

In this paper ω denotes a non-empty set of primes. Every function of the form

$$f : \omega \bigcup \{\omega'\} \to \{\text{group formations}\}$$

is called an ω-local satellite [3]. Following [3] we use $LF_\omega(f)$ to denote the class

$$(G \mid G/G_{\omega d} \in f(\omega') \text{ and } G/F_p(G) \in f(p) \text{ for all } p \in \omega \bigcap \pi(G))$$

where $G_{\omega d}$ is the largest normal in G subgroup all composition factors H/K of which are ωd-group (i.e. $\pi(H/K) \bigcap \omega \neq \varnothing$). By definition $G_{\omega d} = 1$ if for each minimal normal subgroup L in G we have $\pi(L) \bigcap \omega = \varnothing$.

If a formation $\mathfrak{F}$ is such that $\mathfrak{F} = LF_\omega(f)$ for some ω-local satellite f, then we say [3] that $\mathfrak{F}$ is an ω-local formation and f is an ω-local satellite of that formation.

We need the following version of the remarkable Gaschütz–Lubeseder–Schmid Theorem.

Theorem 2.1 ([4], Theorem 1). *A formation $\mathfrak{F}$ is ω-local if and only if it is ω-saturated.*

Let $\{f_i \mid i \in I\}$ be a collection of ω-local satellites. Then $\bigcap_{i\in I} f_i$ is a satellite such that $f(a) = \bigcap_{i\in I} f_i(a)$ for all $a \in \omega \bigcup \{\omega'\}$.

Now let $\{f_i \mid i \in I\}$ be the set of all ω-local satellite of $\mathfrak{F}$. Then the satellite $\bigcap_{i\in I} f_i$ is called the minimal ω-local satellite of $\mathfrak{F}$.

Lemma 2.2 ([3], Lemma 5). *Let f be the minimal ω-local satellite of $\mathfrak{F} = l_\omega \operatorname{form} \mathfrak{X}$. Then*

$$f(a) = \begin{cases} \operatorname{form}(G/G_{\omega d} \mid G \in \mathfrak{X}), & \text{if } a = \omega', \\ \operatorname{form}(G/F_p(G) \mid G \in \mathfrak{X}), & \text{if } p \in \omega \bigcap \pi(\mathfrak{X}), \\ \varnothing, & \text{if } a \in \omega \setminus \pi(\mathfrak{X}). \end{cases}$$

Lemma 2.3 ([3], Lemma 6). *Let f_i be the minimal ω-local satellite of the formation $\mathfrak{F}_i$, $i = 1, 2$. Then $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ if and only if $f_1 \leq f_2$.*

Lemma 2.4 ([5], Lemma 1). *Let G be a monolithic group with the monolith R. If $R \nsubseteq \Phi(G)$, then $\operatorname{form}(G/R)$ is the unique maximal subformation of the formation $\operatorname{form} G$.*

Lemma 2.5 ([1], Corollary 1.2.27). *If $A \in \operatorname{form} \mathfrak{X}$, then*

$$A/\operatorname{Soc}(A) \in \operatorname{form}(G/\operatorname{Soc}(G) \mid G \in \mathfrak{X}).$$

Lemma 2.6 ([3], Lemma 3). *If $A/O_p(A) \in f(p) \bigcap \mathfrak{F}$ where $\mathfrak{F} = LF_\omega(f)$, then $A \in \mathfrak{F}$.*

Theorem 2.7 ([3], Theorem 7). *If $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ is a product of two p-local formations $\mathfrak{M}$ and $\mathfrak{H}$, then $\mathfrak{F}$ is p-local as well.*

3. Results

Lemma 3.1. *Let G be a monolithic group with a non-abelian monolith. Then the formation $l_\omega \operatorname{form} G$ is l_ω-irreducible.*

Proof. Let P be the monolith of the group G. Let f be the minimal ω-local satellite of the formation $\mathfrak{F}$.

Assume that $\pi(R) \bigcap \omega = \varnothing$. And let h be a ω-local satellite such that

$$h(a) = \begin{cases} \operatorname{form}(G/R), & \text{if } a = \omega', \\ f(p), & \text{if } a = p \in \omega. \end{cases}$$

By Lemma 2.3, $\mathfrak{H} = LF_\omega(h) \subset \mathfrak{F}$. Let $\mathfrak{M}$ be a proper ω-local subformation of $\mathfrak{F}$ and let m be the minimal ω-local satellite of $\mathfrak{M}$. Then by Lemma 2.3, $m \leq f$. Assume that $\mathfrak{M} \nsubseteq \mathfrak{H}$. Then, evidently, $m(\omega') \nsubseteq h(\omega')$. By Lemma 2.2,

$$f(\omega') = \operatorname{form}(G/G_{\omega d}) = \operatorname{form} G.$$

And by Lemma 2.4, $h(\omega') = \operatorname{form}(G/R)$ is the unique maximal subformation of $f(\omega')$. Hence $m(\omega') = f(\omega)$, and so $G \in m(\omega') \subseteq \mathfrak{M}$. But then

$$\mathfrak{F} = l_\omega \operatorname{form} G \subseteq \mathfrak{M} \subset \mathfrak{F}.$$

This contradiction shows that $\mathfrak{M} \subseteq \mathfrak{H}$. So $\mathfrak{H}$ is the unique maximal ω-local subformation of $\mathfrak{F}$. Hence $\mathfrak{F}$ is l_ω-irreducible.

Now let $\pi(R) \bigcap \omega \neq \varnothing$. Let h be an ω-local satellite such that

$$h(a) = \begin{cases} \operatorname{form}(G/R), & \text{if } a \in \pi(R) \bigcap \omega, \\ f(a), & \text{if } a \in \omega \setminus \pi(R), \\ f(a), & \text{if } a = \omega'. \end{cases}$$

And let $\mathfrak{H} = LF_\omega(h)$. We shall show that $\mathfrak{H}$ is the unique maximal ω-local subformation of $\mathfrak{F}$. Let $\mathfrak{M} = LF_\omega(m)$ be a proper ω-local subformation of $\mathfrak{F}$ where m is the minimal ω-local satellite of $\mathfrak{M}$. Then by Lemma 2.3, $m \leq f$. Assume that $\mathfrak{M} \not\subseteq \mathfrak{H}$. Then there is a prime $p \in \pi(R) \bigcap \omega$ such that $m(p) \not\subseteq h(p)$. But by Lemma 2.4, $h(p)$ is the unique maximal subformation of

$$f(p) = \operatorname{form}(G/F_p(G)) = \operatorname{form} G.$$

Hence $m(p)$=f(p), and so $G \in m(p) \subseteq \mathfrak{M}$. Therefore

$$\mathfrak{F} = l_\omega \operatorname{form} G \subseteq \mathfrak{M} \subset \mathfrak{F}.$$

This contradiction shows that $\mathfrak{M} \subseteq \mathfrak{H}$. Hence $\mathfrak{H}$ is the unique maximal ω-local subformation of $\mathfrak{F}$. So $\mathfrak{F}$ is l_ω-irreducible.

Proof of Theorem 1.1. In view of Theorem 2.1 we have only to consider the case when $\mathfrak{F}$ is an ω-local formation.

Let $\mathfrak{F}$ be an ω-local soluble formation in which all its ω-local l_ω-irreducible subformations are normally hereditary. Assume that $\mathfrak{F} \not\subseteq \mathfrak{N}_\omega\mathfrak{N}$ and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{N}_\omega\mathfrak{N}$. The group G is monolithic and its monolith $R = G^{\mathfrak{N}_\omega\mathfrak{N}}$. The formation $\mathfrak{N}_\omega\mathfrak{N}$ is local as a product of two local formations $\mathfrak{N}_\omega$ and $\mathfrak{N}$ (see Theorem 2.7). Hence $R \not\subseteq \Phi(G)$. Let M be a maximal subgroup of G such that $RM = G$. Then if $C = C_G(R)$, $C = C \bigcap RM = R(C \bigcap M)$. But $C \bigcap M$ is evidently normal in G. Hence $C \bigcap M = 1$, and so $R = C_G(R) = O_p(G) = F_p(G)$ where $\{p\} = \pi(R)$.

Suppose that $p \notin \omega$. Let $\mathfrak{M} = l_\omega \operatorname{form} G$. We shall show that the formation $\mathfrak{M}$ is l_ω-irreducible. Let f be the minimal ω-local satellite of the formation $\mathfrak{M}$ and let h be a satellite such that $h(\omega') = \operatorname{form} M$ and $h(p) = f(p)$ for all $p \in \omega$. Let $\mathfrak{H} = LF_\omega(h)$. Clearly $\mathfrak{H} \subset \mathfrak{F}$. Let $\mathfrak{H}_1$ be a proper ω-local subformation of $\mathfrak{F}$, h_1 be the minimal ω-local satellite of $\mathfrak{H}_1$. By Lemma 2.3, $h_1 \leq f$. Assume that $\mathfrak{H}_1 \not\subseteq \mathfrak{H}$. Then $h(\omega') \not\subseteq h(\omega')$. But by Lemma 2.4, $h(\omega')$ is the unique maximal subformation of $f(\omega') = \operatorname{form} G$. Hence $h(\omega') = f(\omega')$. By Lemma 2.2,

$$f(\omega') = \operatorname{form}(G/G_{\omega d}) = \operatorname{form} G.$$

Hence $G \in h(\omega') \subseteq \mathfrak{H}$, and so $\mathfrak{F} = \mathfrak{H}$. This contradiction shows that $\mathfrak{H}_1 \subseteq \mathfrak{H}$. Thus $\mathfrak{H}$ is the unique maximal ω-local subformation of $\mathfrak{F}$. Hence $\mathfrak{M}$ is l_ω-irreducible. Therefore by hypothesis, $\mathfrak{M}$ is a normally hereditary formation. If M is a nilpotent group, then by Lemma 3.9 of [6], $O_p(M) = 1$. Hence by Lemma 2.4, the subgroup R is not in the formation form $M = f(\omega')$. But $R \in \mathfrak{M}$, and so

$$R \simeq R/R_{\omega d} \in f(\omega').$$

This contradiction shows that G/R is not a nilpotent group. Let $T = G^{\mathfrak{N}}$. Then $R \subseteq T$ and T/R is a nilpotent ω-group. By hypothesis, $T \in \mathfrak{M}$. By Lemma 3.9 of [6], a prime p

is not in $\pi(T/R)$. Hence $T = [R]H$ where H is a nilpotent ω-group with $p \notin \pi(\mathfrak{H})$. Let Q be a minimal normal subgroup in H and $T_1 = [R]Q$. Then $T_1 \in \mathfrak{M}$. It is clear also that there is a chief factor D/L of T_1 such that $C = C_{T_1}(D/L) \neq T_1$ and $1 \subseteq L \subset D \subseteq R$. Let $K = [D/L](T_1/C)$. Then $K \in \mathfrak{M}$, and so

$$K = K/K_{\omega d} \in f(\omega') = \mathrm{form}\, G = \mathrm{form}(G/G_{\omega d}) = f(\omega').$$

Let $\mathfrak{M}_1 = \mathrm{form}(G/R)$. By Lemma 2.4, $\mathfrak{M}_1$ is the unique maximal subformation of $f(\omega')$. Since $G/R \in \mathfrak{N}_\omega \mathfrak{N}$, we have $K \notin \mathfrak{M}$. Hence

$$\mathrm{form}\, K = \mathrm{form}\, G.$$

Using now Lemma 2.5 we see that

$$\mathrm{form}(G/R) = \mathrm{form}\, K/(D/L) = \mathrm{form}\, Q.$$

This contradiction shows that $p \in \omega$.

Evidently the group G/R is not nilpotent. Let $T/R = (G/R)^{\mathfrak{N}}$. Then $T \neq R$ and it is clear that there is a chief factor H/K of the group G such that $C = C_G(H/K) \neq G$ and $R \subseteq K \subset H \subseteq T$. By Lemma 3.31 [7], the group $E = [H/K](G/C)$ belongs to the formation $\mathrm{form}(G/R)$. If t is the minimal ω-local satellite of $\mathfrak{F}$, then

$$t(p) = \mathrm{form}(G/F_p(G)) = \mathrm{form}(G/R).$$

Hence $E \in t(p)$, and so by Lemma 2.6, $L = [P]E \in \mathfrak{F}$ where P is a simple and faithful E-module over $GF(p)$.

We consider the formation $\mathfrak{M} = l_\omega\, \mathrm{form}\, L_1$. First we show that $\mathfrak{M}$ is l_ω-irreducible. Let m be the minimal ω-local satellite of $\mathfrak{M}$ and let h be a satellite with the following values:

$$h(a) = \begin{cases} \mathrm{form}(G/C), & \text{if } a = p, \\ m(a), & \text{if } a \in \omega \setminus \{p\}, \\ m(\omega'), & \text{if } a = \omega'. \end{cases}$$

Let $\mathfrak{H} = LF_\omega(h)$ and let $\mathfrak{H}_1 = LF_\omega(h_1)$ be a proper ω-local subformation of $\mathfrak{M}$ where h_1 is the minimal ω-local satellite of $\mathfrak{H}_1$. By Lemma 2.3, $h_1 \leq m$. Hence if $\mathfrak{H}_1 \not\subseteq \mathfrak{H}$, then $h_1(p) \not\subseteq h(p)$. By Lemma 2.4, the formation $\mathrm{form}(G/C)$ is the unique maximal subformation of the formation $\mathrm{form}\, E$. But, evidently, $P = F_p(E)$, and so $m(p) = \mathrm{form}\, E$. Therefore $h_1(p) = m(p)$. So by Lemma 2.6, $L \in \mathfrak{H}_1$. But then

$$\mathfrak{M} = l_\omega\, \mathrm{form}\, L \subseteq \mathfrak{H}_1 \subset \mathfrak{M}.$$

This contradiction shows that $\mathfrak{M}$ is a l_ω-irreducible formation. Hence by hypothesis, $\mathfrak{M}$ is normally hereditary. Thus the group $[P](H/K)$ belongs to $\mathfrak{M}$, and so

$$H/K \in m(p) = \mathrm{form}(L/F_p(L)) = \mathrm{form}\, E.$$

But by Lemma 3.9 [6], $p \notin \pi(G/C)$. And by Lemma 2.4, $\mathrm{form}(G/C)$ is the unique maximal subformation in $\mathrm{form}\, E$. This contradiction shows that $\mathfrak{F} \subseteq \mathfrak{N}_\omega \mathfrak{N}$.

Now we suppose that every l_ω-irreducible ω-local subformation of the ω-local formation $\mathfrak{F}$ is hereditary. Assume that $\mathfrak{F} \not\subseteq \mathfrak{N}_\omega \mathfrak{N}$ and let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{N}_\omega \mathfrak{N}$.

and $R = G^{\mathfrak{N}_\omega \mathfrak{N}}$ be the monolith of G. And let $\mathfrak{M} = l_\omega \operatorname{form} G$. We have only to consider the case when R is a non-abelian group.

By Lemma 3.1, the formation $\mathfrak{M}$ is l_ω-irreducible. So by hypothesis, $\mathfrak{M}$ is hereditary.

Since $\pi(R) \not\subseteq \omega$ we may choose a prime $q \in \pi(R) \setminus \omega$. Clearly the group R is not q-nilpotent. Hence by Theorem 4.3.1 of [9], the group R has a q-closed minimal non-nilpotent subgroup H_1 with $q \in \pi(H_1)$. Let $H = H_1/\Phi(H_1)$. Then by Theorems 26.1, 26.2 of [6], the group $H = Q \rtimes Z_r$ where $Q = F_q(H) = C_H(Q)$ is a minimal normal in H subgroup and Z_r is a group of prime order r. By hypothesis, $H \in \mathfrak{M}$. But $q \notin \omega$, and so $H_{\omega d} = 1$. Hence by Lemma 2.2,

$$H \simeq H/H_{\omega d} \in m(\omega') = \operatorname{form}(G/G_{\omega d}) \subseteq \operatorname{form} G.$$

Let $\mathfrak{H} = \operatorname{form}(G/R)$. By Lemma 2.4, $\mathfrak{H}$ is the unique maximal subformation of the formation form G. If $H \in \mathfrak{H}$, then $H^{\mathfrak{N}} \in \mathfrak{N}_\omega$. But $q \notin \omega$. So $H \notin \mathfrak{H}$. Hence

$$\operatorname{form} H = \operatorname{form} G.$$

The contradiction shows that $\pi(R) \subseteq \omega$. Then

$$l \operatorname{form} R = l_\omega \operatorname{form} R = l \operatorname{form} A = l_\omega \operatorname{form} A$$

where A is a composition factor of R. By hypothesis, $R \in \mathfrak{M}$, and so $A \in \mathfrak{M}$. Let $\mathfrak{M}_1 = l_\omega \operatorname{form} A$, m_1 be the minimal ω-local satellite of $\mathfrak{M}_1$. By Lemma 3.1, the formation $\mathfrak{M}_1$ is l_ω-irreducible. Hence it is hereditary. Thus $H \in \mathfrak{M}_1$, and so by Lemma 2.2,

$$Z_r \simeq H/Q = H/F_q(H) \in m_1(q) = \operatorname{form}(A/F_q(A)) = \operatorname{form} A.$$

But in the formation form A all its non-identity groups have the form $A_1 \times \ldots \times A_t$ where $A_1 \simeq \ldots \simeq A_t \simeq A$. This contradiction completes the proof.

Corollary 3.2 ([9], Theorem 2). *Let $\mathfrak{F}$ be a local formation. If every local subformation of $\mathfrak{F}$ is hereditary, then $\mathfrak{F}$ is metanilpotent.*

Corollary 3.3 ([9], Theorem 1). *If every subformation of the formation $\mathfrak{F}$ is hereditary, then $\mathfrak{F}$ is nilpotent.*

Corollary 3.4 ([10]). *Let $\mathfrak{F}$ be a p-local formation. Then $\mathfrak{F} \subseteq \mathfrak{N}_p \mathfrak{N}_{p'}$ if every p-local subformation of $\mathfrak{F}$ is hereditary.*

Corollary 3.5 ([2]). *Let $\mathfrak{F}$ be a ω-local formation. Then $\mathfrak{F} \subseteq \mathfrak{N}_\omega \mathfrak{N}$ if every ω-local subformation of $\mathfrak{F}$ is hereditary.*

Corollary 3.6 ([11], Theorem 4.11). *A formation $\mathfrak{F}$ is nilpotent if every its one-generated formation is hereditary.*

Резюме. Найдены достаточные условия вхождения ω-насыщенной формации в класс $\mathfrak{N}_\omega \mathfrak{N}$.

References

[1] A.N.Skiba, *Algebra of Formations,* Minsk, 1997 (Russian).

[2] Jaradin Jehad J., A.N.Skiba, *Partially local formations with systems hereditary subformations,* Vesti Akad. Navuk Belarus., ser. fiz.-mat. navuk 3 (1996), 13–16 (Russian).

[3] L.A.Shemetkov, A.N.Skiba, *Multiply ω-local formations and Fitting classes of finite groups,* Siberian Advances in Mathematics 10:2 (2000), 1–30 (Russian).

[4] A.N.Skiba, L.A.Shemetkov, *On partially local formations,* Dokl. Akad. Nauk Belarus 39:3 (1995), 9–11 (Russian).

[5] A.N.Skiba, *On minimal local non-p-supersoluble formations,* Voprosy Algebry (Problems in Algebra), № 1 (1985), 105–112 (Russian).

[6] L.A.Shemetkov, *Formations of finite groups,* Nauka, Moscow, 1978 (Russian).

[7] L.A.Shemetkov, A.N.Skiba, *Formations of algebraic systems,* Nauka, Moscow, 1989 (Russian).

[8] S.A.Chunikhin, *Subgroups of finite groups,* Nauka i Tekhnika, Minsk, 1964 (Russian).

[9] A.N.Skiba, *Characterizations of finite metanilpotent groups,* Mat. Zametki 27:3 (1980), 345–351 (Russian).

[10] Jaraden Jehad J., *On p-saturated formations with systems hereditary subformations* Vestnik Belarus University 3 (1995), 52–55 (Russian).

[11] V.A.Vedernikov, *Elements of the class theory,* Smolensk, 1988 (Russian).

Received June 1, 2000

Jordan