Известия Гомельского государственного университета имени
Ф.Скорины, 3(16) Вопросы алгебры (2000), 107–110
2000 Mathematics Subject Classification: 20D10

On a cover-avoidance properties of injectors of finite soluble groups

I.V.Dudkin

Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

Throughout this paper, we will assume without mentioning that all groups considered belong to the class $\mathfrak{S}$ of finite soluble groups. The subgroup R covers (avoids) a section A/B of a group G if $RB \geq A$ (respectively $R \cap A \leq B$). Recall that a class $\mathfrak{F}$ of groups is a Fitting class if $\mathfrak{F}$ is closed under taking normal subgroups and in each group G there is a unique normal subgroup that is maximal with respect to being in $\mathfrak{F}$; that subgroup, the $\mathfrak{F}$-radical, will be denoted by $G_{\mathfrak{F}}$. If $\mathfrak{F}$ is a Fitting class, then an $\mathfrak{F}$-injector of a group G is a subgroup V of G such that, for any subnormal subgroup N of G, $V \cap N$ is a maximal $\mathfrak{F}$-subgroup of N. It is not difficult to see that a $\mathfrak{F}$-injector of a group covers or avoids each chief factor of the group (see VII.2.14 [1], for example). For the first time the problem of description of chief factors covered or avoided by $\mathfrak{F}$-injectors was considered by Hartley [2]. It is well-known that an $\mathfrak{F}$-normalizer of a group covers every $\mathfrak{F}$-central chief factor and avoids every $\mathfrak{F}$-eccentric chief factor of that group where $\mathfrak{F}$ is a local formation. This result in a soluble case was obtained by Carter and Hawkes [3] and further was developed in the work of L.A.Shemetkov [4].

In this paper, using Hartley's idea [2], we dualize the above mentioned Carter-Hawkes result for a Fitting class $\mathfrak{F}$ that is semilocally defined by a full constant Hartley function. It is proved, that a $\mathfrak{F}$-injector of G covers every $\mathfrak{F}$-central chief factor and avoids every $\mathfrak{F}$-eccentric chief factor of the group G. The notion of f-centrality we define following [5].

A map $\mathbb{P} \to \{\text{Fitting classes}\}$ is called a Hartley function or an H-function (proposed by L.A. Shemetkov).

Let f be some H-function, $\pi = supp(f) = \{p \in \pi \mid f(p) \neq \varnothing\}$. We denote by $LR(f)$ a Fitting class

$$\mathfrak{S}_\pi \bigcap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}).$$

A Fitting class $\mathfrak{F}$ is called local [6] if $\mathfrak{F} = LR(f)$ for some H-function f.

Let $\mathfrak{X}$ be some Fitting class. We denote by $SLR(x)$ a Fitting class

$$\bigcap_{p\in\pi(\mathfrak{X})} x(p)\mathfrak{S}_{p'}$$

where x is an H-function. According to [7, 8] we say that a Fitting class $\mathfrak{F}$ is semilocally defined by a H-function f if $\mathfrak{F} = SLR(f)$ for some H-function f.

We use the classification of H-functions offered in [7, 8].

An H-function f is called:

1) constant if $f(p) = f(q)$ for any p and q from π;

2) integrated if $f(p) \subseteq \mathfrak{F}$ for all prime $p \in \mathbb{P}$;

3) full if $f(p)\mathfrak{N}_p = f(p)$ for all $p \in \mathbb{P}$.

Other definitions can be found in [1, 9].

According to Hartley [2], we introduce the following.

Definition 1. *Let $\mathfrak{F} = LR(f)$ where f is an H-function and $p \in \pi(\mathfrak{F})$. A chief p-factor H/K of a group G will be called $f(p)$-covered if $H = K(V_{f(p)} \cap H)$ for some $\mathfrak{F}$-injector V of G, and $f(p)$-avoided if $V_{f(p)} \cap H \leq K$.*

We quote some well-known properties of $\mathfrak{F}$-injectors which we will sumarized below.

Lemma 1. [2, 10]. *Let $\mathfrak{F}$ be a Fitting class, and V an $\mathfrak{F}$-injector of G.*

(1) Any two $\mathfrak{F}$-injectors of the group G are conjugate in G;

(2) If $H \lhd G$ then $V \cap H$ is $\mathfrak{F}$-injector of H and all $\mathfrak{F}$-injectors of H are of this form and $G = HN_G(V \cap H)$;

(3) V covers or avoids each chief factor of G.

Lemma 2. *Let $\mathfrak{F} = LR(f)$ and p be a prime from $\pi(\mathfrak{F})$. Then a ny chief p-factor of G is either $f(p)$-covered or $f(p)$-avoided.*

Proof. Let H/K be a p-chief factor of a group G and V be an $\mathfrak{F}$-injector of G. Since $V \cap H \trianglelefteq V$, we have

$$(V \cap H)_{f(p)} = V_{f(p)} \cap (V \cap H) = V_{f(p)} \cap H.$$

By Lemma 1, $V \cap H$ is an $\mathfrak{F}$-injector of H. Hence, any subgroup conjugated with $(V \cap H)_{f(p)}$ in G is conjugated in H. Therefore $G = HN_G((V \cap H)_{f(p)})$ and hence $K(V \cap H)_{f(p)} \leq G$. Hence every chief p-factor H/K is either $f(p)$-covered or $f(p)$-avoided.

Definition 3. *Let f be an H-function and $p \in \pi = Supp(f)$. A chief p-factor of a group G is called f-central in G if it is covered by the $f(p)\mathfrak{N}_p$-radical of G, and f-eccentric otherwise.*

Lemma 3. *Let $\mathfrak{F} = SLR(f) \neq \varnothing$ for some full H-function f. Then $G \in \mathfrak{F}$ if and only if all chief factors of G are f-central.*

Proof. Let $G \in \mathfrak{F}$ and H/K be a chief p-factor of G. Since H is a normal subgroup of G, H is a $\mathfrak{F}$-group and consequently $p \in \pi(\mathfrak{F})$. But $\mathfrak{F}$ is local. Hence $H \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. We obtain $H/H_{f(p)\mathfrak{N}_p} \in \mathfrak{S}_{p'}$. Let P be a Sylow p-subgroup of H, then $PH_{f(p)\mathfrak{N}_p}/H_{f(p)\mathfrak{N}_p}$ is a Sylow p-subgroup of $H/H_{f(p)\mathfrak{N}_p}$. Since $H/H_{f(p)\mathfrak{N}_p}$ is p'-group, we have $PH_{f(p)\mathfrak{N}_p} = H_{f(p)\mathfrak{N}_p}$ and therefore $P \leq H_{f(p)\mathfrak{N}_p}$. Thus P is a Sylow p-subgroup of $H_{f(p)\mathfrak{N}_p}$, and $Syl_p(H) = Syl_p(H_{f(p)\mathfrak{N}_p})$. Furthermore, if PK/K is a Sylow p-subgroup of H/K, then $H/K = PK/K \leq H_{f(p)\mathfrak{N}_p}K/K \leq H/K$. Hence $H = H_{f(p)\mathfrak{N}_p}K = (G_{f(p)\mathfrak{N}_p} \cap H)K$, i.e. $G_{f(p)\mathfrak{N}_p}$ covers H/K and the chief factor H/K is f-central.

Assume that $G \notin \mathfrak{F}$. We will show, that there exists a chief factor in G which is not covered by $G_{f(p)\mathfrak{N}_p}$. Since $G \notin \mathfrak{F}$, there exists $p \in \pi(\mathfrak{F})$, such that $G \notin f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Consequently there exists a chief p-factor over $G_{f(p)\mathfrak{N}_p}$. Obviously that factor is avoided by $G_{f(p)\mathfrak{N}_p}$. Thus there exist chief factors in G which are not f-central.

Corollary 1. *Let $\mathfrak{F} = LR(f) \supseteq \mathfrak{N}$ for some H-function f. Then $G \in \mathfrak{F}$ if and only if all chief factors of group G are f-central.*

Lemma 4. *Let f_1 and f_2 be different integrated H-functions of a local Fitting class $\mathfrak{F}$. Then the following statements are equivalent:*

(1) a chief p-factor H/K is f_1-central in G;

(2) a chief p-factor H/K is f_2-central in G.

Proof. Without loss of generality we suppose that $f_1 \leq f_2$. This is because $LR(f_1) = LR(f_1 \cap f_2)$ and an H-function $f_1 \cap f_2$ is integrated. If the chief p-factor H/K of G is covered by $G_{f_1(p)\mathfrak{N}_p}$, then it is covered by a Sylow p-subgroup P of $G_{f_1(p)\mathfrak{N}_p}$. We assume that H/K is covered by $G_{f_2(p)\mathfrak{N}_p}$. Since f_2 is an integrated H-function, by Lemma 3 [8], we have $f_2(p)\mathfrak{N}_p \subseteq \mathfrak{F}$. We obtain the following:

$$G_{f_2(p)\mathfrak{N}_p}/(G_{f_2(p)\mathfrak{N}_p})_{f_1(p)\mathfrak{N}_p} = G_{f_2(p)\mathfrak{N}_p}/(G_{(f_2(p)\cap f_1(p))\mathfrak{N}_p}) = G_{f_2(p)\mathfrak{N}_p}/G_{f_1(p)\mathfrak{N}_p} \in \mathfrak{S}_{p'}.$$

Hence $Syl_p(G_{f_1(p)\mathfrak{N}_p}) = Syl_p(G_{f_2(p)\mathfrak{N}_p})$. Therefore $H/K = (G_{f_2(p)\mathfrak{N}_p} \cap H)K/K = (P \cap H)K/K = (G_{f_1(p)\mathfrak{N}_p} \cap H)K/K$. Thus if $G_{f_2(p)\mathfrak{N}_p}$ covers H/K, then $G_{f_1(p)\mathfrak{N}_p}$ covers H/K. The reverse inclusion is obvious because $G_{f_1(p)\mathfrak{N}_p} \leq G_{f_2(p)\mathfrak{N}_p}$.

In view of lemma 3, if f is an integrated H-function locally defining a Fitting class $\mathfrak{F}$, then a f-central (f-eccentric) chief factor of G will be called $\mathfrak{F}$-central in G ($\mathfrak{F}$-eccentric in G, respectively).

Lemma 5. *Let $\mathfrak{F} = SLR(f) \neq \varnothing$ where f is a full H-function. Then for any Fitting class $\mathfrak{X} \subseteq \cap_{p\in\pi(\mathfrak{F})} f(p)$ we have $C_G(G_{\mathfrak{F}}/G_{\mathfrak{X}}) \leq G_{\mathfrak{F}}$.*

Proof. We assume by contradiction, that $C = C_G(G_{\mathfrak{F}}/G_{\mathfrak{X}})$ is not equal to $G_{\mathfrak{F}}$. Since $C \cap G_{\mathfrak{F}} \lhd C$, we can construct a series $1 \trianglelefteq C \cap G_{\mathfrak{F}} \lhd K \trianglelefteq C \trianglelefteq G$ such that $K/C \cap G_{\mathfrak{F}}$ is a chief factor. If $K = C$ then obviously $K \cap G_{\mathfrak{F}} = C \cap G_{\mathfrak{F}}$. If $K < C$, then $K \cap G_{\mathfrak{F}} = C \cap G_{\mathfrak{F}}$ whereas the factor $K/C \cap G_{\mathfrak{F}}$ is chief. Therefore $K/C \cap G_{\mathfrak{F}} = K/K \cap G_{\mathfrak{F}} \cong KG_{\mathfrak{F}}/G_{\mathfrak{F}}$ and $K/K \cap G_{\mathfrak{F}}$ is a nontrivial elementary abelian p-group. Since $(K/K \cap G_{\mathfrak{F}})^{\mathfrak{A}} = 1$ by lemma 1.2 of [9], we have that $K^{\mathfrak{A}}(K \cap G_{\mathfrak{F}})/(K \cap G_{\mathfrak{F}})$ is an identity group. Hence, $K^{\mathfrak{A}} \subseteq K \cap G_{\mathfrak{F}}$. Since $K \subseteq C_G(G_{\mathfrak{F}}/G_{\mathfrak{X}})$, we have $K \subseteq C_G((K \cap G_{\mathfrak{F}})/G_{\mathfrak{X}})$. But $K^{\mathfrak{A}} \subseteq K \cap G_{\mathfrak{F}}$, and so $[K^{\mathfrak{A}}, K] \subseteq [K \cap G_{\mathfrak{F}}, K] \subseteq G_{\mathfrak{X}}$. Hence $K/G_{\mathfrak{X}}$ is a nilpotent group of class at most 2. Let $P/G_{\mathfrak{X}}$ be a non-identity normal Sylow p-subgroup of $K/G_{\mathfrak{X}}$. Since a Sylow p-subgroup covers $K/K \cap G_{\mathfrak{F}}$ we have $P(K \cap G_{\mathfrak{F}}) \supseteq K$. Hence, $PG_{\mathfrak{F}} = KG_{\mathfrak{F}}$. We show, that $P \in \mathfrak{F}$. We note that $G_{\mathfrak{X}} = P \cap G_{\mathfrak{X}} = P_{\mathfrak{X}}$ and $P/P_{\mathfrak{X}} \in \mathfrak{N}_p$. Let $r \in \pi$ and $r \neq p$, then $P/P_{\mathfrak{X}} \in \mathfrak{S}_{r'}$ and $P \in \mathfrak{X}\mathfrak{S}_{r'} \subseteq f(r)\mathfrak{S}_{r'}$. If $r = p$, then $P \in f(p)\mathfrak{N}_p = f(p) \subseteq f(p)\mathfrak{S}_{p'}$. Hence $P \in \bigcap_{p\in\pi} f(p)\mathfrak{S}_{p'} = \mathfrak{F}$. Thus $KG_{\mathfrak{F}} = PG_{\mathfrak{F}}$ is a normal $\mathfrak{F}$-subgroup of the group G. Hence $KG_{\mathfrak{F}} = G_{\mathfrak{F}}$. This contradicts to nontriviality of the factor $K/K \cap G_{\mathfrak{F}}$.

Corollary 2. *Let $\mathfrak{F} = SLR(f) \neq \varnothing$ where f is a constant full H-function. If V is an $\mathfrak{F}$-injector of a group G, then $V_{f(p)} = G_{f(p)}$ for all prime $p \in \pi$.*

Proof. Since $G_{\mathfrak{F}} \trianglelefteq V$ we have $G_{f(p)} = (G_{\mathfrak{F}})_{f(p)} = G_{\mathfrak{F}} \cap V_{f(p)}$. Consequently $[V_{f(p)}, G_{\mathfrak{F}}] \leq G_{f(p)}$ and therefore $V_{f(p)} \leq C_G(G_{\mathfrak{F}}/G_{f(p)})$. Since the function f is constant, $C_G(G_{\mathfrak{F}}/G_{f(p)}) \leq G_{\mathfrak{F}}$, by lemma 4. Hence $V_{f(p)} = G_{f(p)}$ for all primes $p \in \pi(\mathfrak{F})$.

Theorem 1. *Let $\mathfrak{F} = SLR(f) \neq \varnothing$ where f is a full constant H-function and $p \in \pi(\mathfrak{F})$. Then the following statements hold:*

1) $f(p)$-covered chief factors of a group G are exactly the chief factors which are covered by the $f(p)$-radical of G;

2) $f(p)$-injectors of a group G cover every $f(p)$-covered chief factor of G.

Proof. Statement 1 follows immediately from the definition of $f(p)$-covered chief factor and corollary 2.

Let V be a $\mathfrak{F}$-injector of G and $V_{f(p)}$ cover a chief p-factor H/K of G. By corollary 2, $V_{f(p)} = G_{f(p)}$ and $G_{f(p)} \leq V_1$ for some $f(p)$-injector V_1 of G. Hence V_1 covers H/K.

Theorem 2. *Let $\mathfrak{F} = SLR(f) \neq \varnothing$ where f is a full constant H-function. Then an $\mathfrak{F}$-injector of a group G covers each $\mathfrak{F}$-central chief factor of G and avoids each $\mathfrak{F}$-eccentric chief factor of G.*

Proof. Let H/K be an $\mathfrak{F}$-central chief factor and V be an $\mathfrak{F}$-injector of a group G. Then $G_{f(p)}$ covers H/K. But inclusions $G_{f(p)} \leq G_{\mathfrak{F}} \leq V$ hold, and consequently V covers H/K.

We assume that the $\mathfrak{F}$-injector V of the group G covers some chief p-factor H/K of G. Then a Sylow p-subgroup P of V also covers H/K. If p does not belong to $\pi(\mathfrak{F})$ then $V \in \mathfrak{F}$. So we obtain $P = (1)$. Consequently in this case H/K is trivial and we have a contradiction.

Let $p \in \pi(\mathfrak{F})$. Then $V \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'} = f(p)\mathfrak{S}_{p'}$. Hence, $V/V_{f(p)} \in \mathfrak{S}_{p'}$. Consequently $Syl_p(V) = Syl_p(V_{f(p)})$. Therefore $P \leq V_{f(p)}$. By corollary 2, $V_{f(p)} = G_{f(p)} = G_{f(p)\mathfrak{N}_p}$. Hence the chief p-factor H/K is covered by the subgroup $G_{f(p)\mathfrak{N}_p}$, i.e. it is $\mathfrak{F}$-central.

Резюме. Доказано, что если $\mathfrak{F}$ — полулокальный класс Фиттинга, то $\mathfrak{F}$-инъектор конечной разрешимой группы покрывает ее $\mathfrak{F}$-центральные и изолирует $\mathfrak{F}$-эксцентральные главные факторы.

References

[1] K.Doerk, T.Hawkes, *Finite Soluble Groups,* Berlin–New York, Walter de Gruyter, 1992.

[2] B.Hartley, *On Fischer's dualization of formation theory,* Proc. London Math. Soc. 3 (1969), 193–207.

[3] R.W.Carter, T.O.Hawkes, *The $\mathfrak{F}$-normalizers of a finite soluble group,* J.Algebra 5 (1967), 175–202.

[4] L.A.Shemetkov, *Factorizations of non-simple finite groups,* Algebra and Logic 15:6 (1976), 425–445.

[5] H.Lucke, *Zur Theorie der lokalerklauzbaren Fittingklassen,* Diplomarbeit. Mainz, 1985, 79 p.

[6] P.D'Arcy, *Locally defined Fitting classes,* J.Austral. Math. Soc. 20 (1975), 25–32.

[7] N.T.Vorob'ev, *A Hartley method for injectors,* VII Belarus Mat. Conf., 1 (1996), 98–99.

[8] N.T.Vorob'ev, *On local radical classes,* Voprosy algebry (Problems in Algebra) 2 (1986), 41–50 (Russian).

[9] L.A.Shemetkov, *Formations of finite groups,* Nauka, Moscow, 1978 (Russian).

[10] B.Fischer, W.Gaschütz, B.Hartley, *Injectoren endlicher auflösbarer Gruppen,* Math. Z. 102 (1967), 337–339.

Received June 3, 2000

Mathematics Department
Vitebsk State University
210036 Vitebsk
Belarus