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Gaschutz's local method in the theory of Fitting classes of finite soluble groups

N.T.Vorob'ev

Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

Introduction

The idea of localization is a leading one in the theory of groups. During the last three decades a series of deep and serious results was obtained in the theory of finite soluble groups thanks to the systematic development of the local method offered by Gaschutz [1] in 1963. It is shown in [2–7]. The concept of local formation became an important instrument in investigations.

From the second part of the sixties the investigations connected with radical classes (Fitting classes), i.e. group classes closed with respect to normal subgroups and their products, have become popular. The idea of studying such classes goes back to the large-scale program of structural analysis of finite groups introduced in 1938 by Fitting [8]. A bright result in the theory of finite soluble groups was obtained by Fischer, Gaschütz and Hartley [9] in 1967 who found an elegant generalization of Sylow's and Hall's fundamental theorems in terms of Fitting classes. Using the local Gaschütz method in the theory of formations, Shemetkov [10, 11] obtained general results on $\mathfrak{F}$-radicals where $\mathfrak{F}$ is a local Fitting formation.

A dualization of the Gaschütz local method, as a local method of study of finite soluble groups with the help of radicals and Fitting classes, was considered by Hartley [12] in 1969. But for a long time the Hartley's approach and its role in the theory of finite soluble groups have remained vague. There were some separate results of Hartley [12], D'Arcy [13], Schnakenberg [14], Beidleman and Brewster [15]. All this results are devoted to the study of some properties of injectors. In [16] Cossey specially noted the difficulty of the development of the local Hartley's approach in the theory of classes of finite soluble groups.

In this paper we give a short review of the main results of the author in the development of the local method and consider their application to the solution of problems of construction, classification of Fitting classes and description of injectors of finite soluble groups.

All groups considered are finite and soluble, unless otherwise stipulated. In the books [2, 4, 5] one can find all necessary definitions and notations which we do not mention. If $\mathfrak{H}$ is a class of groups, then $\pi(\mathfrak{H})$ denotes the set of prime divisors of orders of groups in $\mathfrak{H}$.

1. The foundations of the local method

The first section of the paper is auxiliary. Its main purpose is finding common regularities of the construction of Fitting classes by the local method.

We remind that the product of Fitting classes $\mathfrak{X}$ and $\mathfrak{Y}$ is the class $\mathfrak{X}\mathfrak{Y}$ of groups G such that $G/G_{\mathfrak{X}} \in \mathfrak{Y}$. It is well-known that $\mathfrak{X}\mathfrak{Y}$ is a Fitting class and the product of Fitting classes is associative.

(1.1) Definitions (it was proposed by L.A. Shemetkov). (a) A f unction $f : \mathbb{P} \to$\{Fitting classes\} is called a *Hartley function,* or shortly, a *H-function.*

(b) The support of the H-function f is defined thus: $\pi = \mathrm{Supp}(f) = \{p \in \mathbb{P} : f(p) \neq \varnothing\}$.

Let $LR(f)$ denote the Fitting class

$$\mathfrak{S}_\pi \cap (\cap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{\pi'}).$$

If $\pi = \varnothing$, then we shall suppose $LR(f) = (1)$ where (1) is a class of groups of order 1.

(c) A class $\mathfrak{F}$ is called a *local Fitting class* (or a *local radical class*) [13] if $\mathfrak{F} = LR(f)$ for some H-function f. In this case we say that $\mathfrak{F}$ is locally defined by a H-function f. We shall also suppose that an empty class is a local Fitting class by definition.

Many classes of groups which one can often come across in investigations are local Fitting classes. Such, for example, are: the class $\mathfrak{S}$ of all soluble groups, the class $\mathfrak{N}_\pi$ of all nilpotent π-groups, the class $\mathfrak{S}_\pi$ of all π-groups, the class $\mathfrak{S}_{p'}\mathfrak{N}_p$ of all p-nilpotent groups, the class $\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$ of all π-closed groups, the class of all φ-dispersive groups, the classes of groups in the forms $\mathfrak{X}\mathfrak{N}$ and $\mathfrak{X}\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$, where $\mathfrak{X}$ is a non-empty Fitting class, the class $K_\pi(\mathfrak{F}) = (G \in \mathfrak{S} :$ if $H \in Hall_\pi(G)$, then $H \in \mathfrak{F})$ [17], where $\mathfrak{F}$ is a local Fitting class.

The following fact was proved in [18].

(1.2) Theorem. *Every subgroup-closed Fitting class is local.*

(1.3) Remarks. Let f_λ be a H-function for $\lambda \in \Lambda$.

(a) $\cap\{LR(f_\lambda) : \lambda \in \Lambda\} = LR(\varphi)$, where $\varphi(p) = \cap\{f_\lambda(p) : \lambda \in \Lambda\}$ for all $p \in \mathbb{P}$.

(b) Hartley [12] considered the function $h : \Sigma \to \{\text{Fitting classes}\}$ where Σ is some subset of a partition of $\mathbb{P}$ and the Fitting class $\cap_{\pi\in\Sigma} h(\pi)\mathfrak{S}_{\pi'}\mathfrak{S}_\pi$ which is a local Fitting class in accordance with (1.1). We denote this class by $LH(h)$ and call it a local Hartley class (see Hartley, 1969, [12], p. 201). In particular, if $h(\pi) = \varnothing$ for some $\pi \in \Sigma$, then we suppose that $LH(h) = \varnothing$.

We produce a classification of H-functions of a Fitting class $\mathfrak{F}$ in the following way.

(1.4) Definitions. If $\mathfrak{F} = LR(f)$, then f is called:

(a) integrated, if $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$;

(b) full, if $f(p) = f(p)\mathfrak{N}_p$ for all $p \in \mathbb{P}$;

(c) full integrated, if f simultaneously is full and integrated.

In the construction of local Fitting classes the following fact [19] proved to be the primary key moment.

(1.5) Lemma. *Every local Fitting class is defined by a full integrated H-function.*

We note that this property is an analogy of the well-known local formation property obtained by Carter, Hawkes [20] and Schmid [21].

We call the Fitting classes product $\mathfrak{F}\mathfrak{H}$ a local product, if $\mathfrak{F}\mathfrak{H}$ is a local Fitting class.

The result by Gaschütz-Shemetkov [3, 22] in the theory of formations concerning description of local functions of products of two local formations was used as an orientation for the following investigations. It was found that the analogy of this result is valid for Fitting classes [23]. Moreover we describe [23] general regularities of the construction of local products of Fitting classes with the help of H-functions (for comparison see the well-known results by Shemetkov L.A. [24] in the formation theory).

(1.6) Theorem. *Let $\mathfrak{F} = LR(f)$ and $\mathfrak{H} = LR(h)$, where f and h are full integrated H-functions, $\pi = \mathrm{Supp}(h)$. Then $\mathfrak{F}\mathfrak{H} = LR(\varphi)$, where φ is a full integrated H-function such that for every prime p*

$$\varphi(p) = \begin{cases} \mathfrak{F}h(p), & \text{if } p \in \pi; \\ f(p), & \text{if } p \in \pi'. \end{cases}$$

We also studied local Fitting classes represented in the form of two Fitting classes product one of which is local. For that purpose we use in [23] a generalization of the locality concept, the idea of which was offered by Shemetkov L.A. [5,24]. We denote by $\mathrm{Fit}\,\mathfrak{X}$ (by $l\,\mathrm{Fit}\,\mathfrak{X}$) the Fitting class (respectively local Fitting class) generated by a set of groups $\mathfrak{X}$.

(1.7) Definition (proposed by L.A.Shemetkov). Let $\varnothing \neq \omega \subset \mathbb{P}$. A Fitting class $\mathfrak{F}$ is called ω-local if $l\,\mathrm{Fit}\,\mathfrak{F} \subseteq \mathfrak{F}\mathfrak{S}_{\omega'}$.

(1.8) Theorem. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be Fitting classes and $\mathfrak{H} = LR(h)$, $\omega = Supp(h)$. Then $\mathfrak{F}\mathfrak{H}$ is local if and only if $\mathfrak{F}$ is ω'-local.*

Let $\{\mathfrak{F}_\lambda : \lambda \in \Lambda\}$ be a set of Fitting classes. By the definition, their local join, denoted by $\vee_l \mathfrak{F}_\lambda$, is the class $l\,\mathrm{Fit}(\cup_{\lambda\in\Lambda}\mathfrak{F}_\lambda)$. Using the obtained properties of local products and the operation of the Fitting classes local join we construct series of examples of the local Fitting classes which consist of all groups represented in the form of the product of their radicals. For example,

$$(\mathfrak{F}\mathfrak{H})^n \vee_l (\mathfrak{H}\mathfrak{F})^n = \{G : G = G_{(\mathfrak{F}\mathfrak{H})^n} \cdot G_{\mathfrak{H}\mathfrak{F})^n}\},$$

where $\mathfrak{F}$, $\mathfrak{H}$ are Q-closed local Fitting classes, $\mathfrak{F} \cap \mathfrak{H} = (1)$, $n \in \mathbb{N}$.

Further we shall consider general regularities of the local Fitting classes constructions by means of the H-functions defined by the properties of the radicals direct products. In contrast with the properties of the direct products of the residuals in the theory of formations [25] for a Fitting class $\mathfrak{F}$, the $\mathfrak{F}$-radical of two groups direct product does not coincide with a direct product of $\mathfrak{F}$-radicals of these groups. Therefore Lockett defined [26] the class $\mathfrak{F}^\star$ which is the smallest Fitting class containing $\mathfrak{F}$ such that for any groups G and H we have $(G \times H)_{\mathfrak{F}^\star} = G_{\mathfrak{F}^\star} \times H_{\mathfrak{F}^\star}$, and the Fitting class $\mathfrak{F}_\star$ which is an intersection of all those Fitting classes $\mathfrak{X}$ for which $\mathfrak{X}^\star = \mathfrak{F}^\star$. In the theory of group classes the Lockett's operators were found very useful in many investigations (see chapters X–XI in [2]) thanks to a number of remarkable properties and the fact that the Fitting classes family $\mathfrak{F} = \mathfrak{F}^\star$ (Lockett's classes) is extensive: Fitting classes closed at least with respect to one of the operations S_F (in particular S), Q, R_0 are Lockett's classes.

(1.9) Definition. Let f be a H-function. We define [27] the H-functions $f_\star$ and $f^\star$ in the following way: $f_\star(p) = (f(p))_\star$ and $f^\star(p) = (f(p))^\star$ for all $p \in \mathbb{P}$.

The fact that every local Fitting class is a Lockett class and can be defined with the help of Lockett's operators was a useful observation [27] for the further research. Namely, we proved [27]:

(1.10) Theorem. If $\mathfrak{F} = LR(f)$, then $\mathfrak{F} = LR(f_\star) = LR(f^\star) = \mathfrak{F}^\star$.

Let f be a H-function and $SLR(f) = \cap_{p\in\pi} f(p)\mathfrak{S}_{p'}$, where $\pi = Supp(f)$ (we suppose that $SLR(f) \neq (1)$ if $\pi \neq \varnothing$).

(1.11) Definition. If $\mathfrak{F} = SLR(f)$ we say that $\mathfrak{F}$ is *semilocally defined by a H-function f*.

Obviously, $\mathfrak{F} = SLR(f)$ for every Fitting class $\mathfrak{F}$ and a H-function f such that $f(p) = \mathfrak{F}$ for all $p \in \mathbb{P}$. There exist the non-local Fitting classes which are semilocally defined by H-functions f with $\varnothing \subset Supp(f) \subset \mathbb{P}$. For example [19], if $\mathfrak{F} = \{G : Soc_3(G) \subseteq Z(G)\}$ then the class $\mathfrak{H} = \{G : \mathfrak{F}\text{-injectors of } G \text{ have } 2'\text{-index in } G\}$ is a non-local Fitting class semilocally defined by a H-function f such that $f(p) = Fit\{G \in \mathfrak{H} : G \cong O^{p'}(H) \text{ for } H \in \mathfrak{H}\}$ for all $p \in \pi(\mathfrak{H})$.

In general, a Fitting class can possess many local and semilocal definitions. Therefore, following Shemetkov's terminology [4], we shall regard any H-function set Ω as partially ordered with the relation $\leq$ which is given in the following way: if $f_1, f_2 \in \Omega$, then $f_1 \leq f_2$ if and only if $f_1(p) \subseteq f_2(p)$ for all $p \in \mathbb{P}$. A problem of minimal and maximal elements in Ω description appears.

Minimal H-functions are the most useful for our further research. Let Ω be the set of all H-functions which locally define $\mathfrak{F}$. By 1.3 (a), Ω always has a minimal element, a local H-functlon f defined as follows: $\underline{f}(p) = \cap\{g(p) : \mathfrak{F} = LR(g)\}$, $p \in \mathbb{P}$.

The notation $\mathfrak{F} = LR(\underline{f})$ will always mean that $\underline{f}$ is the minimal local definition of $\mathfrak{F}$.

Minimal semilocal H-functions of Fitting classes is similarly defined. The notation $\mathfrak{F} = SLR(\underline{f})$ will always mean that $\underline{f}$ is the minimal semilocal definition of $\mathfrak{F}$.

In [19] we proved

(1.12) Proposition. *(a) If $\mathfrak{F} = SLR(\underline{f})$ and $p \in \pi(\mathfrak{F})$ then $\underline{f}(p) = Fit\{G \in \mathfrak{F} : G \cong H^{\mathfrak{S}_p}$ for some $H \in \mathfrak{F}\}$;*

(b) If $\mathfrak{F} = LR(\underline{f})$ and $p \in \pi(\mathfrak{F})$ then $\underline{f}(p) = \mathrm{Fit}\{G \in \mathfrak{F} : G \cong H^{\mathfrak{S}_p\mathfrak{N}_{p'}}$ for some $H \in \mathfrak{F}\}$;

(c) If $\mathfrak{F} = LR(\underline{f})$ and $\mathfrak{H} = LR(\underline{h}$, then $\mathfrak{F} \subseteq \mathfrak{H}$ if and only if $\underline{f} \leq \underline{h}$.

In [27] we obtained a description of minimal and maximal H-functions in terms of Lockett operators * and $_\star$.

(1.13) Theorem. *Every local Fitting class is defined by the unique maximal integrated H-function $\overline{f}$ and the unique minimal local H-function $\underline{f}$ such that $\overline{f}(p) = \overline{f}(p)\mathfrak{N}_p \subseteq \mathfrak{F}$ and $\overline{f}(p)$ is a Lockett class and*

$$\underline{f}(p) = \cap\{\mathfrak{X} : \mathfrak{X} \text{ is a Fitting class and } \mathfrak{X}^\star = \underline{f}^\star(p)\} \text{ for all } p \in \pi(\mathfrak{F}).$$

This result is interesting from the point of view that it is dual to the well-known results in the formation theory [28–35].

(1.14) Remark. The duality of (1.10) and (1.13) in the formations theory of finite groups is valid. In particular, if f is a local definition of a non-soluble formation $\mathfrak{F}$, then $\mathfrak{F}$ is also defined by functions f^0 and f_0 such that $f^0(p) = (f(p))^0$ and $f_0(p) = (f(p))_0$ for all $p \in \pi(\mathfrak{F})$ (by Doerk-Hawkes [25] $(f(p))^0$ is the smallest formation containing $f(p)$ such that $(G \times H)^{(f(p))^0} = G^{(f(p))^0} \times H^{(f(p))^0}$ for all groups G, H and $(f(p))_0 = \cap\{\mathfrak{X} : \mathfrak{X}$ is a formation and $\mathfrak{X}^0 = (f(p))^0\})$.

2. The problems of the structure of Fitting classes

The questions that we solve in this section using the local method are connected with the well-known *Lockett conjecture* in the theory of group classes.

We remind that a Fitting class is called normal if for any group G its $\mathfrak{F}$-radical is a $\mathfrak{F}$-maximal subgroup in G. In the theory of Fitting classes the following Lockett's characterization [26] of normal Fitting classes is known: a Fitting class $\mathfrak{F}$ is normal if and only if $\mathfrak{F}^\star = \mathfrak{S}$. Hence, using Lockett's operators properties [26], it is easy to obtain the relations $\mathfrak{F}_\star \subseteq \mathfrak{F} \subseteq \mathfrak{F}^\star$ and $\mathfrak{F}_\star \subseteq \mathfrak{F}_\star \cap \mathfrak{X} \subseteq \mathfrak{F}^\star$ for any normal Fitting class $\mathfrak{X}$. Lockett formulated the following problem.

(2.1) Problem (Lockett, 1974). *Given a Fitting class, does there exist a normal Fitting class $\mathfrak{X}$ such that $\mathfrak{F} = \mathfrak{F}^\star \cap \mathfrak{X}$?*

Bryce, Cossey [36] and Berger [37] have confirmed the correctness of the Lockett conjecture for primitive saturated formations (local subgroup-closed Fitting classes). Later on Berger and Cossey [38] constructed an example of a non-local Fitting class for which the Lockett conjecture was not correct. In 1979 Bedleman and Hauck [39] confirmed the correctness of the Lockett conjecture for two types of local Fitting classes: $\mathfrak{X}\mathfrak{N}$ and $\mathfrak{X}\mathfrak{S}_\pi\mathfrak{S}_{\pi'}$. We showed

[40, 41] that the Lockett conjecture is valid for any local Fitting class. Namely, we proved the following result.

(2.2) Theorem. *If $\mathfrak{F} = LR(f)$, then $\mathfrak{F}_\star = \mathfrak{F}^\star \cap \mathfrak{S}_\star$ where $\mathfrak{S}_\star$ is the minimal normal Fitting class.*

(2.3) Remark. For arbitrary local Fitting class a generalized variant of the Lockett conjecture is valid. We say that a Fitting class $\mathfrak{F}$ satisfies the Lockett conjecture with respect to a Fitting class $\mathfrak{H}$ if $\mathfrak{F} \subseteq \mathfrak{H}$ and $\mathfrak{F}_\star = \mathfrak{F}^\star \cap \mathfrak{H}_\star$ (see X.1.19 [2]. We note that this equality was established earlier only for some special cases. In particular, it was established by Brison [42] for $\mathfrak{F} = \mathfrak{S}_\pi (\pi \subseteq \mathbb{P})$, it was done by Bryce and Cossey [36] for local subgroup-closed Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$, it was done by Doerk and Hawkes (see X.6.10 in [2]) for a local Fitting class $\mathfrak{F}$ in the form $\mathfrak{X}(\cap_{\lambda\in\Lambda} \mathfrak{S}_{\pi_\lambda} \mathfrak{S}'_{\pi_\lambda})$ and a local subgroup-closed Fitting class $\mathfrak{H}$.

In 1996 Gallego [43] confirmed the correctness of the Lockett conjecture for non-soluble local Fitting classes.

The corollaries from our results connected with the lattices of Fitting classes of Lockett's sections and subsections are of an independent interest. We remind that if $\mathfrak{F}$ is a Fitting class then $Locksec\mathfrak{F} = \{\mathfrak{X} : \mathfrak{X}$ is a Fitting class and $\mathfrak{X}^\star = \mathfrak{F}^\star\}$ is called a Lockett section of $\mathfrak{F}$, and $Locksub\mathfrak{F} = \{\mathfrak{D} : \mathfrak{D} \in Locksec\mathfrak{F}$ and $\mathfrak{D} \subseteq \mathfrak{F}\}$ is called a Lockett subsection of $\mathfrak{F}$. Taking into account X.4.16, 4.19 in [2], we obtain

(2.4) Corollaries. *(a) For any local Fitting subclass $\mathfrak{F}$ of a Fitting class $\mathfrak{H}$ the lattice of Fitting classes $Locksub(\mathfrak{F} \vee \mathfrak{H}_\star)$ is complete, modular and atomic.*

(b) The lattice of normal Fitting subclasses of a normal Fitting class generated by the local Fitting class is complete, modular and atomic.

We note [44] that an analogy of the Bryant-Bryce-Hartley theorem on the finiteness of the lattice of subformations of one-generated formations is true for normal Fitting classes. Namely, if $\mathfrak{F} = \operatorname{Fit} G$, then the lattice of all normal Fitting subclasses of a normal Fitting class generated by $\mathfrak{F}$ is finite. In particular, the lattice of all normal Fitting subclasses of a one-generated normal Fitting class is finite.

Using (2.2), we also solve the following two Lausch problems from "Kourovka Notebook" [45].

(2.5) Problem (Lausch, [45], 8.30). *Let $\mathfrak{X}$, $\mathfrak{Y}$ be Fitting classes of soluble groups which satisfy the Lockett condition, i.e. $\mathfrak{X} \cap \mathfrak{S}_\star = \mathfrak{X}_\star$, $\mathfrak{Y} \cap \mathfrak{S}_\star = \mathfrak{Y}_\star$ where $\mathfrak{S}$ denotes the Fitting class of all soluble groups and the lower star the bottom group of the Lockett section determined by the given Fitting class. Does $\mathfrak{X} \cap \mathfrak{Y}$ satisfy the Lockett condition?*

(2.6) Problem (Lausch, [45], 9.18). *Let $\mathfrak{S}_\star$ be the smallest normal Fitting class. Are there Fitting classes which are maximal in $\mathfrak{S}_\star$ (with respect to inclusion)?*

We say that for a Fitting class $\mathfrak{F}$ the Lockett condition is realized in a Fitting class $\mathfrak{X}$ if $\mathfrak{F} \subseteq \mathfrak{X}$ and $\mathfrak{F}_\star = \mathfrak{F} \cap \mathfrak{X}_\star$. We note that the Fitting class $\operatorname{Fit} S_3$ generated by the symmetric permutation group S_3 is an example of this class.

(2.7) Theorem. *(a) If $\mathfrak{F}$, $\mathfrak{H}$ are local Fitting classes with the Lockett condition in a Fitting class $\mathfrak{X} \subseteq \mathfrak{S}$, then $\mathfrak{F} \cap \mathfrak{H}$ is a Fitting class with the Lockett condition in $\mathfrak{X}$;*

(b) there exist non-soluble Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ such that $\mathfrak{F}_\star = \mathfrak{F} \cap \mathfrak{E}_\star$ and $\mathfrak{H}_\star = \mathfrak{H} \cap \mathfrak{E}_\star$ but $(\mathfrak{F} \cap \mathfrak{H}) \neq (\mathfrak{F} \cap \mathfrak{H}) \cap \mathfrak{E}_\star$ ($\mathfrak{E}$ is a class of all finite soluble and non-soluble groups).

From (2.7) (a) for $\mathfrak{X} = \mathfrak{S}$ we have [40] a positive answer to the question (2.5) for local Fitting classes, from (2.7) (b) we have a negative answer to this question in the class of all soluble and non-soluble finite groups.

A Fitting class $\mathfrak{F}$ is called the maximal Fitting subclass of a Fitting class $\mathfrak{X}$ if $\mathfrak{F} \subset \mathfrak{X}$ and from $\mathfrak{F} \subseteq \mathfrak{M} \subseteq \mathfrak{X}$ where $\mathfrak{M}$ is a Fitting class it always follows that $\mathfrak{M} \in \{\mathfrak{F}, \mathfrak{X}\}$. By Cossey's result [46], if $\mathfrak{F}$ is a maximal Fitting subclass in $\mathfrak{S}$, then $\mathfrak{F}$ is normal. The following theorem [47] gives a negative answer to (2.6).

(2.8) Theorem. *There are no any maximal Fitting subclasses in $\mathfrak{S}_\star$.*

(2.9) Remarks. Using the characterizations of normal Fitting classes by Lockett [26] and Cossey [46] we deduce that if $\mathfrak{F}$ is a maximal Fitting subclass in some normal Fitting class $\mathfrak{X}$ then $\mathfrak{F}$ is normal.

3. The problems of factorizations

A series of well-known results in the theory of formations by Shemetkov L.A. [22, 24], Skiba A.N. [48], in the theory of normal Fitting classes by Beidleman [49], in the theory of Lockett classes by Hauck [50], Brison [42] were devoted to the research of class products.

We note the following problem.

(3.1) Problem ([45], 11. 25 (a)). *Are there any local products (different from classes $\mathfrak{S}$ and $\mathfrak{E}$) of Fitting classes where each of them is non-local and is not a formation?*

If $\mathfrak{X}$ is a non-trivial normal Fitting class then according to Cossey's theorem [46] $\mathfrak{S} = \mathfrak{X}\mathfrak{S}_\star$ where $\mathfrak{X}$ and $\mathfrak{S}_\star$ are non-local and it stimulated the formulation of this question. At first, using the Fitting class conctruction proposed by Berger and Cossey [38], we give [51] a positive answer to this question. In [19] we proved that a Fitting class $\mathfrak{F}$ coincides with $SLR(f)$ for some H-function f with $\varnothing \subset \pi = Supp(f)$ if and only if $\mathfrak{F}\mathfrak{S}_{\pi'} = \mathfrak{F}$.

(3.2) Theorem. *There exists a continual set of local Fitting classes $\mathfrak{F} \neq \mathfrak{S}$ such that $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2$, where $\mathfrak{F}_i$ is semilocally defined by a H-function f_i with $\varnothing \subset \mathrm{Supp}(f) \subset \mathbb{P}$ and $\mathfrak{F}_i$ is not a formation $(i = 1, 2)$.*

The following dual problem in the theory of formations is natural.

(3.3) Problem (L.A.Shemetkov, A.N.Skiba, [45], 9.58). *Are there local products of non-local formations of finite groups?*

We found a number of examples (in $\mathfrak{S}$) of local formations representable in the form of the two non-local formation product. For this purpose we use the concept of a functor which was introduced by Barnes and Kegel [52]. We remind that a mapping τ of $\mathfrak{S}$ into a set of group classes is called a functor if for any group G the following conditions are satisfied:

1) $\tau(G)$ is a class of conjugate subgroups of G;

2) $U^\varphi \in \tau(G^\varphi)$ for any group $U \in \tau(G)$ and any homomorphism φ of G.

We denote by τ_π a functor such that $\tau_\pi(G)$ is a class of conjugate Hall's π-subgroups in G. Let $l\,\mathrm{form}\,\mathfrak{X}$ be a local formation generated by $\mathfrak{X}$, and let $\mathfrak{A}$ be a formation of all abellian groups. In particular, we proved [53, 54] the following result.

(3.4.) Theorem. *Let $\varnothing \subset \pi \subset \mathbb{P}$ and $\mathfrak{F} = \mathfrak{S}_\pi\mathfrak{A}_{\pi'}$. Then the following statements are satisfied:*

1) If $\mathfrak{X}$ is a non-empty formation, τ is a functor and $\mathfrak{H} = \{G \mid \tau(G) \in \mathfrak{S}_\pi\mathfrak{X}\}$ then the formation $\mathfrak{F}\mathfrak{H}$ is local, $\mathfrak{F}$ is non-local and $\mathfrak{F}\mathfrak{H} = (l\,\mathrm{form}\,\mathfrak{F})\mathfrak{H}$;

2) If $\mathfrak{H} = \{G|\tau_\pi(G) \in \mathfrak{A}$ then the formation $\mathfrak{F}\mathfrak{H} = (l\,\mathrm{form}\,\mathfrak{F})\mathfrak{H}$ is local but $\mathfrak{F}$ and $\mathfrak{H}$ are non-local formations.

From (3.4) it follows that there exists a continual set of local products of non-local formations $\mathfrak{F}$ and $\mathfrak{H}$ such that $\mathfrak{F}\mathfrak{H} = (l\,\mathrm{form}\,\mathfrak{F})\mathfrak{H}$ and there exists a continual set of non-local formation products $\mathfrak{F}\mathfrak{H}$ such that $\mathfrak{F}\mathfrak{H} = (l\,\mathrm{form}\,\mathfrak{F})\mathfrak{H}$.

4. The problems of classification

We use the multiple locality conception proposed by A.N.Skiba [6].

(4.1) Definition. *Every Fitting class is 0-multiply local. We call a Fitting class $\mathfrak{F}$:*

(1) n-multiply local ($n \in \mathbb{N}$) if $\mathfrak{F}$ is defined by a H-function such that all its non-empty values are $(n-1)$-multiply local;

(2) totally local or primitive if $\mathfrak{F}$ is n-multiply local for all $n \in \mathbb{N}$.

Hawkes proved [55] that every primitive saturated formation (or totally local formation by terminology of A.N.Skiba [6]) is a subgroup-closed Fitting formation, and posed the following problem.

(4.2) Problem (Hawkes, [55]). *Are the primitive saturated formations exactly subgroup-closed Fitting formations?*

Bryce and Cossey [56] confirmed this Hawkes conjecture. L.A. Shemetkov proposed the following analogy of the Hawkes problem.

(4.3) Problem. *Are the totally local Fitting classes exactly subgroup-closed Fitting classes?*

We denote by $S\,\mathrm{Fit}\,\mathfrak{X}$ a subgroup-closed Fitting class generated by a set of groups $\mathfrak{X}$. In [57] we classified the subgroup-closed Fitting classes by means of Hartley functions and give [57] a positive answer to (4.3).

(4.4) Theorem [57]. *Let $\mathfrak{F}$ be a non-empty Fitting class.*

(a) $\mathfrak{F}$ is subgroup-closed if and only if $\mathfrak{F} = LR(f)$ for a H-function f such that

$$f(p) = \begin{cases} (S\,\mathrm{Fit}\{G \in \mathfrak{F} : G = O^{p'}(G)\})\mathfrak{N}_p, & \text{if } p \in \pi(\mathfrak{F}) \\ \varnothing, & \text{if } p \in \pi'(\mathfrak{F}). \end{cases}$$

(b) $\mathfrak{F}$ is totally local if and only if $\mathfrak{F}$ is subgroup-closed.

We also describe the procedure of construction of the family of non-subgroup-closed local Fitting classes and show that theorem (4.4) is not valid in an unsoluble case.

Further we consider the classification of local Fitting classes using the Frattini duality. The idea of investigation of the Frattini duality goes back to the works by Ito [58] and Gaschütz [59] in which they studied the subgroup $\Psi_0(G)$ of G, the subgroup generated of all minimal subgroups of G.

By the famous Gaschütz-Lubeseder-Schmid theorem [60, 61], a non-empty formation $\mathfrak{F}$ is local if and only if $\mathfrak{F}$ is E_ϕ-closed. The operator E_ϕ is called the Frattini operator and is defined in the following way: $E_\phi\mathfrak{X} = (G : \exists N \trianglelefteq G,\ N \subseteq \Phi(G) \text{ and } G/N \in \mathfrak{X})$. It orients towards searching operators which are dual to the Frattini operator and are useful in the theory of Fitting classes. Such operators were introduced by Doerk and Hauck [62, 63]. But it was found that an E^{ψ_0}-closed Fitting class $\mathfrak{F}(E^{\psi_0}\mathfrak{F} = \{G : \exists K \lhd\lhd G,\ K \in \mathfrak{F} \text{ and } \Psi_0(G) \subseteq K\})$ is exactly a class of π-groups ($\pi \subseteq \mathbb{P}$), and therefore for the characterization of Fitting classes Doerk and Hauck [63] proposed the use of the Frattini duality in the following variant. Let τ be a closure operator and $\Psi_\tau(G)$ be the least normal subgroup of a group G such that $\tau(\Psi_\tau(G) \cap M) \supseteq \tau(M)$ for all $M \lhd\lhd G$.

A Fitting class $\mathfrak{F}$ is called E^{Ψ_τ}-closed or τ-saturated if from $\Psi_\tau(G) \in \mathfrak{F}$ it always follows that $G \in \mathfrak{F}$.

The following problem of the characterization of τ-saturated Fitting classes was posed by Doerk and Hawkes.

(4.5) Problem (Doerk, Hawkes, [2], p.829) *Given a closure operation τ such that $S_n \leq \tau$ in the universe $\mathfrak{S}$, which Fitting classes are τ-saturated?*

Let $m \in \mathbb{N}$ and let τ_m be an operator comparing for each class of groups $\mathfrak{X}$ the intersection of all those m-multiply local Fitting classes which are formations containing $\mathfrak{X}$. The following theorem [64] gives an answer to (4.5) for a countable set of Fitting classes and is a characterization of the local Fitting classes by means of formations.

(4.6) Theorem. *Let $\mathfrak{F}$ be a m-multlply local Fitting class. $\mathfrak{F}$ is a formation if and only if $\mathfrak{F}$ is τ_m-saturated.*

5. The local method for injectors

In this section we consider characterizations of injectors in groups for local Fitting classes. For the first time this question was considered for concrete cases by Hartley in [12].

We remind that if $\mathfrak{F}$ is a Fitting class then a subgroup V of a group G is called its $\mathfrak{F}$-injector [2] if $V \cap N$ is $\mathfrak{F}$-maximal in N for any normal subgroup N of G.

The elegant characterization of injectors for the local radical class of nilpotent groups ($\mathfrak{N}$-injectors) was obtained by Fischer [65] who proved that $\mathfrak{N}$-injectors of a group are exactly all those its $\mathfrak{N}$-maximal subgroups which contain a Fitting radical of this group.

The examples show that $\mathfrak{F}$-injectors characterization which is analogous to the Fischer's characterization, for arbitrary local Fitting classes is not realized. In this connection the search of such characterization can be realized for some families of local Fitting classes only.

Let f be a H-function, $\pi = Supp(f) \neq \varnothing$ and $G_\mathfrak{F}$ be a $\mathfrak{F}$-radical of G. We call the H-function f constant if $f(p) = f(q)$ for all $p, q \in \pi$.

(5.1) Theorem [66]. *Let $\mathfrak{F} = SLR(f)$ for some full integrated constant H-function and $\pi = Supp(f) \neq \varnothing$. A subgroup V is an $\mathfrak{F}$-injector of a group G if and only if $V/G_\mathfrak{F}$ is a Hall π'-subgroup of $G/G_\mathfrak{F}$.*

We note that $\mathfrak{F}$-injectors in (5.1) are exactly $\mathfrak{F}$-maximal subgroups of G containing $G_\mathfrak{F}$.

The analogous $\mathfrak{H}$-injectors characterization was obtained by Hartley [12] for a special case $\mathfrak{H} = \mathfrak{X}\mathfrak{N}$ as a generalization of Fischer's result [65]. We consider the characterization of injectors for local Hartley classes in a general case.

Let $\Sigma = (\pi_\lambda : \lambda \in \Lambda)$, where $\pi_{\lambda_1} \cap \pi_{\lambda_2} = \varnothing$ for $\lambda_1 \neq \lambda_2$ and $\mathbb{P} = \cup_{\lambda \in \Lambda} \pi_\lambda$. Let $\mathfrak{H} = LH(h)$ be a local Hartley class (see (1.3) above). The function h defining $\mathfrak{H}$ can be always taken such that $h(\pi_\lambda) \subseteq \mathfrak{H}$ and $h(\pi_\mu) \subseteq h(\pi_\lambda)\mathfrak{S}'_{\pi_\lambda}$, for all different λ and μ from Λ such that $h(\pi_\mu)$, $h(\pi_\lambda)$ are non empty Fitting classes. Let G be a group and G_h be a product of $h(\pi_\lambda)$-radicals of G for all $\lambda \in \Lambda$ such that $h(\pi_\lambda) \neq \varnothing$. In this terminology we proved [66] the following theorem.

(5.3) Theorem. *Let $\mathfrak{H} = LH(h)$ be a local Hartley class.*

(1) V is a $\mathfrak{H}$-injector of G if and only if V/G_h is a $\mathfrak{D}$-injector of G/G_h, where $\mathfrak{D} = \cap_{\lambda \in \Lambda} \mathfrak{S}_{\pi'_\lambda}\mathfrak{S}_{\pi_\lambda}$.

(2) The $\mathfrak{H}$-injectors of G are exactly all those $\mathfrak{H}$-maximal subgroups from G which contain its $\mathfrak{H}$-radical.

Резюме. Дается обзор результатов по применению локального метода Гашюца в теории классов Фиттинга конечных разрешимых групп.

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Vitebsk State University
210036 Vitebsk, Belarus

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