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Some results in a class of locally finite groups

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

The Frattini subgroup $\phi(G)$ of a group G is defined to be the intersection of G and its maximal subgroups. The following result for finite groups is well known:

Theorem A. *If G is a finite group, then $\phi(G)$ is nilpotent and the following conditions are pairwise equivalent:*

(i) G is nilpotent.
(ii) $G/\phi(G)$ is nilpotent.
(iii) $G' \leq \phi(G)$.
(iv) Every maximal subgroup of G is normal in G.

The situation is not longer true in infinite groups, mainly due to the fact of G having insufficient maximal subgroups or even none at all. It is clear that in these cases, condition (iii) in Theorem A is a very weak property. Let G be the standard wreath product of Prüfer groups of type p^∞ and q^∞. It is easy to show that $G = \phi(G)$. However G is not locally nilpotent if $p \neq q$. Thus $G' \leq \phi(G)$ does not imply local nilpotence and hence nilpotence.

In the first part of this survey we use the Frattini-like subgroup $\mu(G)$ introduced by Tomkinson in 1973 [9] to extend the above result to a class of locally finite-soluble groups.

On the other hand, the concepts of formations and Fitting classes were introduced in finite soluble groups by Gaschütz and by Fischer, Gaschütz and Hartley. Both theories form an important and a well-established part of the theory of finite soluble groups (see [6]).

Probably one of the most celebrated results connecting these two theories is a theorem of Bryce and Cossey establishing that a subgroup-closed Fitting formation of finite soluble groups is saturated ([4]).

The above theories have been successfully extended to various classes of infinite groups, specially locally finite-soluble, and there is a vast literature on the subject (see [5]). The definition of saturated formation has usually been done in a local way, mainly due to the fact that an infinite group may have a few maximal subgroups while in the case of Fitting classes, and working on a fixed class of infinite groups, it is possible to use several definitions. This is due to the fact that subnormality condition can be replaced by seriality, ascendancy or descendancy.

In this note we study saturated formations and Fitting classes based on certain infinite groups. These groups are elements of the class $c\overline{\mathfrak{L}}$ of radical locally finite groups satisfying $min-p$ for every prime p. Dixon has developed in this class a satisfactory formation theory and interesting results on Fitting classes (see [5]). He uses the more general definition of Fitting classes, depending on the concept of serial subgroup, although in this universe serial subgroups are in fact ascendant.

We use standard notation and terminology. It was taken from [5], [7].

Let M be a subgroup of a group G and consider the properly ascending chains

$$M = M_0 < M_1 < \ldots < M_\alpha = G$$

from M to G. Define $m(M)$ to be the least upper bound of the types α of all such chains.

Clearly $m(M) = 1$ if and only if M is a maximal subgroup of G.

A proper subgroup M of G is said to be a major subgroup of G if $m(U) = m(M)$ whenever $M \leq U \leq G$.

The intersection of all major subgroups of G is denoted by $\mu(G)$, which coincides with the Frattini subgroup of G, $\phi(G)$, if G is finitely generated. In [9, Proposition 2.1] Tomkinson shows that every proper subgroup is contained in a major subgroup of the group G. Hence $\mu(G)$ is always a proper subgroup of G.

Following [11], a group G is semiprimitive if it is the semidirect product $G = [D]M$, where M is a finite soluble group with trivial core and D is a divisible abelian group such that every proper M-invariant subgroup of D is finite.

The next result concerning major subgroups and semiprimitive groups appears to be crucial in the proof of one of our main theorems.

Theorem B [1; Theorem 1]. *Let G be a $c\overline{\mathfrak{L}}$-group and let M be a major subgroup of G. Then:*

(a) If M is a maximal subgroup of G, then $G/Core_G(M)$ is a finite soluble primitive group.

(b) If M is not a maximal subgroup of G, then $G/Core_G(M)$ is a semiprimitive group.

Let $\mathfrak{B}$ be the class of groups in which each proper subgroup has a proper normal closure. Then we have the following result:

Lemma 1 [2]. *Let G be a semiprimitive $\mathfrak{B}$-group. Then G is abelian.*

As a consequence of the lemma 1 and Theorem B we obtain:

Theorem 1 [2]. *Let G be a $c\overline{\mathfrak{L}}$-group in the class $\mathfrak{B}$. Let M be a major subgroup of G. Then:*

(i) If M is a maximal subgroup of G, then $G/Core_{(}M) \simeq C_p$ for some prime p.

(ii) If M is not a maximal subgroup of G, then $G/Corec(M) \simeq C_{p^\infty}$ for some prime p.

Now we can formulate one of our main results. It shows that in the class $c\overline{\mathfrak{L}}$, $\mathfrak{B}$-groups are to infinite groups as nilpotent groups to finite groups. Moreover we obtain a complete characterization of the $\mathfrak{B}$-groups G, through the Frattini-like subgroup $\mu(G)$, analogously to the one of Theorem A for nilpotent groups and the Frattini subgroup.

Theorem 2 ([2], [3]). *Let G be a group in the class $c\overline{\mathfrak{L}}$. The following statements are pairwise equivalent:*

(i) G is a $\mathfrak{B}$-group.

(ii) $G/\mu(G)$ is a $\mathfrak{B}$-group.

(iii) $G' \leq \mu(G)$.

(iv) Every major subgroup of G is a normal subgroup of G.

(v) G is a locally nilpotent group and each Sylow subgroup of G is nilpotent.

(vi) G is locally nilpotent and $G^0 \leq Z(G)$.

(vii) Every subgroup of G is descendant.

Notice that in the above theorem, condition (v) cannot be weakened to the one G is a locally nilpotent group as the following example shows. Let $G = D_{2^\infty}[C_{2^\infty}]\langle\alpha\rangle$ the dihedral group, where α acts on C_{2^∞} such that $x^\alpha = x^{-1}$ for each $x \in C_{2^\infty}$. Then G is a locally nilpotent group in the class $c\overline{\mathfrak{L}}$. However G does not belong to the class $\mathfrak{B}$. Notice that for instance $\langle\alpha\rangle$ is a major subgroup of the group G which is not normal in G.

As a consequence of the theorem we obtain the following result:

Theorem 3 [2]. *$\mathfrak{B}$ is a subgroup-closed $c\overline{\mathfrak{L}}$-formation.*

The above theorem will allow us to show the existence of the $\mathfrak{B}$-radical in every group G in the class $c\overline{\mathfrak{L}}$.

Sometimes when a finiteness restriction is placed upon a locally nilpotent group, the group is forced to become nilpotent. For instance, it easy to see that a finite $\mathfrak{B}$-group is a nilpotent group. Here is a less trivial result of the same kind.

Theorem 4 [2]. *Let G be a group in the class $c\overline{\mathfrak{L}}$. Assume G satisfies the minimal condition on subgroups. Then:*

G is a $\mathfrak{B}$-group if and only if G is a nilpotent group.

Our aim now is to continue the study of the class $\mathfrak{B}$ in the universe of $c\overline{\mathfrak{L}}$, as it enjoys very interesting properties of nilpotent type.

It is a well-known result that in a finite group G, the Frattini subgroup $\phi(G)$ is nilpotent. This result can be generalized in the class $c\overline{\mathfrak{L}}$.

Lemma 2 [2]. *Let G be a locally soluble group satisfying $min-p$ for every prime p. Then $\mu(G)$ is a residually finite subgroup of G whose Sylow subgroups are finite.*

Theorem 5 [2]. *Let G be a group in the class $c\overline{\mathfrak{L}}$. Then $\mu(G)$ is a $\mathfrak{B}$-group.*

Recall that the product of two normal nilpotent subgroups is nilpotent -this is Fitting's theorem [7; 5.2.8]. The corresponding statement holds for locally nilpotent groups and is of great importance. Moreover in any group G there is a unique maximal normal locally nilpotent subgroup (called the Hirsch-Plotkin radical) containing all normal locally nilpotent subgroups of G. In this sense we obtain analogous results in the class $\mathfrak{B}$ (in the universe $c\overline{\mathfrak{L}}$) defining the corresponding radical subgroup in this class, the subgroup $\delta(G)$. Furthermore using the characterization of Theorem 2, we can conclude that in fact, $\delta(G)$ is the Fitting subgroup of G. This fact is of considerable interest. Note that in general the Fitting subgroup in an infinite group gives little information about the structure of the group. However in this case it plays an important role as it inherits the properties of the $\mathfrak{B}$-radical.

Theorem 6 [2]. *Let G be a group in the class $c\overline{\mathfrak{L}}$. Assume H and K are two normal $\mathfrak{B}$-subgroups of G. Then the product HK is a $\mathfrak{B}$-group.*

Lemma 3 [2]. *Let G be a group in the class $c\overline{\mathfrak{L}}$. If G is a Chernikov group with $G = \bigcup_{i\in I} G_i$, such that $G_i \trianglelefteq G$ for each $i \in I$ and $\{G_i, i \in I\}$ is a totally ordered set, then G is a $\mathfrak{B}$-group.*

Theorem 7 ([2], [3]). *In any group $G \in c\overline{\mathfrak{L}}$, there is a unique maximal normal $\mathfrak{B}$-subgroup, denoted by $\delta(G)$, containing all normal $\mathfrak{B}$-subgroups of G. In fact, a subgroup H of G is descendant if and only if $H < \delta(G)$.*

This theorem yields that the arbitrary product of normal $\mathfrak{B}$-subgroups of a group also belongs to the class $\mathfrak{B}$. In particular, that the subgroup $\delta(G)$ is the product of all normal $\mathfrak{B}$-subgroups of the group G, that is, it is the $\mathfrak{B}$-radical of G.

This fact will allow us to prove that the class $\mathfrak{B}$ is in fact a Fitting class in the universe $c\overline{\mathfrak{L}}$.

A $c\overline{\mathfrak{L}}$-Fitting class, is a subclass $\mathfrak{F}$ of $c\overline{\mathfrak{L}}$ satisfying the properties:

(i) If $G \in \mathfrak{F}$ and H is a normal subgroup of G, then $H \in \mathfrak{F}$.

(ii) If $G = \langle H_i \mid i \in I \rangle \in c\overline{\mathfrak{L}}$ and for each $i \in I$ the subgroup H_i is a normal $\mathfrak{F}$-subgroup of G, then $G \in \mathfrak{F}$.

Notice that applying Theorem 3 the class $\mathfrak{B}$ (in the universe $c\overline{\mathfrak{L}}$) clearly satisfies condition (i) of the definition above. Moreover by Theorem 7 also condition (ii) is true. Therefore $\mathfrak{B}$ is a $c\overline{\mathfrak{L}}$-Fitting class.

Consequently $c\overline{\mathfrak{L}}$ is a subgroup-closed $c\overline{\mathfrak{L}}$-Fitting formation.

Bryce and Cossey show that a subgroup-closed Fitting formation of finite soluble groups is saturated (see [4]). Thus we can formulate the following question: Does the Bryce and Cossey's theorem hold in the class $c\overline{\mathfrak{L}}$?

The answer to this question is negative as the class $\mathfrak{B}$ shows.

In [1] the following definition is introduced.

A $c\overline{\mathfrak{L}}$-formation $\mathfrak{F}$ is said to be E_μ-closed if $\mathfrak{F}$ enjoys the following properties:

(a) A $c\overline{\mathfrak{L}}$-group G is in $\mathfrak{F}$ if and only if $G/\mu(G)$ is in $\mathfrak{F}$.

(b) A semiprimitive group G is an $\mathfrak{F}$-group if and only if it is the union of an ascending chain $\{Gi : i \in \mathcal{V}\}$ of finite $\mathfrak{F}$-subgroups.

Moreover the following is true.

Theorem C [l; Theorem A]. *Let $\mathfrak{F}$ be a $c\overline{\mathfrak{L}}$-formation. Then $\mathfrak{F}$ is E_μ-closed if and only if $\mathfrak{F}$ is a saturated $c\overline{\mathfrak{L}}$-formation.*

Applying this result, to see the class $\mathfrak{B}$ is not saturated, it will be enough to prove that in $c\overline{\mathfrak{L}}$, the class $\mathfrak{B}$ is not E_μ-closed.

Example *Consider $G = D_{2^\infty} = [G_{2^\infty}]\langle\alpha\rangle$ the aforementioned dihedral group and the subgroups of C_{2^∞},*

$$\Omega_i(C_{2^\infty}) := \{g \in C_{2^\infty} : o(g) \mid 2^i\} \simeq G_{2^i}$$

for every natural number i. Then the group G can be expressed as $G = \bigcup_{i\geq 1} G_i$, where $G_i = \Omega_i(C_{2^\infty})\langle\alpha\rangle$, for each $i \geq 1$ and $\{G_i; i \geq 1\}$ is an ascending chain. Notice that G_i is a finite 2-group for every $i \geq 1$. Hence G_i is a nilpotent group and then a $\mathfrak{B}$-group for each $i \geq 1$. However G is not a $\mathfrak{B}$-group. Therefore we have a semiprimitive group which is the union of an ascending chain of finite $\mathfrak{B}$-subgroups but it is not in the class $\mathfrak{B}$. This proves that in $c\overline{\mathfrak{L}}$ the class $\mathfrak{B}$ is not E_μ-closed and so it is not a saturated formation.

We finish by giving a criteria for a subgroup-closed Fitting formation to be a Fitting class.

Theorem 8 [3] *Let $\mathfrak{F}$ be a subgroup-closed $c\overline{\mathfrak{L}}$-Fitting formation. Then $\mathfrak{F}$ is saturated if and only $\mathfrak{F}$ satisfies the following property:*

(α) If G is a Chernikov group which is the union of an ascending chain $\{G_i \mid i \geq 0\}$ of finite subgroups, then G belongs to $\mathfrak{F}$ if and only if G_i belongs to $\mathfrak{F}$ for all i.

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