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On one-generated $\mathfrak{X}$-local formations of finite groups

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Dedicated to the memory of Sergey Antonovich Chunikhin (1905–1985).

In this paper, all groups considered are supposed to be finite.

Recall that a *formation* is a class of groups satisfying the following two conditions:

1. If $G \in \mathfrak{F}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{F}$.

2. If N and M are normal subgroups of G such that $G/M, G/N \in \mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$.

Given a formation $\mathfrak{F}$ and a group G, the *$\mathfrak{F}$-residual* $G^{\mathfrak{F}}$ of G is the smallest normal subgroup N of G such that G/N belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be *saturated* when $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G.

Gaschütz [7] introduced the concept of *local formation*, which enabled him to construct a rich family of saturated formations.

Definition 1. • A *formation function* f assigns to every $p \in \mathbb{P}$ a (perhaps empty) formation $f(p)$.

• If f is a formation function, then the local formation $\mathrm{LF}(f)$ defined by f is the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$ for all $p \in \pi(H/K)$.

• A formation $\mathfrak{F}$ is said to be local if there exists a formation function f such that $\mathfrak{F} = \mathrm{LF}(f)$.

Theorem 2 (Gaschütz-Lubeseder-Schmid [5, IV,4.6]). *The family of local formations coincides with the family of saturated formations.*

This result was proved by Gaschütz and Lubeseder in the soluble universe and later generalised by Schmid to the general finite universe.

Baer followed another approach to extend way the theorem of Gaschütz and Lubeseder to the finite universe. He used a different notion of local formation, in which non-abelian chief factors were treated with more flexibility than abelian ones. This led him to find a new family of formations, the Baer-local formations, containing the local ones.

Definition 3. • A *Baer function* assigns to every simple group J a class of groups $f(J)$ such that $f(C_p)$ is a formation for every $p \in \mathbb{P}$.

• If f is a Baer function, then the *Baer-local formation* or *Baer formation* $\mathrm{BLF}(f)$ defined by f is the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(J)$, where J is the composition factor of H/K.

• A formation $\mathfrak{F}$ is said to be *Baer-local* if there exists a Baer function f such that $\mathfrak{F} = \mathrm{BLF}(f)$.

Shemetkov introduced in [11] the concept of *composition formation*. This notion is equivalent to the one of Baer-local formation.

Definition 4. A formation $\mathfrak{F}$ is said to be *solubly saturated* when, for every group G, the condition $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$, where $G_{\mathfrak{S}}$ denotes the soluble radical of G.

Theorem 5 (Baer, [5, IV, 4.17]). *A formation $\mathfrak{F}$ is solubly saturated if and only if $\mathfrak{F}$ is Baer-local.*

With the aim of presenting a common generalisation of the Gaschütz-Lubeseder-Schmid and Baer theorems, Förster introduced in [6] the concept of $\mathfrak{X}$-local formation, where $\mathfrak{X}$ is a class of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$, where $\pi(\mathfrak{X}) := \{p \in \mathbb{P} :$ there exists $G \in \mathfrak{X}$ such that $p \in \pi(G)\}$ and $\operatorname{char}\mathfrak{X} := \{p \in \mathbb{P} : C_p \in \mathfrak{X}\}$.

Here we explain the concept of $\mathfrak{X}$-local formation. Let $\mathfrak{J}$ denote the class of all simple groups. For any subclass $\mathfrak{Y}$ of $\mathfrak{J}$, we write $\mathfrak{Y}' := \mathfrak{J} \setminus \mathfrak{Y}$. Denote by $\mathrm{E}\,\mathfrak{Y}$ the class of groups whose composition factors belong to $\mathfrak{Y}$. It is clear that $\mathrm{E}\,\mathfrak{Y}$ is a Fitting class, and so each group G has a largest normal $\mathrm{E}\,\mathfrak{Y}$-subgroup, the $\mathrm{E}\,\mathfrak{Y}$-radical $\mathrm{O}_{\mathfrak{Y}}(G)$. A chief factor which belongs to $\mathrm{E}\,\mathfrak{Y}$ is called a *$\mathfrak{Y}$-chief factor*. If p is a prime, we write $\mathfrak{Y}_p$ to denote the class of all simple groups $S \in \mathfrak{Y}$ such that $p \in \pi(S)$. The class of all π-groups, where π is a set of primes, is denoted by $\mathfrak{E}_\pi$.

Definition 6 ([6]). • An *$\mathfrak{X}$-formation function* f assigns to each $X \in \operatorname{char}(\mathfrak{X}) \cup \mathfrak{X}'$ a (possibly empty) formation $f(X)$.

- If f is an $\mathfrak{X}$-formation function, then $\mathrm{LF}_{\mathfrak{X}}(f)$ is the class of all groups G satisfying the following two conditions:

  1. If H/K is an $\mathfrak{X}_p$-chief factor of G, then $G/\,C_G(H/K) \in f(p)$.
  2. If G/L is a monolithic quotient of G such that $\operatorname{Soc}(G/L)$ is an $\mathfrak{X}'$-chief factor of G, then $G/L \in f(E)$, where E is the composition factor of $\operatorname{Soc}(G/L)$.

  The class $\mathrm{LF}_{\mathfrak{X}}(f)$ is a formation ([6]).

- A formation $\mathfrak{F}$ is said to be *$\mathfrak{X}$-local* if there exists an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. In this case we say that f is an *$\mathfrak{X}$-local definition of $\mathfrak{F}$* or that f *defines* $\mathfrak{F}$.

If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, an $\mathfrak{X}$-formation function is simply a formation function and the $\mathfrak{X}$-local formations are exactly the local formations. If $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, an $\mathfrak{X}$-formation function is a Baer function and the $\mathfrak{X}$-local formations are exactly the Baer-local ones (see [5, IV, 4.9]). Moreover, every formation is $\mathfrak{X}$-local for $\mathfrak{X} = \emptyset$. It can be checked that if $\mathfrak{X}$ and $\overline{\mathfrak{X}}$ are two classes of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$, $\pi(\overline{\mathfrak{X}}) = \operatorname{char}\overline{\mathfrak{X}}$, and $\overline{\mathfrak{X}} \subseteq \mathfrak{X}$, then every $\mathfrak{X}$-local formation is $\overline{\mathfrak{X}}$-local.

Förster also introduced in [6] an $\mathfrak{X}$-Frattini subgroup $\Phi^*_{\mathfrak{X}}(G)$ for every group G. He defined $\mathfrak{X}$-saturation in the obvious way and he proved that the $\mathfrak{X}$-saturated formations are exactly the $\mathfrak{X}$-local ones. From this one can deduce at once the theorems of Gaschütz-Lubeseder-Schmid and Baer. However, Förster's definition of $\mathfrak{X}$-saturation is not the natural one if our aim is to generalise the concepts of saturation and soluble saturation. Since $\mathrm{O}_{\mathfrak{J}}(G) = G$ and $\mathrm{O}_{\mathbb{P}}(G) = G_{\mathfrak{S}}$, we would expect the $\mathfrak{X}$-Frattini subgroup of a group G to be defined as $\Phi(\mathrm{O}_{\mathfrak{X}}(G))$. In general $\Phi^*_{\mathbb{P}}(G)$ does not coincide with $\Phi(G_{\mathfrak{S}})$, as we can see in [2, Example 2.4]. Hence the proof of Baer's theorem does not follow immediately from Förster's result.

In [3] another $\mathfrak{X}$-Frattini subgroup $\Phi_{\mathfrak{X}}(G)$ in every group G is introduced. It is smaller than Förster's one. The definition of this subgroup has been modified in [4, Chapter III] in the following way.

Definition 7. 1. Let p be a prime number. We say that a group G belongs to the class $A_{\mathfrak{X}_p}(\mathfrak{P}_2)$ provided that G is monolithic and there exists an elementary abelian normal p-subgroup N of G such that

(a) $N \le \Phi(G)$ and G/N is a primitive group with a unique non-abelian minimal normal subgroup, i.e., G/N is a primitive group of type 2,

(b) $\mathrm{Soc}(G/N) \in \mathrm{E}\,\mathfrak{X} \setminus \mathfrak{E}_{p'}$, and

(c) $\mathrm{C}^h_G(N) \le N$, where

$$\mathrm{C}^h_G(N) := \bigcap\{\mathrm{C}_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ below } N\}.$$

2. The *$\mathfrak{X}$-Frattini subgroup* of a group G is the subgroup $\Phi_{\mathfrak{X}}(G)$ defined as

$$\Phi_{\mathfrak{X}}(G) := \begin{cases} \Phi(\mathrm{O}_{\mathfrak{X}}(G)) & \text{if } G \notin A_{\mathfrak{X}_p}(\mathfrak{P}_2) \text{ for all } p \in \operatorname{char}\mathfrak{X}, \\ \Phi(G) & \text{otherwise.} \end{cases}$$

The following theorem is proved in [3, Theorem A] and [4, 3.2.14].

Theorem 8. *Let $\mathfrak{X}$ be a class of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. A formation $\mathfrak{F}$ is $\mathfrak{X}$-saturated if and only if $\mathfrak{F}$ is $\mathfrak{X}$-local.*

Moreover, since $\Phi_{\mathbb{P}}(G) = \Phi(G_{\mathfrak{S}})$, Baer's theorem is a direct consequence of this result.

At the moment of writing, it is not known whether or not there exist groups in $A_{\mathfrak{X}_p}(\mathfrak{P}_2)$ such that $\Phi(\mathrm{O}_{\mathfrak{X}}(G)) \neq \Phi(G)$. Hence the following question remains open:

Let $\mathfrak{F}$ be a formation containing all groups G with $G/\Phi(\mathrm{O}_{\mathfrak{X}}(G)) \in \mathfrak{F}$. Is $\mathfrak{F}$ $\mathfrak{X}$-local?

A different approach to the notion of local formation is the concept of *ω-local formation*, where ω is a non-empty set of primes.

Definition 9. Let ω be a non-empty set of prime numbers.

- An *ω-local satellite* f assigns to every element of $\omega \cup \{\omega'\}$ a (perhaps empty) formation.
- The symbol $G_{\omega d}$ is used to denote the largest normal subgroup N of G such that $\omega \cap \pi(H/K) \neq \emptyset$ for every composition factor H/K of N (if $\omega \cap \pi(\mathrm{Soc}(G)) = \emptyset$, then we set $G_{\omega d} = 1$).
- If f is an ω-local satellite, then $\mathrm{LF}_\omega(f)$ denotes the class of groups G satisfying the following two conditions:
  1. if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in f(p)$ for every $p \in \pi(H/K) \cap \omega$, and
  2. $G/G_{\omega d} \in f(\omega')$.
- A formation $\mathfrak{F}$ is *ω-local* when there exists an ω-local satellite f such that $\mathfrak{F} = \mathrm{LF}_\omega(f)$. In this case, f is called an *ω-local satellite* of $\mathfrak{F}$.

Definition 10. • Let p be a prime number. A formation $\mathfrak{F}$ is said to be *p-saturated* if $G \in \mathfrak{F}$ whenever $G/(\mathrm{O}_p(G) \cap \Phi(G)) \in \mathfrak{F}$.

- If ω is a set of primes, we say that $\mathfrak{F}$ is *ω-saturated* if $\mathfrak{F}$ is p-saturated for every prime $p \in \omega$.

In [13], Skiba and Shemetkov proved the following theorem:

Theorem 11. *A formation $\mathfrak{F}$ is ω-saturated if and only if $\mathfrak{F}$ is ω-local.*

These formations appear in a natural way when the saturation of formation products is considered.

Given two classes $\mathfrak{Y}$ and $\mathfrak{Z}$ of groups, a product class can be defined by setting

$$\mathfrak{Y}\mathfrak{Z} = (G \in \mathfrak{E} \mid \text{there is a normal subgroup } N \text{ of } G \text{ such that } N \in \mathfrak{Y} \text{ and } G/N \in \mathfrak{Z}),$$

where $\mathfrak{E}$ denotes the class of all finite groups. This product class turns out to be useful in the theory of classes of groups, especially when certain formations are considered. However this class product is not in general a formation when $\mathfrak{Y}$ and $\mathfrak{Z}$ are formations. Fortunately, there is a way of modifying the above definition to ensure that the class product of two formations is again a formation. If $\mathfrak{F}$ and $\mathfrak{G}$ are formations, the *formation product* or *Gaschütz product* of $\mathfrak{F}$ and $\mathfrak{G}$ is the class $\mathfrak{F} \circ \mathfrak{G}$ defined by

$$\mathfrak{F} \circ \mathfrak{G} := (X \in \mathfrak{E} \mid X^{\mathfrak{G}} \in \mathfrak{F}).$$

It is known that $\mathfrak{F} \circ \mathfrak{G}$ is again a formation and if $\mathfrak{F}$ is closed under taking subnormal subgroups, then $\mathfrak{F}\mathfrak{G} = \mathfrak{F} \circ \mathfrak{G}$ (see [5, IV, 1.7 and 1.8]).

Definition 12.
- A formation $\mathfrak{F}$ is said to be a *one-generated Baer-local formation* if there exists a group G such that $\mathfrak{F}$ is the smallest Baer-local formation containing G.

- Let $\mathfrak{X}$ be a class of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$. A formation $\mathfrak{F}$ is said to be a *one-generated $\mathfrak{X}$-local formation* if there exists a group G such that $\mathfrak{F}$ is the smallest $\mathfrak{X}$-local formation containing G.

In [9], Skiba posed the following question:

If $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a one-generated Baer-local formation, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial formations, is $\mathfrak{F}$ a Baer-local formation?

It is announced in the 1999 edition of the same book [10] that Skiba has answered the question negatively. An example can be found in [8, page 224]: Let G be a group with a unique non-abelian normal subgroup $R = O^p(G)$. Let $A = G \wr C_p$, $\mathfrak{F} = \operatorname{form}(A)$ (the smallest formation containing A), and $\mathfrak{G} = \mathfrak{S}_p \operatorname{form}(G)$. By [5, A, 18.5], A has a unique minimal subgroup, the base group $R^\natural$. By [12, 18.2], every simple group in $\mathfrak{F}$ has order p. Hence by [8, 4.5.25], $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a one-generated Baer-local formation and, since $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, $\mathfrak{F}$ is not a Baer-local formation.

We note that in the known examples of that situation, the equalities $\mathfrak{H} = \mathfrak{G}$ and $\mathfrak{H} = \mathfrak{S}_p\mathfrak{H}$ for a prime p hold, where $\mathfrak{S}_p$ denotes the class of all p-groups. Consequently the following question arises naturally:

Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a one-generated Baer-local formation, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial formations. Is $\mathfrak{F}$ a Baer formation provided that $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{H} \neq \mathfrak{S}_p\mathfrak{H}$ for every prime p?

The authors gave in [2, Theorem 1] an affirmative answer to a more general question in the context of $\mathfrak{X}$-local formations.

Theorem 13. *Let $\mathfrak{X}$ be a class of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char}\mathfrak{X}$. Let $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ be an $\mathfrak{X}$-saturated formation generated by a group G. If $\mathfrak{F}$ and $\mathfrak{G}$ are non-trivial and either $\mathfrak{H} \neq \mathfrak{G}$ or $\mathfrak{S}_p\mathfrak{H} \neq \mathfrak{H}$ for all primes $p \in \operatorname{char}\mathfrak{X}$, then $\mathfrak{F}$ is $\mathfrak{X}$-saturated.*

Moreover, in [1], it has been proved that the formation $\mathfrak{F}$ in Theorem 13 is π-local, where $\pi = \operatorname{char}\mathfrak{X}$. It should be remarked that, in general, $\mathfrak{X}$-local formations are not necessarily π-local for $\pi = \operatorname{char}\mathfrak{X}$.

Example 14. Let us consider the formation $\mathfrak{F} := \mathrm{E}\mathfrak{Y}$, where $\mathfrak{Y} := (A_n \mid n \geq 5)$, i. e., the formation of all finite groups whose composition factors are isomorphic to an alternating group of degree $n \geq 5$. It is clear that $\mathfrak{F}$ is a Baer formation. In particular, $\mathfrak{F}$ is $\mathfrak{X}$-saturated for every $\mathfrak{X} \subseteq \mathbb{P}$.

Assume that $\mathfrak{F}$ is p-saturated for a prime p. If $p \geq 5$, set $k := p$; otherwise, set $k := 5$. As $p \mid |A_k|$, by [5, B,11.8] there exists a group E with a normal elementary abelian p-subgroup $A \neq 1$ such that $A \leq \Phi(E)$ and $E/A \cong A_k$. We have that $E/(\mathrm{O}_p(E) \cap \Phi(E)) = E/(\mathrm{O}_p(E) \cap A) = E/A \in \mathfrak{F}$. Therefore $E \in \mathfrak{F}$, a contradiction.

Therefore $\mathfrak{F}$ is not ω-saturated for any set ω of primes. Moreover, by setting $\mathfrak{X} := (C_2)$ and $\omega := \{2\}$, we have that $\mathfrak{F}$ is $\mathfrak{X}$-saturated, but not 2-saturated.

Note that an analogous result to Theorem 13 was proved by Vishnevskaya in [14] for p-saturated formations. She shows that the p-saturated formation $\mathfrak{H}$ generated by a finite group cannot be the Gaschütz product $\mathfrak{F} \circ \mathfrak{G}$ of two non-p-saturated formations provided $\mathfrak{H} \neq \mathfrak{G}$. Although in general there does not exist a class of simple groups $\mathfrak{X}(\omega)$ such that the $\mathfrak{X}(\omega)$-saturated formations are exactly the ω-saturated formations (see [2, Section 3]), the arguments used in the proof of Theorem 13 still hold for ω-saturated formations. It leads to an alternative proof of Vishnevskaya's result.

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Abstract. In this survey, some generalisations of local formations are presented and characterised in terms of Frattini-like subgroups. Some results on factorisations of one-generated Baer-local and $\mathfrak{X}$-local formations are also shown.

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