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On c-supplemented primary subgroups of finite groups

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1. Introduction. There has been much interest in the past in investigating the relationship between the properties of some primary subgroups of a finite group G and the structure of G ([1–3]). In this aspect the concept of a c-supplemented subgroup in a finite group was introduced by Wang in [9] and he proved that a finite group G is soluble if and only if every Sylow subgroup of G is c-supplemented in G. As an application of the above result some well-known results were generalized by using the concept of c-supplementation. Thus, c-supplementation provides a useful tool for the investigation of the structure of finite groups which is shown in [10].

In this paper, we shall continue to study the c-supplemented subgroups in a finite group G. Some theorems on soluble groups and p-nilpotent groups are obtained by considering their c-supplemented subgroups. Some results in [10] are extended and generalized.

All the groups considered in this paper are finite. Most of the notations are standard and can be found in [4] and [8]. We denote a semi-product of a subgroup H and K by $G = [H]K$, where H is normal in G.

Let π be a set of primes. We say that $G \in E_\pi$ if G has a Hall π-subgroup. We say that $G \in C_\pi$ if any two Hall π-subgroups of G are conjugate in G. We say that $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a Hall π-subgroup of G.

Definition 1.1. A subgroup H of G is called c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \le H_G$, where $H_G = Core_G(H)$ is the largest normal subgroup of G contained in H. Here, K is called a c-supplement of H in G.

2. Preliminaries. For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 2.1 ([10],Lemma 2.1). *Let G be a group. Then*

(1) *If H is c-supplemented in G, $H \le M \le G$, then H is c-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \le H$. Then H is c-supplemented in G if and only if H/N is c-supplemented in G/N.*

(3) *Let π be a set of primes. Let N be a normal π'-subgroup and let H be a π-subgroup of G . If H is c-supplemented in G, then HN/N is c-supplemented in G/N. If furthermore N normalizes H, then the converse also holds.*

(4) *Let $H \le G$ and $L \le \Phi(H)$. If L is c-supplemented in G, then $L \trianglelefteq G$ and $L \le \Phi(G)$.*

Lemma 2.2 ([5], the main theorem). *Suppose that a finite group G has a Hall π-subgroup, where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.3 ([9], Theorem 3.3). *Let R be a soluble minimal normal subgroup of a group G, R_1 be a maximal subgroup of R. If R_1 is c-supplemented in G, then R is a cyclic group of prime order.*

Lemma 2.4 ([7], Lemma 2.6). *Let N be a normal subgroup of a group G ($N \neq 1$). If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which is contained in $F(N)$.*

Lemma 2.5 ([10], Theorem 3.3). *Let G be a finite group and let N be a normal subgroup of G such that G/N is supersoluble. If every maximal subgroup of every Sylow subgroup of N is c-supplemented in G, then G is supersoluble.*

Lemma 2.6. *Let G be a finite group and p be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Assume that the order of G is not divisible by p^3. Then G is p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Since every proper subgroup and every proper quotient group also satisfy the hypothesis of the lemma, the minimal choice of G implies that G is a minimal non-p-nilpotent group but every proper subgroup and every proper quotient group of G is p-nilpotent. Therefore $G = [P]Q$ with Q cyclic (see [8]). Since both $\Phi(P)$ and $\Phi(G)$ are in $Z(G) = 1$, we have that P is an elemenary abelian Sylow p-subgroup and Q is a cyclic group of order q. $Q \cong G/P$ and $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. Hence, q divides $p(p+1)(p-1)$. Since $p \neq q$ and $(|G|, p^2 - 1) = 1$, we have G is p-nilpotent, by Burnside p-nilpotent theorem, a contradiction.

Final contradiction completes our proof.

Lemma 2.7 ([6]). *Let G be a finite group and U any p-subgroup of G. If $N_G(U)/C_G(U)$ is a p-subgroup, then G is p-nilpotent.*

Lemma 2.8. ([11]). *Let P be an elementary abelian p-group with $|P| = p^n$, where p is a prime. Then $|Aut(P)| = k_n \cdot p^{n(n-1)/2}$, where $k_n = \prod_{i=1}^{n}(p^i - 1)$.*

Lemma 2.9. ([11]). *Let G be a group of order p^n, where p is a prime. Then $|Aut(G)|$ is the factor of the order of $Aut(P)$, where P is an elementary abelian p-group of order p^n.*

Lemma 2.10. *Let G be a finite group and p be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. If G/L is p-nilpotent and $p^3 \nmid |L|$, then G is p-nilpotent.*

Proof. By the hypothesis and Lemma 2.6, we know that L is p-nilpotent and L has a normal p-complement $L_{p'}$. Since $L_{p'}$ char L and L is normal in G, we have that $L_{p'} \trianglelefteq G$. Therefore $G/L \cong (G/L_{p'})/(L/L_{p'})$ is p-nilpotent. There exists a Hall p'-subgroup $(H/L_{p'})/(L/L_{p'})$ of $(G/L_{p'})/(L/L_{p'})$ and $H/L_{p'} \trianglelefteq G/L_{p'}$. By Schur-Zassenhaus Theorem, we have that $H/L_{p'} = [L/L_{p'}]H_1/L_{p'}$, where $H_1/L_{p'}$ is a Hall p'-subgroup of $H/L_{p'}$. Then by Lemma 2.6, we have $H_1/L_{p'} \trianglelefteq H/L_{p'}$ and $H_1/L_{p'}$ char $H/L_{p'} \trianglelefteq G/L_{p'}$. Therefore $H_1/L_{p'} \trianglelefteq G/L_{p'}$. Hence, $G/L_{p'}$ is p-nilpotent. Thus, G is p-nilpotent.

3. Main results. Theorem 3.1. *Let G be a finite group and p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If there exists a normal subgroup N of G such that G/N is p-nilpotent and every maximal subgroup of every Sylow subgroup of N is c-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and choose G to be a counterexample of minimal order. Moreover, we have

(1) *G is soluble, G has a minimal normal subgroup $L \leq N$ and L is an elementary abelian r-group, where r is the largest prime number in $\pi(N)$.*

By the hypothesis, every maximal subgroup of every Sylow subgroup of N is c-supplemented in G, thus, it is c-supplemented in N, by Lemma 2.1. Applying Lemma 2.5 for the case $G = N$, we get that N is supersoluble and hence G is soluble. So, for the largest prime number r in $\pi(N)$, a Sylow r-subgroup R of N is normal in N. Obviously, R is a characteristic subgroup of N. Therefore, R is normal in G as N is normal in G. Thus, G has a minimal normal subgroup $L \leq N$ and L is an elementary abelian r-group.

(2) *G/L is p-nilpotent, $L \nleq \Phi(G)$ and $C_N(L) = L = F(N)$, $L = R \in Syl_p(N)$.*

In fact, $(G/L)/(N/L) \cong G/N$ is p-nilpotent. Let R_1/L be a maximal subgroup of a Sylow r-subgroup of N/L. Then R_1 is a maximal subgroup of a Sylow r-subgroup R of N. By the hypothesis of the theorem, R_1 is c-supplemented in G. By Lemma 2.1, R_1/L is c-supplemented in G/L. Let Q_1/L be a maximal subgroup of a Sylow q-subgroup of N/L, where $q \neq r$. It is clear that $Q_1 = Q_1^*L$, where Q_1^* is a maximal subgroup of a Sylow q-subgroup of N. By the hypothesis, Q_1^* is c-supplemented in G. Hence, Q_1^*L/L is c-supplemented in G/L, by Lemma 2.1. We have proved that G/L satisfies the hypothesis of the theorem. Hence, G/L is p-nilpotent by the choice of G. We have that L is a p-group, otherwise, if $p \nmid |N|$, then G is p-nilpotent since G/N is p-nilpotent, a contradiction. If $p \neq r$, then G is p-nilpotent, since G/L is p-nilpotent, a contradiction. Since the class of all p-nilpotent groups is a saturated formation, we can easily prove that L is the unique minimal normal subgroup of G which is contained in N, $L \not\leq \Phi(G)$. By Lemma 2.4, $F(N) = L$. The solubility of N implies that $L \leq C_N(F(N)) \leq F(N)$, and $C_N(L) = L = F(N)$, as L is an abelian group. Since $R \trianglelefteq G$ and $R \leq F(N)$, thus, $L = R \in Syl_p(G)$.

(3) *G is p-nilpotent.*

Let P_1 be a maximal subgroup of L, then P_1 is c-supplemented in G, by (2), and hence $|L| = p$, by Lemma 2.3. We have that $LH/L \trianglelefteq G/L$, since G/L is p-nilpotent, where H is a Hall p'-subgroup of G. Since $(|G|, p-1) = 1$, we know that LH is p-nilpotent. It follows from $H char HL \trianglelefteq G$ that $H \trianglelefteq G$. Therefore, G is p-nilpotent.

The final contradiction completes our proof.

Corollary 3.2. *Let G be a finite group. If every maximal subgroup of every Sylow subgroup of G is c-supplemented in G, then G has a Sylow tower of the supersoluble type.*

Lemma 3.3. *Let G be a finite group and p be a prime divisor of $|G|$ with $(|G|, p^2 - 1) = 1$. Assume that every second maximal subgroup of a Sylow p-subgroup of G is c-supplemented in G. Then $G/O_p(G)$ is soluble and p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Furthermore, we have

(1) $O_p(G) = 1$.

If $O_p(G) = P$, then $G/O_p(G)$ is a p'-group and, of course, it is p-nilpotent, a contradiction. If $O_p(G) = P_1$, where P_1 is the maximal subgroup of P, then $G/O_p(G)$ is p-nilpotent, since $(|G|, p^2 - 1) = 1$ and $|G/O_p(G)|_p = p$, a contradiction. If $O_p(G) = P_2$, where P_2 is the second maximal subgroup of P, then $p^3 \nmid |G/O_p(G)|$. Hence, $G/O_p(G)$ is p-nilpotent, by Lemma 2.6. If $1 < O_p(G) < P_2$, then $G/O_p(G)$ satisfies the hypothesis and the minimal choice of G implies that $G/O_p(G) \cong G/O_p(G)/O_p(G/O_p(G))$ is p-nilpotent, a contradiction.

(2) *$|G|$ is divisible by p^3.*

If $p^3 \nmid |G|$, then G is p-nilpotent, by Lemma 2.6, a contradiction.

(3) *For every second maximal subgroup P_1 of a Sylow subgroup P of G, the c-supplement of P_1 is p-nilpotent.*

Let P be a Sylow p-subgroup of G and P_1 be a second maximal subgroup of P. By the hypothesis, P_1 is c-supplemented in G. So, there exists a subgroup K_1 of G such that $P_1 \cap K_1 \leq (P_1)_G \leq O_p(G) = 1$. Now $|K_1|_p = p^2$, Lemma 2.6 implies that K is p-nilpotent.

(4) *G is p-nilpotent.*

Let $N = N_G(K_{1_{p'}})$ and $K_1 = K_{1_p}K_{1_{p'}}$. By (3), $K_1 \leq N$. So, we have $G = P_1K_1 = P_1N$. If $N = G$, then G is p-nilpotent, a contradiction. Let $P_1 \leq \overline{P_1} \leq P$, where $\overline{P_1}$ is a maximal subgroup of a Sylow subgroup P of G. Hence, $G = P_1K_1 = \overline{P_1}K_1 = \overline{P_1}N$. If $\overline{P_1} \leq N$, then G is p-nilpotent, a contradiction. So, we may assume $\overline{P_1} \cap N < \overline{P_1}$. We may choose a maximal subgroup P_2 of $\overline{P_1}$ such that $\overline{P_1} \cap N \leq P_2$. It is clear that P_2 is a second maximal subgroup of P. By (3), P_2 is c-supplemented in G and the c-supplement K_2 of P_2 is

p-nilpotent. We denote $K_2 = K_{2_p}K_{2_{p'}}$. Since $(|G|, p^2 - 1) = 1$, Lemma 2.2 or the odd order Theorem implies that $G \in C_{p'}$. Now both $K_{1_{p'}}$ and $K_{2_{p'}}$ are Hall p'-subgroups of G, these two subgroups are conjugate in G. Let $K_{1_{p'}} = (K_{2_{p'}})^g$. Since $G = P_2K_2$ and $K_2 \leq N_G(K_{2_{p'}})$, we may choose $g \in P_2$. We also have that $K_2{}^g$ normalizes $K_{2_{p'}}{}^g = K_{1_{p'}}$, hence, $K_2{}^g \leq N$. Now $G = G^g = (P_2K_2)^g = P_2N$. Therefore $\overline{P_1} = \overline{P_1} \cap P_2N = P_2(\overline{P_1} \cap N) = P_2$, contrary to the condition.

The final contradiction completes our proof.

Theorem 3.4. *Let N be a normal subgroup of G and p be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Assume that G/N is p-nilpotent and every second maximal subgroup (if exists) of every Sylow subgroup of N is c-supplemented in G. Then G is p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Then

(1) *G is soluble.*

By the hypothesis, every second maximal subgroup of every Sylow subgroup of N is c-supplemented in G, thus, is c-supplemented in N, by Lemma 2.1. By Lemma 3.3, we have that N is soluble, and hence, G is soluble. Let L be a minimal normal subgroup of G which is contained in N . Then L is an elementary abelian r-group for some prime r.

(2) *G/L is p-nilpotent and L is the unique minimal normal subgroup of G which is contained in N. Furthermore, $L = F(N) = C_N(L)$.*

In fact, $(G/L)/(N/L) \cong G/N$ is p-nilpotent. Let R_1/L be a second maximal subgroup of a Sylow r-subgroup of N/L. Then R_1 is a second maximal subgroup of a Sylow r-subgroup R of N. By the hypothesis of the theorem, R_1 is c-supplemented in G. By Lemma 2.1, R_1/L is c-supplemented in G/L. Let Q_1/L be a second maximal subgroup of a Sylow q-subgroup of N/L, where $q \neq r$. It is clear that $Q_1 = Q_1^*L$, where Q_1^* is a second maximal subgroup of a Sylow q-subgroup of N. Since Q_1^* is c-supplemented in G, we have that Q_1^*L/L is c-supplemented in G/L, by Lemma 2.1. We have proved that G/L satisfies the hypothesis of the theorem, and hence, G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we have that L is the unique minimal normal subgroup of G which is contained in N, $L \not\leq \Phi(G)$. By Lemma 2.4, $F(N) = L$. The solubility of N implies that $L \leq C_N(F(N)) \leq F(N)$ and $C_N(L) = F(N) = L$.

(3) *L is a Sylow p-subgroup of N.*

By (1), we have known that G is soluble. If $p \nmid |N|$, then, it is easy to see that G is p-nilpotent, since G/N is p-nilpotent, a contradiction. Thus, $p \mid |N|$. If $p \neq r$, then obviously, G is p-nilpotent by (2), a contradiction. Therefore $p = r$ and L is an elementary abelian p-subgroup of G which is contained in N. Let D be a Hall p'-subgroup of N. Then, LD/L is a Hall p'-subgroup of N/L. Since N/L is p-nilpotent, we have $LD/L \trianglelefteq N/L$ and hence $LD \trianglelefteq N$. Let P be a Sylow p-subgroup of N. Assume that $L < P$. Then $PD = PLD$ is a subgroup of N. Since every second maximal subgroup of a Sylow subgroup of PD is c-supplemented in G, by Lemma 2.1, every second maximal subgroup of every Sylow subgroup of PD is also c-supplemented in PD. Therefore, PD satisfies the hypothesis for G. If $PD < G$, then, by the minimal choice of G, we have that PD is p-nilpotent, in particular, $D \trianglelefteq PD$.Hence, $LD = L \times D$ and $D \leq C_N(L) = L$, a contradiction. Now we may assume that $G = PD = N$ and $L < P$. Since N/L is p-nilpotent, $LD \trianglelefteq G = N$. By the Frattini argument, $G = LN_G(D)$. Since L is the unique minimal normal subgroup of G, D is not normal in G and $L \cap N_G(D) = 1$. Therefore, $G = [L]N_G(D)$. Let P_2 be a Sylow p-subgroup of $N_G(D)$. Then LP_2 is a Sylow p-subgroup of G. Choose a second maximal subgroup P_1 of LP_2 such that $P_2 \leq P_1$. Otherwise, if P_2 is a maximal subgroup of LP_2, then $|L| = p$

and hence G is p-nilpotent, by Lemma 2.10, a contradiction. Clearly, $L \nleq P_1$ and hence $(P_1)_G = 1$. By our hypothesis, P_1 is c-supplemented in G. There exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_G = 1$. Now $|K|_p = |G : P_1|_p = p^2$. By the hypothesis and Lemma 2.6, we have that K is p-nilpotent. It follows that K has a normal p-complement which is in fact a Hall p'-subgroup D_1 of G. By the solubility of G, there exists an element $g \in L$ such that $D_1^g = D$. Since $P_1 < P_1^* \trianglelefteq LP_2$, where P_1^* is the maximal subgroup of LP_2 which contains P_1, we have that $G = P_1K = P_1^*K = (P_1^*K)^g = P_1^*K^g$ and $P_1 \cap K = 1$. Since $K^g \cong K$ has a normal p-complement and $D = D_1^g \trianglelefteq K^g$, it follows that $K^g \leq N_G(D)$. Since $LP_2 = LP_2 \cap G = LP_2 \cap P_1^*K^g = P_1^*(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \nleq P_2$, otherwise, $LP_2 \leq P_1^*P_2 = P_1^*$, a contradiction. Therefore, P_2 is a proper subgroup of $P_3 =< P_2, LP_2 \cap K^g >$ while P_3 is a subgroup of a Sylow p-subgroup LP_2. Now, both P_2 and K^g are contained in $N_G(D)$ and we have that P_3 is a p-subgroup of $N_G(D)$ which contains a Sylow subgroup P_2 as a proper subgroup, a contradiction.

(4) *G is p-nilpotent.*

Let L_1 be a second maximal subgroup of L. If $|L| \leq p^2$, then G is p-nilpotent by Lemma 2.10, a contradiction. If $|L| \geqslant p^3$, then $L_1 \neq 1$, hence, L_1 is c-supplemented in G. There exists a subgroup K of G such that $L_1K = G$ and $L_1 \cap K \leq (L_1)_G = 1$. It follows that $L = L_1(L \cap K)$, hence, $L \cap K \trianglelefteq G$. Since L is a unique minimal normal subgroup of G contained in N, we have $L \cap K = L$ or $L \cap K = 1$. If $L \cap K = 1$, then $L = L_1$, a contradiction. Therefore, $L \cap K = L$, hence, $G = K$. This leads to that G is p-nilpotent.

The final contradiction completes our proof.

Corollary 3.5. *Let G be a finite group and p be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Assume that every second maximal subgroup of every Sylow subgroup of G is c-supplemented in G. Then G is p-nilpotent.*

Corollary 3.6. *Let G be a finite group and p be a prime divisor of $|G|$ such that $(|G|, p^2 - 1) = 1$. Assume that every second maximal subgroup of every Sylow subgroup of G' is c-supplemented in G. Then G is p-nilpotent.*

Lemma 3.7. *Let G be a finite group with $(|G|, 21) = 1$. Assume that every third maximal subgroup (if exists) of a Sylow 2-subgroup of G is c-supplemented in G, then $G/O_2(G)$ is 2-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Let P be a Sylow 2-subgroup of G. Furthermore, we have

(1)$O_p(G) = 1$.

If $O_2(G) = P$, then $G/O_2(G)$ is a $2'$-group and of course, it is 2-nilpotent, a contradiction. If $O_2(G) = P_1$, where P_1 is a maximal subgroup of P, then $G/O_2(G)$ is 2-nilpotent, by Lemma 2.6, and $|G/O_2(G)|_2 = 2$, a contradiction. If $O_2(G) = P_2$, where P_2 is a second maximal subgroup of P, then $2^3 \nmid |G/O_2(G)|$. Hence, $G/O_2(G)$ is 2-nilpotent by Lemma 2.6, a contradiction. If $1 < O_2(G) < P_2$, then $G/O_2(G)$ satisfies the hypothesis and the minimal choice of G implies that $G/O_2(G) \cong G/O_2(G)/O_2(G/O_2(G))$ is 2-nilpotent, a contradiction.

(2) *$|G|$ is divisible by 2^4.*

If $2^3 \nmid |G|$ and $(|G|, 21) = 1$, then G is 2-nilpotent by Lemma 2.4, a contradiction. If $2^3 \mid |G|$ and $2^4 \nmid |G|$, then $|G_2| = 2^3$. Next we consider $N_G(U)/C_G(U)$, where U is any 2-subgroup of G. If $U = P$, then $N_G(P)/C_G(P)$ is isomorphic to the subgroup of $Aut(P)$. By Lemma 2.8 and Lemma 2.9, we have $N_G(P)/C_G(P)$ is a 2-subgroup. If $U \neq P$, it is easy to know that $N_G(U)/C_G(U)$ is also 2-group, according to Lemma 2.8 and Lemma 2.9 . Then by Lemma 2.7, it is clear that G is 2-nilpotent in this case, a contradiction.

(3) *For every third maximal subgroup P_3 of a Sylow 2-subgroup P of G, the c-supplement of P_3 in G is 2-nilpotent.*

By the hypothesis, P_3 is c-supplemented in G. So, there exists a subgroup K_3 of G such that $G = P_3K_3$ and $K_3 \cap P_3 \leq (P_3)_G$. By (2), we know that K_3 is 2-nilpotent, since $K_3 \cap P_3 \leq (P_3)_G \leq O_p(G) = 1$.

(4) *G is 2-nilpotent.*

Let $N = N_G((K_3)_{2'})$ and $K_3 = (K_3)_2(K_3)_{2'}$. By (3), $K_3 \leq N$. So we have $G = P_3K_3 = P_3N$. If $N = G$, then G is 2-nilpotent, a contradiction. Let $P_3 \leq P_2 \leq P_1 \leq P$, where P_2 is a second maximal subgroup of P and P_1 is a maximal subgroup of P. Hence, $G = P_3K_3 = P_2K_3 = P_2N$. If $P_2 \leq N$, then G is 2-nilpotent, a contradiction. So, we may assume that $P_2 \cap N < P_2$. We may choose a maximal subgroup $P_3{}^*$ of P_2 such that $P_2 \cap N \leq P_3{}^*$. It is clear that $P_3{}^*$ is the third maximal subgroup of P. By (3), P_3^* is c-supplemented in G and the c-supplement K_3^* of P_3^* is 2-nilpotent. We denote $K_3{}^* = (K_3{}^*)_2(K_3{}^*)_{2'}$. Lemma 2.2 implies that $G \in C_{2'}$. Now, both $(K_3)_{2'}$ and $(K_3{}^*)_{2'}$ are the Hall $2'$-subgroups of G, these two subgroups are conjugate in G. Let $(K_3)_{2'} = ((K_3{}^*)_{2'})^g$. Since $G = P_3{}^*K_3{}^*$ and $K_3{}^* \leq N_G((K_3{}^*)_{2'})$, we may choose $g \in P_3{}^*$. We also have that $(K_3{}^*)^g$ normalizes $((K_3{}^*)_{2'})^g = (K_3)_{2'}$; hence, $(K_3{}^*)^g \leq N$. Now, $G = G^g = (P_3{}^*K_3{}^*)^g = P_3{}^*N$. Therefore, $P_2 = P_2 \cap P_3{}^*N = P_3{}^*(P_2 \cap N) = P_3{}^*$, contrary to the choice of G.

The final contradiction completes our proof.

Theorem 3.8. *Let G be a finite group with $(|G|, 21) = 1$. Assume that there exists a normal subgroup N of G such that G/N is 2-nilpotent and every third maximal subgroup (if exists) of every Sylow subgroup of N is c-supplemented in G. Then G is 2-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Then:

(1) *G is soluble and G has a minimal normal subgroup L such that $L \leq N$ and L is an elementary abelian r-group, for some prime r.*

By the hypothesis, every third maximal subgroup of every *Sylow* subgroup of N is c-supplemented in G; thus, it is c-supplemented in N, by Lemma 2.1. By the choice of G and Lemma 3.7, we have that N is soluble; hence, G is soluble. Let L be a minimal normal subgroup of G which is contained in N. Then L is an elementary abelian r-group for some prime r.

(2) *G/L is 2-nilpotent and L is the unique minimal normal subgroup of G which is contained in N. Furthermore, $L = F(N) = C_N(L)$.*

In fact, $(G/L)/(N/L) \cong G/N$ is 2-nilpotent and $(|G/L|, 21) = 1$. Let R_1/L be a third maximal subgroup of a Sylow r-subgroup of N/L. Then R_1 is a third maximal subgroup of a Sylow r-subgroup R of N. By the hypothesis of the theorem, R_1 is c-supplemented in G. By Lemma 2.1, R_1/L is c-supplemented in G/L. Let Q_1/L be a third maximal subgroup of a Sylow q-subgroup of N/L, where $q \neq r$. It is clear that $Q_1 = Q_1^*L$, where Q_1^* is a third maximal subgroup of a Sylow q-subgroup of N. By the hypothesis, Q_1^* is c-supplemented in G. Hence, Q_1^*L/L is c-supplemented in G/L, by Lemma 2.1. We have proved that G/L satisfies the hypothesis of the theorem and hence G/L is 2-nilpotent. Since the class of all 2-nilpotent groups is a saturated formation, we have that L is the unique minimal normal subgroup of G which is contained in N, $L \nleq \Phi(G)$. By Lemma 2.4, $F(N) = L$. The solubility of N implies that $L \leq C_N(F(N)) \leq F(N)$ and $C_N(L) = F(N) = L$.

(3)*L is a Sylow 2-subgroup of N.*

By (1), we have known that G is soluble. If $2 \nmid |N|$, then G is 2-nilpotent by (2), a contradiction. If $2 \neq r$, then G is 2-nilpotent by (2), a contradiction. Therefore, L is an elementary abelian 2-subgroup of G which is contained in N. Let D be a Hall $2'$-subgroup of N. Obviously, LD/L is a Hall $2'$-subgroup of N/L. Since N/L is 2-nilpotent, we have that $LD/L \trianglelefteq N/L$. So, $LD \trianglelefteq N$. Let P be a Sylow 2-subgroup of N. Assume that $L < P$. Then

$PD = PLD$ is a subgroup of N. Since every third maximal subgroup of a Sylow subgroup of PD is c-supplemented in G, by Lemma 2.1, every third maximal subgroup of every Sylow subgroup of PD is also c-supplemented in PD. Therefore, PD satisfies the hypothesis for G. If $PD < G$, the minimal choice of G implies that PD is 2-nilpotent, in particular, $D \trianglelefteq PD$. Hence, $LD = L \times D$ and $D \leq C_N(L) = L$, a contradiction. Now we assume that $G = PD = N$ and $L < P$. Since N/L is 2-nilpotent, $LD \trianglelefteq N = G$. By the Frattini argument, $G = LN_G(D)$. Since L is the unique minimal normal subgroup of G, D is not normal in G and $L \cap N_G(D) = 1$. Therefore, $G = [L]N_G(D)$. Let P_2 be a Sylow 2-subgroup of $N_G(D)$. Then LP_2 is a Sylow 2-subgroup of G. Choose a third maximal subgroup P_3 of LP_2 such that $P_2 \leq P_3$. Otherwise, if P_2 is a maximal subgroup of LP_2, then $|L| = 2$ and hence G is 2-nilpotent by Lemma 2.10, a contradiction. If P_2 is a second maximal subgroup of LP_2, then $|L| = 2^2$ and hence G is 2-nilpotent by Lemma 2.10 and (2), a contradiction. Clearly, $L \nleq P_3$ and hence $(P_3)_G = 1$. By our hypothesis, P_3 is c-supplemented in G. There exists a subgroup K of G such that $G = P_3K$ and $K \cap P_3 \leq (P_3)_G = 1$. It follows that K has a normal 2-complement which is in fact a Hall $2'$-subgroup D_1 of G. By the hypothesis and Lemma 2.2, there exists an element $g \in L$ such that $D_1^g = D$. Since $P_2 \leq P_3 < P_2^* < P_1 < LP_2$, where P_1 is a maximal subgroup of LP_2 which contains P_2^*, P_2^* is a second maximal subgroup of LP_2 which contains P_3, we have that $G = P_3K = P_1K = (P_1K)^g = P_1K^g$. Since $K^g \cong K$ has a normal 2-complement D and $D = D_1^g \leq K^g$, it follows that $K^g \leq N_G(D)$. Since $LP_2 = LP_2 \cap G = LP_2 \cap P_1K^g = P_1(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \nleq P_2$. Otherwise, $LP_2 \leq P_1P_2 = P_1$, a contradiction. Therefore, P_2 is a proper subgroup of $P_4 =< P_2, LP_2 \cap K^g >$, where P_4 is a subgroup of the Sylow 2-subgroup LP_2. Now both P_2 and K^g are contained in $N_G(D)$ and we have that P_4 is a 2-subgroup of $N_G(D)$ which contains a Sylow subgroup P_2 as a proper subgroup, a contradiction. Hence, L is a Sylow 2-subgroup of N.

(4) *G is 2-nilpotent.*

If $|L| \leq 4$, then G is 2-nilpotent by Lemma 2.10 and (2), a contradiction. If $|L| = 2^3$, then G is 2-nilpotent by Lemma 3.7. Let L_1 be a nontrivial third maximal subgroup of L.Then L_1 is c-supplemented in G. There exists a subgroup K of G such that $L_1K = G$ and $K \cap L_1 \leq (L_1)_G$. It follows that $L = L_1(L \cap K)$ and $L \cap K \trianglelefteq G$. Since L is a unique minimal normal subgroup of G contained in N, we have $L \cap K = L$ or $L \cap K = 1$. If $L \cap K = 1$, then $L = L_1$, a contradiction. Therefore, $L \cap K = L$, and hence $G = K$. This leads to that G is 2-nilpotent.

The final contradiction completes our proof.

Corollary 3.9. *Let G be a finite group with $(|G|, 21) = 1$. If every third maximal subgroup of every Sylow subgroup of G is c-supplemented in G, then G is 2-nilpotent.*

Corollary 3.10. *Let G be a finite group with $(|G|, 21) = 1$. If every third maximal subgroup of every Sylow subgroup of G' is c-supplemented in G, then G is 2-nilpotent.*

Abstract. A subgroup H is called c-supplemented in a group G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is contained in H_G. In this paper we investigate the influence of c-supplementation of some primary subgroups in finite groups. Some recent results are generalized.

References

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