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Groups with some minimal conditions for non-normal subgroups

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The groups, satisfying the minimal conditions for non-normal, abelian non-normal, non-abelian non-normal subgroups, are considered in the present article.
Keywords: minimal conditions, non-normal, abelian non-normal, non-abelian non-normal subgroups, Chernikov, Artinian, Dedekind, weakly graded groups.

Группы, удовлетворяющие условиям минимальности для ненормальных, абелевых ненормальных, неабелевых ненормальных подгрупп, рассматриваются в настоящей статье.
Ключевые слова: условия минимальности, ненормальные, абелевы ненормальные, неабелевы ненормальные подгруппы, черниковские, артиновы, дедекиндовы, слабо ступенчатые группы.

Remind that a (non-abelian) group with all subgroups normal is called Dedekind (respectively Hamiltonian). The remarkable R. Baer's Theorem [1] states that the group G is Dedekind iff G is abelian or $G=Q \times E \times R$ with quaternion, elementary abelian 2- and abelian 2′- subgroups Q, E, R respectively. The remarkable S.N. Chernikov's Theorem [2] states that the group G, having a series with finite factors, satisfies the minimal condition for non-normal subgroups (briefly, satisfies $min-\bar{n}$) iff it is Chernikov or Dedekind. Further, remind that the group G is called weakly graded, if for $g, h \in G$, the subgroup $<g, g^h>$ possesses a subgroup of finite index$\neq 1$ whenever g is of infinite order, or g is a p-element$\neq 1$ with some odd prime p and also $[g^p,h]=1$ and $<g,h>$ is periodic (N.S. Chernikov, see, for instance, [3]).

The class of weakly graded groups is very large and includes the classes of binary and locally graded, locally finite and locally solvable groups, groups, having a series with finite factors; binary finite, linear, 2-groups, periodic Shunkov groups, … . Further, let $\mathcal{L}$ be the minimal local class of groups closed with respect to forming subgroups, series (and, at the same time, to forming subcartesian products) and containing the class of weakly graded groups. The class $\mathcal{L}$ is extremely large.

The following new theorem holds.

Theorem 1 (N.S. Chernikov). *Let G be a group with $min-\bar{n}$. Then G′ is Artinian. Further, G is Chernikov or Dedekind iff G′∈$\mathcal{L}$.*

The Ol'shanskiy's Examples of infinite simple Artinian groups (see, for instance, [4]) are, of course, the examples of non-Chernikov non-Dedekind groups with $min-\bar{n}$. The group is called primitive graded, it for $g,h \in G$, the subgroup $<g, g^h>$ possesses a subgroup of finite index $\neq 1$ whenever g is a p-element $\neq 1$ with some odd p and also $[g^p,h]=1$ and $<g,h>$ is periodic (N.S. Chernikov, 2013). The periodic group is primitive graded iff it is weakly graded.

Corollary 1 (N.S. Chernikov). *The group G with primitive graded G′ satisfies $min-\bar{n}$ iff it is Chernikov or Dedekind.*

Remind that an infinite non-abelian group, satisfying the minimal condition for abelian non-normal subgroups, is called an *I*-group (S.N. Chernikov [5]). Let, as usual, *min-ab* be the minimal condition for abelian subgroups. S.N.Chernikov has disclosed the structure of non-periodic *I*-groups and has established the important properties of periodic *I*-groups without *min-ab* (see in [5] his Theorems 4.6, 4.10, 4.11). The following new theorem gives the description of periodic *I*-groups without *min-ab*. Below A_p is a Sylow *p*-subgroup of the group A and *e(A)* is its exponent.

Theorem 2 (N.S. Chernikov). *For the periodic group G the following statements are equivalent.*
(i) G is an I-group without min-ab.

(ii) Either G is Hamiltonian non-Chernikov, or for some $b \in G$ and some Dedekind non-Chernikov subgroup $A \lhd G$ with Chernikov A_2 and with $1 < m = |G:A| < \infty$, $G = A<b>$ and $A \cap <b> \subseteq Z(A)$ and b induces on any A_p an automorphism of certain order $n_p \mid m$, raising each element of A_p to some power, and also: 1) $n_2 = 1$, if A is Hamiltonian; 2) for every $t \in <b>$, every finite subgroup $F \subseteq A$ and any $f \in F$ with $|<f>| = e(F)$, and for an integer r with $f^t = f^r$ and every $a \in F$, $a^t = a^r$ and $C_{<t>}(f) = C_{<t>}(F)$; 3) $n_p = sp^k$ with some natural $s \mid p-1$ and integer $k \geq 0$; in particular, $1 \leq n_p \leq p-1$ if $p \nmid n_p$, and $n_2 = 2^k, k \geq 0$; 4) if $p \mid n_p$ and $p \neq 2$, then A_p is finite; 5) if A_2 is infinite and $[A_2, b] \neq 1$, then for any $a \in A_2$, $a^b = a^{-1}$; 6) if $p \nmid n_p$ and for $t \in <b>$ $[A_p, t] \neq 1$, then for every $a \in A_p \setminus \{1\}$, $a^t \neq a$; 7) the subgroup $<A_p, 1: p \mid n_p$ and $p \neq 2>$ (respectively $<A_p, 1: n_p \neq m>$) is finite (respectively Chernikov).

Now let $min-\overline{abn}$ be the minimal condition for non-abelian non-normal subgroups. The infinite non-abelian groups with normal infinite non-abelian subgroups present some special case of groups with $min-\overline{abn}$. These infinite groups, are called $\overline{IH}$-groups (S.N.Chernikov, see, for instance, [5]). S.N.Chernikov has obtained a series of principal results relating to the $\overline{IH}$-groups (see, for instance, [5, Chapter 6] and [6, §§3-5]). N.S.Chernikov (see, for instance, Theorem 3.2 [6]) gives an affirmative answer to the following S.N.Chernikov's question (see [7, P.20]): Are the binary graded $\overline{IH}$-groups solvable?

The following two principal propositions hold.

Theorem 3 (N.S.Chernikov [3, Theorem 3]). *The non-abelian group $G \in \mathcal{L}$ satisfies the condition $min-\overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

Proposition 1 (N.S.Chernikov). *The $\overline{IH}$-group G is solvable iff $G \in \mathcal{L}$.*

The following result is an immediate consequence of Proposition 1.

Corollary 2 (N.S.Chernikov, see Theorem 3.2[6]). *The binary graded $\overline{IH}$-groups are solvable.*

Proof of Theorem 1. Let G be non-Artinian. It is easy to see: G contains some non-Artinian subgroup H such that any its non-Artinian subgroup is normal in G; H has some descending series $H = H_0 \supset H_1 \supset H_2 \supset ... \supset H_\gamma (= \bigcap_{\alpha < \gamma} H_\alpha) \supseteq 1$ such that H_γ is Artinian. Obviously all $H/H_\alpha, \alpha < \gamma$, are Dedekind. So by R.Baer's Theorem [1], $|(H/H_\alpha)'| \leq 2, \alpha < \gamma$. Therefore $|(H/H_\gamma)'| \leq 2$. Put $(H/H_\gamma)' = L/H_\gamma$. Then L is Artinian and H/L is abelian non-Artinian. Since G/L does not satisfy $min-ab$ and satisfies $min-\overline{n}$, it is Dedekind (see Corollary 2[2]). Put $(G/L)' = K/L$. Then $G' \leq K$ and $|K/L| \leq 2$. So K and G' are Artinian. Suppose that $G' \in \mathcal{L}$. The class $\mathcal{L}$ coincides with the minimal local class of groups, closed with respect to forming series and containing the class of weakly graded groups (see Proof of Theorem 3[3]). Consequently the class $\mathcal{L}$ coincides with the minimal local class of groups, closed with respect to forming series, subcartesian products and containing the class of weakly graded groups. So by Theorem C (iii) [8], G' is Chernikov. In consequence of S.N.Chernikov's Theorem [2], G is Chernikov or Dedekind.

Proof of Corollary 1. If G satisfies $min-\overline{n}$, then by Theorem 1, G' is Artinian and so G' is periodic weakly graded and G is Chernikov or Dedekind.

Proof of Theorem 2. Let (i) hold and A be the subgroup of G generated by all its abelian non-Artinian subgroups. Clearly A is non-Chernikov. In view of S.N.Chernikov's Theorem 4.11[5], A is Dedekind, all its subgroups are normal in G, and for some $g \in G$, $G = A<g>$. If $G = A$, then G is Hamiltonian non-Chernikov. Let $G \neq A$ and $K = \{a \in A: |<a>| \leq 2\}$. Since $<a> \lhd G$, $a \notin Z(G)$. Therefore K is an elementary abelian central 2-subgroup of G. Since $<g>K \not\subseteq A$ and $<g>K$ is abelian, K is Artinian and so finite. Therefore A_2 is Chernikov.

Put $b=g$ if A is abelian. Let A be Hamiltonian. Then by R.Baer's Theorem [1], $A=Q\times E\times R$ with Q, E and R as above. Clearly $R\underline{\subseteq}Z(A)$, $E\underline{\subseteq}K(\underline{\subseteq}Z(G))$ and $A_2=Q\times E$, $A=A_2\times R$. If $[g,\ A_2]=1$, then obviously $A\cap<g>\underline{\subseteq}Z(A)$. Put again $b=g$. Let now $[g,\ A_2]\neq1$. For Sylow 2- and 2'-subgroups $<h>$ and $<c>$ of $<g>$ resp., $<g>=<h>\times<c>$. For distinct subgroups $<u>$, $<v>\underline{\subseteq}Q$ of order 4, $Q=<u,v>$. Since $<u>,<v>\underline{\triangleleft}G$ and $2/|<c>|$, $[c,u]=[c,v]=1$. Then, because of $E\underline{\subseteq}Z(G)$, $[c,\ A_2]=1$ and $[h,\ A_2]=[h,Q]\neq1$. Clearly, we may assume that $[h,u]\neq1$ and $[h,v]\neq1$. Then $u^h=u^{-1}$ and $v^h=v^{-1}$. Consequently $u^{huv}=u$ and $v^{huv}=v$, and $[huv,\ A_2]=1$. Since $<uv>\underline{\triangleleft}G$, huv is a 2-element. Clearly $[huv,c]=1$. Put now $b=(huv)c$. Then $G=A<b>$. Since $[huv,\ A_2]=[c,\ A_2]=1$, we have $[b,\ A_2]=1$ and, obviously, $A\cap<b>\underline{\subseteq}Z(A)$. The assertion 1 of the statement (ii) is correct.

Consider the assertion 2 of (ii). Show that $a^t=a^r$. First, let F be abelian. For some subgroup D, $F=<f>\times D$. Since $<f>$, $<fd>$, $D\underline{\triangleleft}G$, for some integer s: $(fd)^t=(fd)^s=f^sd^s=f^td^t$; so $f^{s-r}=d^{t-s}\in D$ and $f^{s-r}=1$. Since obviously $<fd>\cap D=1$, $(fd)^{s-r}=f^{s-r}d^{s-r}=$ $=d^{s-r}\in<fd>\cap D=1$.

Thus $(fd)^t=(fd)^s=(fd)^{s-r+r}=(fd)^{s-r}(fd)^r=(fd)^r$. Since $F=<fd:d\in D>$, for every $a\in F$, $a^t=a^r$.

Let F be Hamiltonian. Then for some abelian subgroup F^* of F, $<a>F_{2'}\underline{\subseteq}F^*$ and $|F:F^*|=2$. Take $f^*\in F^*$ with $|<f^*>|=e(F^*)$. It is easy to see: $<f^*>\underline{\sim}<f>$. Let $<f>_2=<f_2>$, $<f>_{2'}=<f_{2'}>$, $<f^*>_2=<f_2^*>$, $<f^*>_{2'}=<f_{2'}^*>$. In view of the assertion 1, $f_2^t=f_2$ and $f_2^{*t}=f_2^*$. Since $<f>F_2'$ is abelian, $f_{2'}^t=f_{2'}^r$ and $f_{2'}^{*t}=f_{2'}^{*r}$ (see above). Consequently, because of $f^t=f^r$, we have $f^{*t}=f^{*r}$. Since F^* is abelian and $a\in F^*$, $a^t=a^r$ (see above).

Let $w\in C_{<t>}(f)$. Since $f^w=f$, as above for every $a\in F$, $a^w=a$. Thus $C_{<t>}(f)=C_{<t>}(F)$. Consider the assertion 3 of (ii). Let $n_p\neq1$. For each $w\in <b>\backslash C_{<b>}(A_p)$, there exists $a_w\in A_p$ such that $[w,\ a_w]\neq1$. Let F be the subgroup generated by all a_w. Then F is finite and $C_{<b>}(F)=C_{<b>}(A_p)$. In view of the assertion 2, for some $f\in F$, $C_{<b>}(F)=C_{<b>}(<f>)$. So b induces on $<f>$ an automorphism of order n_p. Thus $n_p\big| |Aut<f>|$. But $|Aut<f>|=(p-1)p^s$, $s\underline{\geq}0$.

Consider the assertions 4 and 5 of (ii). Let $p\neq2$ and $p\mid n_p$, or $p=2$ and $[A_2,t]\neq1$ with $t=b$. In the first case, for some p-element $t\in<b>$, $[A_p,t]\neq1$. Put $K=\{u\in A_p:|<u>|\underline{\leq}p\}$. Since $u\triangleleft G$, $K<t>$ is clearly an abelian p-subgroup. Since clearly $K<t>\underline{\not\subseteq}A$, $K<t>$ is Artinian. So K is finite. Then A_p is Chernikov (see, for instance, Theorem 7.9 [6]). In the second case, A_p is also Chernikov.

Let A_p be infinite. Then it contains some quasicyclic subgroup N. Let $a,d\in A_p$ and also $[d,t]\neq1$. Let $f\in N$ and also $|<a>|$, $|<d>|<|<f>|$. Then $e(<a><d><f>)=|<f>|$. In view of the assertion 2, $f^t\neq f$. Since $[N,t]\neq1$ and $N\underline{\triangleleft}G$, $N<t>$ is not a p-subgroup with odd p. Then $p=2$ and $f^b=f^{-1}$. In view of the assertion 2, $a^b=a^{-1}$.

Consider the assertion 6. Note that $p\neq2$: otherwise $n_p=2^k$, $k>0$, (see the assertion 3) and $p\mid n_p$ which is a contradiction. It is enough to consider the case when $|<a>|=p$. For some $d\in A_p$, $d^t\neq d$. Let $c\in<d>$ and $|<c>=p$. Since $p\nmid n_p$, $p/\big|<t>:C_{<t>}(<d>)\big|$. Consequently for some natural r such that $1<r<p$, $c^t=c^r$. By virtue of the assertion 2, $a^t=a^r\neq1$.

Consider the assertion 7. In view of the assertion 4 $|A_p|<\infty$, if $p\mid n_p$ and $p\neq2$. Since $n_p\mid n$, the set of all p, for which $p\mid n_p$, is finite or empty. Thus the subgroup $<A_p,1:p\mid n_p$ and $p\neq2>$ is finite. Further, assume that $<A_p,1:n_p\neq m>$ is not Chernikov. Then by Kurosh'es Theorem (see, for instance, Proposition 4.2.11 [9]), it is not Artinian. If $n_p\neq m$, then for some $w\in<b>\backslash A$, $[w,\ A_p]=1$. Since $<w>A_p$ is abelian and $<w>A_p\underline{\not\subseteq}A$, $<w>A_p$ is Artinian. It is easy to see: there exist some infinite set π of primes and some $w\in<b>\backslash A$ such that for each $p\in\pi$, $A_p\neq1$ and $<w>A_p$ is abelian. Then $<w>(\underset{p\in\pi}{\times}A_p)$ is abelian non-Artinian and also $<w>(\underset{p\in\pi}{\times}A_p)\underline{\not\subseteq}A$, which is a contradiction.

Now let (ii) hold. Obviously G is non-abelian and contains some Dedekind non-Chernikov subgroup. In view of R.Baer's Theorem [1], this subgroup has an abelian subgroup of finite index. By virtue of Kurosh'es Theorem, the last is non-Artinian. Thus G does not satisfy *min-ab*. So if G is Hamiltonian non-Chernikov, then (i) holds. Assume that G is not such group. Let U be its abelian subgroup. If $U \subseteq A$, then obviously $U \trianglelefteq G$. Let $U \not\subseteq A$ and $v \in U \backslash A$. Then $v = dt$ with $d \in A$ and $t \in \langle b \rangle \backslash A$. By virtue of the assertion 7, the subgroup $L = A_2 \langle A_p, 1: p \mid n_p \rangle \langle\langle A_p, 1: n_p \neq m \rangle$ is Chernikov. Further, for every odd p such that $p \nmid n_p$ and $n_p = m$, and for every $a \in A_p$, $a^t \neq a$ if $a \neq 1$ (see the assertion 6). So $a^v = a^{dt} = a^t \neq a$. Consequently, $U \cap A_p = 1$. Therefore $U \cap A = U \cap L$. So $U \cap A$ is Chernikov. Then U is Chernikov. Thus (ii) necessitates (i).

Proof of Proposition 1. Indeed, let $G \in \mathcal{L}$. Since G obviously satisfies $min-\overline{abn}$, it is almost solvable (see Theorem 3). Then in consequence of Proposition 6.5 [5], it is solvable.

In conclusion, mention the following interesting result arising from Theorems 4.11, 4.10 [5].

Proposition 2 (N.S.Chernikov). *A group satisfies the minimal condition for abelian non-normal subgroups iff every its abelian subgroup is Chernikov or normal.*

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