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On some relationships between the upper and lower central series in finite groups

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N.Yu. Makarenko obtained the bound on special rank of $(k+1)$-th term $\gamma_{k+1}(G)$ of the lower central series of a group G as a function of special rank of the factor-group of group G modulo the k-th term $\zeta_k(G)$ of the upper central series. In this paper we obtained the bound on section p-rank of $\gamma_{k+1}(G)$ as a function of section p-rank of the factor-group $G/\zeta_k(G)$. It follows as corollary Makarenko's theorem.
Keywords: finite group, special rank, section p-rank, hypercenter, nilpotent residua.

Н.Ю. Макаренко получила ограничения на специальный ранг $(k+1)$-го члена $\gamma_{k+1}(G)$ нижнего центрального ряда как функцию от специального ранга фактор-группы группы G по k-му члену $\zeta_k(G)$ верхнего центрального ряда. В этой работе мы получаем ограничения на секционный p-ранг $\gamma_{k+1}(G)$ как функцию от секционный p-ранга фактор-группы $G/\zeta_k(G)$. Как следствие, отсюда получается результат Н.Ю. Макаренко.
Ключевые слова: конечная группа, специальный ранг, секционный p-ранг, гиперцентр, нильпотентный резидуал.

One of the oldest outstanding results in the Theory of Groups is a classical theorem due to I. Schur [1]. Schur was the first to study the relationships between the properties of derived subgroup and the central factor-group in finite groups. Here appeared the important group, which now called the Schur multiplicator (all definitions see, for example, in a book [2]). I. Schur proved the following result: if G is a finite group then $[G,G]\cap\zeta(G)$ is isomorphic to a subgroup of $M(G/\zeta(G))$. Here $M(H)$ denotes the Schur multiplicator of a group H. In connection with this result, the following question arises: is there a function f such that $|[G,G]|\le f(t)$ where $t=|G/\zeta(G)|$? J. Wiegold has obtained here the best result. He proved that $|[G,G]|\le w(t)=t^m$ where $m=\frac{1}{2}(\log_p t-1)$ and p is the least prime dividing t [3]. Later J. Wiegold has proved that this bound is attained if and only if $t=p^n$, where p is a prime [4]. When t has more than one prime divisor the picture is less clear.

R. Baer in [5] and [6] considered the relationships between orders of the factors and terms of the upper and lower central series.

One of the important numerical invariants of a group is special rank. We say that a group G has *finite special rank* $r(G)=r$, if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property.

The general concept of special rank (and also the term special rank itself) has been introduced by A.I. Maltsev [7].

A. Lubotzky, A. Mann [8] considered the relationships between special rank of finite group G and special rank of its Schur multiplicator. They proved that there exists a function f_1 such that $r(M(G))\le f_1(r(G))$. This result implies that for finite group G there exists a function f_2 such that $r([G,G])\le f_2(r(G/\zeta(G)))$. Another proof of this result (without the using of Schur multiplicator) was given in paper [9]. Moreover, this result has been extend to very wide class of infinite groups.

N.Yu. Makarenko has obtained a following rank analogue of Baer's theorem. She proved [10] that there exists a function f_3 such that $r(\gamma_{k+1}(G))\le f_3(r(G/\zeta_k(G),k))$. As usually, by $\zeta_j(G)$ (respectively $\gamma_j(G)$) we denote j-th term of the upper (respectively lower) central series of a group G.

Let p be a prime. We say that a group G has *finite section p-rank* $r_p(G) = r$ if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

We can see that the concept of section p-rank generalizes the concept of special rank. Indeed if a group G has a finite special rank r, then G has finite section p-rank for every prime p and $r_p(G) \le r$.

First our result is a following analogue of a Baer's theorem.

Theorem A. *Let G be a finite group. If $G/\zeta_k(G)$ has section p-rank r, then there exists a function $\tau(r,k)$ such that $\gamma_{k+1}(G)$ has section p-rank at most $\tau(r,k)$.*

Later we will show that $r(G) \le m+1$ where $m = \max\{r_p(G) \mid p \in \Pi(G)\}$. This shows that Theorem A gives also the extension of Makarenko's result.

The second our result shows the relationships between the section p-rank and order of the factor-group by upper hypercenter and the section p-rank and order of the nilpotent residual. We note that these functions are simpler than the function obtained in Theorem A. Moreover, these functions do not depend of the length of upper central series.

Theorem B. *Let G be a finite group, Z be an upper hypercenter of G and L be the nilpotent residual of G. Suppose that the factor-group G/Z has order t and section p-rank r, where p is a prime. Then $|L| \le t^{m+1}$ where $m = \frac{1}{2}(\log_2 t - 1)$ and $r_p(L) \le r + \lambda(r)$, where $\lambda(r) = \frac{r(3r+1)}{2} + r^2\iota(\log_2 r)$.*

Here for a real number α we denote by $\iota(\alpha)$ the smallest integer not less than α.

We will start from some properties of section p-rank.

If H is a normal subgroup of G, then in general $r_p(G) \ne r_p(G/H) + r_p(H)$. For example, if G is a non-abelian p-group of order p^3 and $C = \zeta(G)$, then $r_p(C) = 1$, $r_p(G/C) = 2$, but G does not include the elementary abelian p-sections of order p^3.

However clearly $r_p(G) \le r_p(G/H) + r_p(H)$.

Lemma 1. *Let p be a prime and G be a finite p-group. Then $r_p(G) = r(G)$.*

Proof. We noted above that $r_p(G) \le r(G)$. Let $r_p(G) = s$. Choose an arbitrary finite subgroup K of G. Then a section $K/Fratt(K)$ is elementary abelian, so that $|K/Fratt(K)| \le p^s$. But the number of generators of K coincides with the number of generators of $K/Fratt(K)$. It follows that K has at most s generators. Hence $r(G) \le s = r_p(G)$. And we have an equation $r_p(G) = r(G)$.

Lemma 2. *Let G be a group. Then $r(G) \le m+1$ where $m = \max\{r_p(G) \mid p \in \Pi(G)\}$.*

Proof. Let F be an arbitrary subgroup of G, $q \in \Pi(F)$ and S_q be the Sylow q-subgroup of F. Then Lemma 1 shows that S_q has a special rank at most $r_q(G)$. In particular, S_q has at most $r_q(G)$ generators. Since it is valid for each prime $q \in \Pi(F)$, F has at most $m+1$ generators [11, Theorem 1]. It follows that G has a special rank at most $m+1$.

If A is an abelian p-group and n is a positive integer, then, as usually, put

$$\Omega_n(A) = \{a \in A \mid |a| \text{ divides } p^n\}.$$

$\Omega_n(A)$ is a characteristic subgroup of A for each positive integer n, and it called the *n-th layer of* A. The subgroup $\Omega_1(A)$ is called the *lower layer of* A.

Lemma 3. *Let G be a finite group and A be a normal abelian p-subgroup of G where p is a prime. Suppose that G satisfies the following conditions:*

(i) $G/C_G(A)$ is an abelian p'-group;

(ii) $\zeta(G) \cap A$ includes a subgroup B such that A/B has section p-rank r.

Then the factor-group $G/C_G(A)$ has special rank at most r.

Proof. Put $L/B=\Omega_1(A/B)$, then L/B has order p^r. As usual, we can consider L/B as F_pG-module (here F_p is a finite field of p elements). Since $G/C_G(B/C)$ is a finite p'-group, B/C is a semisimple F_pG-module (see, for example, [12, Corollary 5.15]), that is $L/B=S_1/B\times\ldots\times S_m/B$ where S_j/B is a simple F_pG-submodule (in other words, minimal G-invariant subgroup of L/B), $1\le j\le m$ and $m\le r$. Then $G/C_G(S_j/B)$ is a cyclic group (see, for example, [13, Corollary 2.2]). Obvious equation $C_G(S_1/B)\cap\ldots\cap C_G(S_m/B)=C_G(L/B)$ together with Remak's theorem yields the embedding

$$G/C_G(L/B)\hookrightarrow G/C_G(S_1/B)\times\ldots\times G/C_G(S_m/B)$$

which shows that $G/C_G(L/B)$ has special rank at most $m\le r$.

Let $n>1$ and $k=p^n$. The mapping $aB\to(aB)^k$, $aB\in\Omega_{n+1}(A/B)$, is a $\mathbb{Z}G$-endomorphism of $\Omega_{n+1}(A/B)$ in $\Omega_1(A/B)$ with kernel $\Omega_n(A/B)$. Therefore the factor $\Omega_{n+1}(A/B)/\Omega_n(A/B)$ is isomorphic to some $\mathbb{Z}G$-submodule of $\Omega_1(A/B)$. Since it is true for each positive integer n, the subgroup $H/C_G(A)=C_G(\Omega_1(A/B))/C_G(A)$ acts trivially in every factors of the series

$$\langle 1\rangle=B_0\le B=B_1\le B_2\le\ldots\le B_{k+1}=A$$

where $B_2/B=\Omega_1(A/B)$, $B_3/B=\Omega_2(A/B)$, ..., $B_{k+1}/B=\Omega_k(A/B)$. Since $H/C_G(A)$ is a p'-group, $H=C_G(A)$. It follows that $G/C_G(A)$ has special rank at most $m\le r$.

Corollary 1. *Let G be a finite group and A be a normal abelian p-subgroup of G where p is a prime. Suppose that G satisfies the following conditions:*

(i) $G/C_G(A)$ is a p'-group;

(ii) $\zeta(G)\cap A$ includes a subgroup B such that A/B has section p-rank r.

Then the factor-group $G/C_G(A)$ has special rank at most $1+\frac{r(5r+1)}{2}$.

Proof. Lemma 3 shows that every abelian subgroup of $G/C_G(A)$ has special rank at most r. Let $F/C_G(A)$ be an arbitrary subgroup of $G/C_G(A)$, $q\in\Pi(F/C_G(A))$ and $S_q/C_G(A)$ be the Sylow q-subgroup of $F/C_G(A)$. Then $S_q/C_G(A)$ has a special rank at most $\frac{r(5r+1)}{2}=\rho(r)$ (see, for example, [14, 25.2.5]). In particular, $S_q/C_G(A)$ has at most $\rho(r)$ generators. Since it is valid for each prime $q\in\Pi(F/C_G(A))$, $F/C_G(A)$ has at most $\rho(r)+1$ generators [11, Theorem 1]. It follows that $G/C_G(A)$ has a special rank at most $\rho(r)+1$.

Corollary 2. *Let G be a finite group and A be a normal abelian p-subgroup of G where p is a prime. Suppose that G satisfies the following conditions:*

(i) $G/C_G(A)$ has section p-rank t;

(iii) $\zeta(G)\cap A$ includes a subgroup B such that A/B has section p-rank r.

Then the factor-group $G/C_G(A)$ has special rank at most $t+\frac{r(5r+1)}{2}$.

Proof. Put $V=G/C_G(A)$. Let F be an arbitrary subgroup of V, $q\in\Pi(F)$ and R_q be the Sylow q-subgroup of F. If $q\ne p$, then Lemma 3 shows that every abelian subgroup of V has special rank at most r. Then R_q has a special rank at most $\frac{r(5r+1)}{2}=\rho(r)$ (see, for example, [14, 25.2.5]). In particular, R_q has at most $\rho(r)$ generators. If $q=p$, then R_p has finite special rank t [15, Lemma 2.2]. In particular, R_p has at most t generators. Let $m=\max\{t,\rho(r)\}$. Then F has at most $m+1$ generators [11, Theorem 1]. It follows that $G/C_G(A)$ has a special rank at most $m+1\le\rho(r)+t=t+\frac{r(5r+1)}{2}$.

Lemma 4. *Let G be a finite group and A be a normal abelian subgroup of G. Suppose that G satisfies the following conditions:*

(i) $G/C_G(A)$ has section p-rank t;

(ii) $A/(\zeta(G)\cap A)$ has section p-rank r.

Then $[A,G]$ has finite section p-rank and $r_p([A,G])\le t^2+\frac{tr(5r+1)}{2}$.

Proof. Let g be an arbitrary element of G. We defined again the endomorphism $\xi_g : G\to G$ by the rule $\xi_g(a)=[a,g], a\in A$. The inclusion $\zeta(G)\cap A\le C_A(g)=Ker(\xi_g)$ shows that $A/Ker(\xi_g)$ has section p-rank at most t. The isomorphism $A/Ker(\xi_g)\cong_G \mathrm{Im}(\xi_g)=[A,g]$ implies that $r_p([A,g])\le t$.

By Corollary 2 to Lemma 3 $G/C_G(A)=\langle x_1C_G(A),...,x_mC_G(A)\rangle$ for some elements $x_1,...,x_m$, where $m\le t+\frac{r(5r+1)}{2}$. Then $[A,G]=[A,x_1]\cdot ...\cdot[A,x_m]$ (see, for example, [15, Lemma 3.7]). It follows that $[A,G]$ has section p-rank at most $tm\le t\left(t+\frac{r(5r+1)}{2}\right)=t^2+\frac{tr(5r+1)}{2}$.

Proof of Theorem A.
Let

$$\langle 1\rangle = Z_0\le Z_1\le ...\le Z_{k-1}\le Z_k=Z$$

be the upper central series of G. We will apply an induction by k. If $k=1$, then the central factor-group G/Z_1 has section rank r. Proposition 3.3 of paper [15] shows that there exists a function λ such that $\gamma_2(G)=[G,G]$ has section p-rank at most $\lambda(r)$.

Assume now that $k>1$ and we have already found a function $\tau(r,k-1)$ such that $r_p(\gamma_k(G/Z_1))\le\tau(r,k-1)$. Put $K/Z_1=\gamma_k(G/Z_1)$ and $L=\gamma_k(G)$. Then $L\le K$ and by induction hypothesis $r_p(K/Z_1)\le\tau(r,k-1)$. An application of Proposition 3.3 of paper [15] shows that $D=[K,K]$ has section p-rank at most $\lambda(\tau(r,k-1))$. By the properties of upper and lower central series $[L,Z]=\langle 1\rangle$, so that the factor-group $G/C_G(L)$ has section p-rank at most r.

The factor-group K/D is abelian, in particular, LD/D is also abelian. We have

$$(LD/D)(LD/D\cap Z_1D/D)=(LD/D)((LD\cap Z_1D)/D)\cong LD/(LD\cap Z_1D)\cong$$
$$(LD)(Z_1D)/(Z_1D)=(LZ_1D)/(Z_1D)\cong L/(L\cap Z_1D),$$

which shows that $(LD/D)(LD/D\cap Z_1D/D)$ is an epimorphic image of $L/(L\cap Z_1)$. Clearly $r_p(L/(L\cap Z_1))=r_p(LZ_1/Z_1)\le r_p(K/Z_1)$. In particular $(LD/D)(LD/D\cap Z_1D/D)$ has section p-rank at most $\tau(r,k-1)=t$. Furthermore, $C_G(L)D/D\le C_{G/D}(LD/D)$, so that $(G/D)/C_{G/D}(LD/D)$ has section p-rank at most r. By Lemma 4, the subgroup $V/D=[LD/D,G/D]$ has section p-rank at most $t^2+\frac{tr(5r+1)}{2}$. The center of G/V includes LV/V. Since $(G/V)/(LV/V)$ is nilpotent of class at most k, $\gamma_{k+1}(G)\le V$. It follows that $\gamma_{k+1}(G)$ has finite section p-rank at most

$$\lambda(t)+t^2+\frac{tr(5r+1)}{2}=\lambda(\tau(r,k-1))+(\tau(r,k-1))^2+\frac{r\tau(r,k-1)(5r+1)}{2}.$$

We note that $\lambda(n)=\frac{n(3n+1)}{2}+n^2\iota(\log_2 n)$ [15, Proposition 3.3].

For the proof of Theorem B we will need some module-theoretical concepts and results. First of all, we need the analogue of upper central series in modules. Let G be a group, R be a ring and A be an RG-module. Put

$$\zeta_{RG}(A)=\{a\in A\,|\,a(g-1)=0 \text{ for each element } g\in G\}=C_A(G).$$

Clearly $\zeta_{RG}(A)$ is an RG-submodule of A. It is called the *RG-center of A*.

Starting from the RG-center, we can construct the upper RG-central series of A:

$$\langle 0\rangle=\zeta_{RG,0}(A)\le\zeta_{RG,1}(A)\le\zeta_{RG,2}(A)\le ...\zeta_{RG,\alpha}(A)\le\zeta_{RG,\alpha+1}(A)\le ...\zeta_{RG,\gamma}(A),$$

defined by the rule $\zeta_{RG,1}(A)=\zeta_{RG}(A)$ is an RG-center of A, and recursively

$\zeta_{RG,\alpha+1}(A)/\zeta_{RG,\alpha}(A)=\zeta_{RG}(A/\zeta_{RG,\alpha}(A))$ for all ordinals α, $\zeta_{RG,\lambda}(A)=\bigcup_{\mu<\lambda}\zeta_{RG,\mu}(A)$ for the limit ordinals λ and $\zeta_{RG}(A/\zeta_{RG,\gamma}(A))=\langle 0\rangle$. The last term $\zeta_{RG,\gamma}(A)$ of this series is called the *upper* RG*-hypercenter of* A and the ordinal γ is called the RG*-central length of a module* A.

If the upper RG-hypercenter of A coincides with A, then A is called RG*-hypercentral*.

If A is an RG-hypercentral module and γ is finite, then we will say that A is RG*-nilpotent*.

Let G be a group, R be a ring and A be an RG-module. Let B, C be RG-submodules of A and $B \le C$. Then the factor C/B is called G*-central* (respectively G*-eccentric*), if $G = C_G(C/B)$ (respectively $G \ne C_G(C/B)$).

We say that the RG-module A is G*-hypereccentric*, if A has an ascending series of RG-submodules

$$\langle 0\rangle = A_0 \le A_1 \le \ldots A_\alpha \le A_{\alpha+1} \le \ldots A_\gamma = A$$

whose factors $A_{\alpha+1}/A_\alpha$ are G-eccentric simple RG-modules for all $\alpha < \gamma$.

We say that the RG-module A has Z*-decomposition*, if $A = C \times E$ where C is the upper RG-hypercenter of A and E is an RG-submodule, which is G-hypereccentric (see, [12, Chapter 10]). We remark that this decomposition is unique (of course, if it exists). Moreover, C includes every RG-submodule, which is RG-hypercentral, and E includes every RG-submodule, which is RG-hypereccentric [12, Chapter 10].

Proof of theorem B.

Let

$$\langle 1\rangle = Z_0 \le Z_1 \le \ldots \le Z_{k-1} \le Z_k = Z$$

be the upper central series of G. Every subgroup Z_j is G-invariant and all factors Z_j/Z_{j-1} are G-central, $1 \le j \le n$. By L.A. Kaluzhnin theorem [16] the factor-group $G/C_G(Z)$ is nilpotent of nilpotency class at most $k-1$. Put $C = C_G(Z)$. An inclusion $Z \le C_G(C)$ shows that $G/C_G(C)$ has a section p-rank at most r. Clearly $C \cap Z \le \zeta(C)$, so that an isomorphism $C/(Z\cap C) \cong CZ/Z$ follows that $C/(Z \cap C)$ has order at most t and section p-rank at most r. An application of Proposition 3.3 of paper [15] shows that $D=[C,C]$ has finite section p-rank at most $\lambda(r)$. Furthermore, $|D| \le w(t) = t^m$ where $m = \frac{1}{2}(\log_2 t - 1)$ [3]. The subgroup D is G-invariant and C/D is abelian. We can consider C/D as a $\Box H$-module where $H = (G/D)/C_{G/D}(C/D)$. An obvious inclusion $C \le C_G(C/D)$ shows that H is nilpotent. We have $(C \cap Z)D/D \le \zeta_{\Box H,k}(C/D)$, and $(C/D)/((C\cap Z)D/D) \cong C/(C \cap Z)D$ has order at most t and section p-rank at most r. By Corollary 2.5 of paper [17] C/D has the Z-decomposition $C/D = B/D \times E/D$ where B/D is the upper $\Box H$-hypercenter of C/D and E/D is the $\Box H$-submodule, which is G-hypereccentric. As we noted above, B/D includes every $\Box H$-nilpotent $\Box H$-submodule, in particular, $(C\cap Z)D/D \le B/D$. This inclusion shows that $|E/D| \le t$ and $r_p(E/D) \le r$. Then $|E| \le tw(t)$ and $r_p(E) \le r + \lambda(r)$. By the choice of E the factor-group G/E is nilpotent, so that E includes a nilpotent residual L of G.

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