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On transitivity of normality and related topics

L.A. KURDACHENKO[1], I.YA. SUBBOTIN[2]

A group G is said to be a T-group if the property «to be a normal subgroup» is transitive in G. If every subgroup of G is a T-group, then G is called a $\overline{T}$-group. The authors consider some new developments in the infinite group theory related to the transitivity of normality.

Keywords: *T*-group, $\overline{T}$-group, locally nilpotent residual, transitively normal subgroup, (weakly) pronormal subgroup, NE-subgroup, weakly normal subgroup, hypercyclic group, generalized radical, (weakly) abnormal subgroup, nearly pronormal subgroup

Группа G называется T-группой, если в G свойство «быть нормальной подгруппой» является транзитивным в G. Если каждая подгруппа из G является T-группой, то G называется $\overline{T}$-группой. Обсуждаются некоторые новые результаты в теории бесконечных групп, связанные с транзитивностью нормальности.

Ключевые слова: *T*-группа, $\overline{T}$-группа, локально нильпотентный корадикал, транзитивно нормальная подгруппа, (слабо) пронормальная подгруппа, NE-подгруппа, слабо нормальная подгруппа, гиперциклическая группа, обобщенный радикал, (слабо) абнормальная подгруппа, частично пронормальная подгруппа

There are two frequently used kinds of subgroup properties-the properties that are external to the group, i.e. the properties stating that a subgroup belongs to some class **X** of groups (for example, to be an abelian subgroup, nilpotent subgroup and so on), and the properties that are internal to the group, i.e. the properties which define the place of the subgroup in the arrangement of subgroups in a group. These properties based on the relationships and mutual arrangement of subgroups. The examples of such properties are «to be a normal subgroup», «to be a subnormal subgroup», «to be conjugates», «to be a permutable subgroup" and so on. The property «to be a normal subgroup» is one of the most fundamental properties. There are many interesting generalizations of normality and the properties which have their roots in normality. In this survey, we want to consider some of these properties and to show their influence on the structure of a group. These properties do not include the condition «to be a finite group», however most of them first have appeared and studied in finite groups. It seems very natural since finite group theory is very well developed. The situation is much more complicated in infinite groups.

Note at once, that the property «to be a normal subgroup» is not transitive. Therefore it was very natural to study the groups, in which this property is transitive. Such groups are called ***T-groups***, while a group G is said to be a ***$\overline{T}$-group*** if every subgroup of G is a T-group. It should be noted that T-groups have been investigating for a quite long period of time. The structure of finite soluble T-groups has been described by W. Gaschütz [6]. In particular, he proved that every finite soluble T-group is a$\overline{\text{T}}$-group. We note that every finite $\overline{\text{T}}$-group is metabelian. The general structure of infinite soluble T- and $\overline{\text{T}}$-groups have been investigated by D.J.S. Robinson [19]. In particular, he has obtained the following most completed yet result regarding structure of locally soluble $\overline{\text{T}}$-groups.

Theorem 1. (D.J.S. Robinson [19])**.** *Let G be a locally soluble $\overline{T}$-group.*

(i) If G is not periodic, then G is abelian.

(ii) If G is periodic and L is the locally nilpotent residual of G, then satisfies the following conditions:

(a) G/L is a Dedekind group;

(b) $\Pi(L) \cap \Pi(G/L) = \varnothing$;

(c) $2 \notin \Pi(L)$;

(d) every subgroup of L is G-invariant.

Let G be a group and $R_{\mathfrak{LN}}$ be a family of all normal subgroups H of G such that G/H is locally nilpotent. Then the intersection $\cap R_{\mathfrak{LN}} = G_{\mathfrak{LN}}$ is called a ***locally nilpotent residual of G***. It is not difficult to prove that if G is a locally finite, then $G/G_{\mathfrak{LN}}$ is locally nilpotent. Note that if G is a finite or countable group, then the locally nilpotent residual has a complement, but in general case this is not true.

Consideration of the $\overline{T}$-groups naturally leads to the following type of subgroups.

A subgroup H of a group G is said to be ***transitively normal in G,*** if H is normal in every subgroup K ≥ H such that H is subnormal in K. (L.A. Kurdachenko, I.Ya. Subbotin, [11]).

This property is quite strong, and therefore the transitively normal subgroups exert big influence on the structure of a group. There are many important types of subgroups, which are transitively normal. We illustrate this statement by the following well-known examples. First consider the pronormal subgroups.

A subgroup H of a group G is said to be ***pronormal in G*** if for every g ∈ G the subgroups H and H^g are conjugate in the subgroup $< H, H^g >$ (See for example, the book of L.A Shemetkov [22, p.179]. Such important subgroups of finite soluble groups as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal subgroups. Every pronormal subgroup is transitively normal. But converse is not true. For example, for each set π of primes, every maximal π-subgroup is transitively normal. In particular, every Sylow p-subgroup is transitively normal for each prime p. But in an infinite group a Sylow p-subgroup in general is not pronormal. Every maximal locally nilpotent subgroup is also transitively normal.

T.A. Peng [18] has obtained the first characterization of $\overline{T}$-group via pronormal subgroups.

Theoerm 2 (T.A. Peng [18]). *Let G be a finite group. Then every subgroup of G is pronormal if and only if G is a $\overline{T}$-group. Every cyclic subgroup of G is pronormal if and only if G is a $\overline{T}$–group.*

The infinite case is much more complicated here.

Recall that a group is called ***locally graded***, if every it's non-identity finitely generated subgroup has a proper subgroup of finite index.

Theorem 3 (N.F. Kuzennyj, I.Ya. Subbotin, [13]). *Let G be a locally soluble group or a periodic locally graded group. Then the following conditions are equivalent:*

(i) every cyclic subgroup of G is pronormal in G;

(ii) G is a soluble $\overline{T}$-group.

The infinite groups whose subgroups are pronormal firstly have been considered by N.F. Kuzennyj, I.Ya. Subbotin in [12].

Theorem 4 (N.F. Kuzennyj, I.Ya. Subbotin, [12]). *Let G be a group whose subgroups are pronormal, and L be the locally nilpotent residual of G.*

(i) If G is periodic and locally graded, then G is a soluble $\overline{T}$-group, in which L is a complement to every Sylow Π(G/L)-subgroup.

(ii) If G is not periodic and locally soluble, then G is abelian.

The next characterization of T-groups was connected with a following property.

A subgroup H of a group G is called ***an NE-subgroup*** if $N_G(H) \cap H^G = H$. (Shirong Li, [23]).

Theorem 5 (Li, Y. [16]) *If every subgroup of a finite group G is NE-subgroups, then G is a $\overline{T}$-groups.*

If every primary subgroup of a finite group G is NE-subgroups, then G is a $\overline{T}$-group.

M. Bianchi, A.G.B. Mauri, M. Herzog and L. Verardi [3] have offered the following generalization of NE-subgroups.

A subgroup H of a group G is called ***an $\mathcal{H}$-subgroup*** if $N_G(H) \cap H^g \le H$ for all elements g ∈ G.

Theorem 6. (M. Bianchi, A.G.B. Mauri, M. Herzog and L. Verardi, [3]). *If every subgroup of a finite group G is* $\mathcal{H}$-subgroups, *then G is a $\overline{T}$-groups. If every primary subgroup of a finite group G is* $\mathcal{H}$-subgroups, *then G is a $\overline{T}$-groups.* K.H. Müller [17] has introduced the following generalization of pronormal subgroups. A subgroup H of a group G is called **weakly normal in G** if for each element g such that $H^g \le N_G(H)$ we have $g \in N_G(H)$.

In [2] A. Ballester-Bolinches and R. Esteban-Romero have proved that every pronormal subgroup is weakly normal. They also have proved the following result.

Theorem 7 (A. Ballester-Bolinches and R. Esteban-Romero [2]). *If every subgroup of a finite group G is weakly normal in G, then G is a* $\overline{T}$*-groups. If every primary subgroup of a finite group G is weakly normal in G, then G is a* $\overline{T}$*-group.*

Each of these subgroups is transitively normal. Our goal is to consider the widest possible generalization of these results. We start from following criteria of hypercyclicity of a group.

Theorem 8 (P. Csorga and M. Herzog [5]) *Let G be a finite group, whose cyclic subgroup of order 4 and of all prime orders are* $\mathcal{H}$*-subgroups. Then G is supersoluble.*

Yangming Li proved a similar result for NE-subgroups. More precisely

Theorem 9 (Li, Y. [16]) *Let G be a finite group, whose cyclic subgroup of order 4 and of all prime orders, are NE-subgroups. Then G is supersoluble.*

The following generalization of these results for finite and infinite groups has been obtained by L.A. Kurdachenko and J. Otal, 2013.

Let **X** be a class of groups. Recall that a group G is said to be a ***hyper-X-group*** if G has an ascending series of normal subgroups whose factors belong to the class **X**. In particular, a group ***is hypercyclic***, if it has an ascending series of normal subgroups whose factors are cyclic.

Theorem 10 (L.A. Kurdachenko, J. Otal, [8]). *Let G be a locally finite group. If every cyclic subgroup of order 4 and of all prime orders is transitively normal in G, then G is hypercyclic. Moreover, if L is a locally nilpotent residual of G, then L is hypercentral,* $2 \notin \Pi(L)$ *and* $\Pi(L) \cap \Pi(G/L) = \varnothing$.

Theorem 11 (L.A. Kurdachenko, J. Otal, [8]). *Let G be a locally finite group and L be the locally nilpotent residual of G. If every primary cyclic subgroup of G is transitively normal in G, then G is a* $\overline{T}$*-groups.*

We note that this result cannot be extended on an arbitrary periodic group. Indeed, A.Yu. Olshanskij has constructed an example of an infinite p-group G, where p is big enough prime, whose all proper subgroups have order p. Clearly every subgroup of such group G is transitively normal. For the non-periodic case we also need some restriction: A.Yu. Olshanskij has constructed an example of an infinite torsion-free group G, whose proper subgroups are cyclic. Clearly every subgroup of such group G is transitively normal.

Recall that a group G is called a ***generalized radical***, if G has an ascending series whose factors are locally nilpotent or locally finite. We will consider locally generalized radical group.

Theorem 12 (L.A. Kurdachenko, J. Otal, [8]). *Let G be a non-periodic locally generalized radical group. If every cyclic subgroup of G is transitively normal in G, then either G is abelian or* $G = R\langle b \rangle$ *where R is abelian,* $b^2 \in R$ *and* $a^b = a^{-1}$ *for each element* $a \in R$*. Moreover, in the second case the following conditions holds: (i) if* $b^2 = 1$*, then the Sylow 2-subgroup D of R is elementary abelian; (ii) if* $b^2 \neq 1$*, then either D is elementary abelian or* $D = E \times \langle v \rangle$ *where E is elementary abelian and* $\langle b, v \rangle$ *is a quaternion group. Conversely, if a group G has the above structure, then every cyclic subgroup is transitively normal*

Consider now some other kinds of transitively normal subgroups.

A subgroup H of a group G is called ***abnormal*** *in G* if $g \in \langle H, H^g \rangle$ for each element g of G. The notion of an abnormal subgroup belongs to P. Hall (1937), while the term "abnormal subgroup" itself belongs to R. Carter ([4]).

By its meaning, the abnormality is an antagonist to the normality: a subgroup of a group is simultaneously normal and abnormal only if it coincides with the group. All maximal non-normal subgroups are trivial examples of abnormal subgroups. More interesting here is the well-known J. Tits example: the subgroup $T_n(K)$ of all triangular matrices is abnormal in the general linear group $GL_n(F)$ over a field F. Every Carter subgroup of a finite soluble group is abnormal. The next type of subgroup appears from the following characterization of abnormal subgroups. But first we need the following simple concept.

Let G be a group and H be a subgroup of G. A subgroup K is called ***intermediate for*** H, $H \le K \le G$.

The following criterion of abnormality is well-known (see for example the survey [1].

Let G be a group and H a subgroup of G. Then H is abnormal in G if and only if the following two conditions hold:

(i) If K is an intermediate subgroup for H, then K is self-normalizing.

(ii) If K, L are two intermediate subgroup for H such that K and L are conjugate, then K = L.

Let G be a group. A subgroup H is called ***weakly abnormal in G*** *if every intermediate subgroup for H is self-normalizing* (M.S. Ba, Z.I. Borevich, [1]).

Note the following characterization of the weakly abnormal subgroups.

A subgroup H of a group G is weakly abnormal in G if and only if $x \in H^{<x>}$ *for each element* $x \in G$ (M.S. Ba, Z.I. Borevich, [1]).

In general, not every weakly abnormal subgroup is abnormal. There are examples of weakly abnormal subgroups in some finite groups which is not abnormal. But, the following interesting result has been obtained by D. Taunt.

Theorem 13 [21]. *If G is a finite soluble groups, then every weakly abnormal subgroup is abnormal*.

The widest generalization of this statement to infinite groups has been obtained by L.A. Kurdachenko, A. A. Pypka and I.Ya. Subbotin [14].

A group G is called an ***Ñ-group*** if G satisfies the following condition:

If M, L are subgroup of G such that M is maximal in L, then M is normal in L.

We remark that the property "to be an Ñ-group" is local. In particular, every locally nilpotent group is an Ñ-group, but converse is not true (J. S. Wilson, [24]).

Theorem 14 (L.A. Kurdachenko, A.A. Pypka and I.Ya. Subbotin, [14]). *Let G be a hyper-Ñ-group and H be a subgroup of G. If H is weakly abnormal in G, then G is abnormal in G.*

Corollary 15. (L.A. Kurdachenko and I.Ya. Subbotin, [10]). *Let G be a radical group and H be a subgroup of G. If H is weakly abnormal in G, then G is abnormal in G.*

Corollary 16. (F. de Giovanni, G. Vincenzi, [7]). *Let G be a hyperabelian group and H be a subgroup of G. If H is weakly abnormal in G, then G is abnormal in G.*

Corollary 17 (M.S. Ba, Z.I. Borevich, [1]). *Let G be a soluble group and H be a subgroup of G. If H is weakly abnormal in G, then G is abnormal in G.*

As for abnormality we can consider the following weak variation of pronormality (see [1])

Let G be a group. A subgroup H is called ***weakly pronormal*** in G (or ***has the Frattini property***) if for every subgroups K, L such that $H \le K$ and K is normal in L we have $L = N_L(H)K$.

There is the following characterization of the weakly abnormal subgroups.

Theorem 17 (M.S. Ba, Z.I. Borevich, [1]). *Let G be a group and H be a subgroup of G. Then H is weakly pronormal in G if and only if the subgroups H and* H^x *conjugate in* $H^{<x>}$ *for each element* $x \in G$.

The inclusion $< H, H^x > \le H^{<x>}$ shows that every pronormal subgroup is weakly pronormal. The converse statement is not true. In particular, every pronormal subgroup has the Frattini property. We note the following two properties of pronormal subgroups (see, for example, [1]).

Let G be a group and H be a pronormal subgroup of G. Then H is abnormal in G if and only if $H = N_G(H)$***. Let G be a group and H be a subgroup of G. If H is pronormal in G, then*** $N_G(H)$ ***is abnormal in G.***

For finite soluble groups, T.A. Peng obtained the following characterization of pronormal subgroups.

Theorem 18 (T.A. Peng, [18]). *Let G be a finite soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

This characterization of pronormal subgroups of finite groups could be extended on infinite groups in the following way.

We recall that a group G is called an ***N-group*** or ***group with normalizing condition***, if $N_G(H) \ne H$ for every subgroup H of G.

We note that a group G is an N-group if and only if every subgroup of G is ascendant.

Theorem 19 (L.A. Kurdachenko and I.Ya. Subbotin, [10]). *Let G be a hyper-N-group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property*.

Corollary 20 (F. de Giovanni, G. Vincenzi, [7])**.** *Let G be a hyperabelian group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property*.

Corollary 21 (see, for example, [1]). *Let G be a soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

If H is a pronormal subgroup of a group G and L is an intermediate subgroup for H, then H is pronormal in L. So $N_L(H)$ is abnormal in L.

J.S. Rose in [20] had introduced the following important class of subgroups which are antipodes to normal subgroups and are natural generalization of abnormal subgroups.

A subgroup H of a group G is called ***contranormal in G*** if $H^G = G$.

Abnormal subgroups are contranormal. However not every contranormal subgroup is abnormal.

A subgroup H of a group G is called ***nearly pronormal*** if $N_L(H)$ is contranormal in every subgroup L, which includes H.

As we can observe, every pronormal subgroup is nearly pronormal but the converse statement is not true. Nevertheless, for some classes of generalized soluble groups the nearly pronormality coincides with pronormality.

Theorem 22 (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin, [14]). *Le G be a hyper-N-group. Then every nearly pronormal subgroup of G is pronormal in G.*

Corollary 23 (G.J. Wood, [25]). *Let G be a finite soluble group. Suppose that a subgroup H satisfies the following condition: If K is a subgroup, including H, then $N_K(H)$ is abnormal in K. Then K is pronormal in G.*

With the help of nearly pronormal subgroups the following characterization of $\overline{T}$-groups has been obtained.

Theorem 24 (L.A. Kurdachenko, A. Russo, G. Vincenzi, [9]). Let G be a locally radical group. (i) If every cyclic subgroup of G is nearly pronormal, then G is a soluble *$\overline{T}$-group*.(ii) If every subgroup of G is nearly pronormal, then every subgroup of G is pronormal in G.

Theorem 25 (L. Kurdachenko, N.Semko (Jr.), and I. Subbotin, [15]). Let G be a locally soluble periodic group all of whose non-finitely generated subgroups are transitively normal. If G is a non- Chernikov group, then G is a $\overline{T}$-group. If G is a Chernikov group, then either G is a Dedekind group or the divisible part Y of G is a quasicyclic group and G/Y is a finite $\overline{T}$-group.
We observe that the conditions of this theorem are also sufficient.

Corollary 26 (L. Kurdachenko, N. Semko (Jr.), and I. Subbotin, [15]). *Let G be a locally nilpotent group all of whose non-finitely generated subgroups are transitively normal. Then every non-finitely generated subgroup of G is normal in G.*

Corollary 27 (L. Kurdachenko, N.Semko (Jr.), and I. Subbotin, [15]). *Let G be a radical group all of whose non-finitely generated subgroups are transitively normal. Suppose that G is not periodic. If the locally nilpotent radical of G is not a minimax group, then G is abelian.*

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[1] Днепропетровский национальный университет им. О. Гончара

[2] Национальный университет, г. Ла Джолла (США)

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