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Finite groups with permutable 2-maximal and 3-maximal subgroups

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1. Introduction

Throughout this paper, all groups are finite. Subgroups A and B of a group G are called permutable if $AB = BA$.

The presence of systems of certain pairwise permutable subgroups in a group determines essentially the structure of the group. For example, by classical theorem of Hall [1, 9.2.1] a group is soluble if and only if it has a set of mutually permutable Sylow subgroups, one for each prime dividing the group order. It is clear, that every group in which any two maximal subgroups are permutable is always nilpotent. Let us note another observation in this direction. It was proved in [2] that a soluble group G is supersoluble if and only if every maximal subgroup P_1 of every Sylow subgroup of G is $F(G)$-permutable with every maximal subgroup M of G, (i.e. $P_1M^x = M^xP_1$ for some $x \in F(G)$).

Polyakov has obtained in [3] some new criteria for the solubility and the supersolubility of groups by considering the conditions of permutability for maximal, 2-maximal and 3-maximal subgroups. The purpose here is deriving local analogs of the results of this work. Besides, we have extended the results of Polyakov, giving evaluation of the p-length of the groups investigated. In this note we use some methods of the paper [4], where results of Polyakov were extended in another trend.

2. Preliminaries

A saturated formation is a class $\mathfrak{F}$ of groups, closed under homomorphic images, such that $G \in \mathfrak{F}$, whenever G is a group and A, B are normal subgroups of G with $G/A \in \mathfrak{F}$, $G/B \in \mathfrak{F}$, and either $A \leqslant \Phi(G)$, or $A \cap B = 1$.

The group G is called p-supersoluble if its every chief pd-factor is a cyclic group. If the group G is p-supersoluble for all prime p, then G is called supersoluble.

Now we cite some well-known properties of p-supersoluble groups.

Lemma 2.1 ([5], p. 35). *The class of p-supersoluble groups is a saturated formation.*

Lemma 2.2 ([6; VI, 9.1]). *Let G be a supersoluble group. If p and q are the smallest and largest prime divisors of the order of G respectively, then a Sylow q-subgroup is normal in G, and G is p-nilpotent.*

The following lemma is well-known.

Lemma 2.3. *Let G be a p-supersoluble group. If $O_{p'}(G) = 1$, then G is supersoluble.*

Let G be a p-soluble group and

$$1 = P_0 \subseteq N_0 \subset P_1 \subset N_1 \subset P_2 \ldots \subset P_l \subseteq N_l = G,$$

where N_k/P_k is the largest normal p'-subgroup of G/P_k, and P_k/N_{k-1} is the largest normal p-subgroup of G/N_{k-1}. Then the smallest integer l such that $N_l = G$ is called p-length of G and is written $l_p(G)$.

Lemma 2.4 ([5; I, Corollary 5.6.1]). *Let $\mathfrak{F}$ be a class of a p-soluble group with p-legth $\leqslant t$, where t is some natural number. Then $\mathfrak{F}$ is a saturated formation.*

Lemma 2.5 ([5; I, Lemma 3.9]). *If H/K is a chief factor of a group G and if p is a prime divisor of $|H/K|$, then $O_p(G/C_G(H/K)) = 1$.*

Lemma 2.6 ([7; 1.20]). *Let $G = AB$ be the product of subgroups A, B of G. Then for all $x, y \in G$*

1. *$A^x = A^z$ and $B^y = B^z$ for some $z \in G$,*
2. *$G = A^x B^y$.*

Definition 2.7. *A subgroup H of a group G is called i-maximal subgroup if G has a chain of subgroups*

$$H = G_i \leqslant G_{i-1} \leqslant \ldots G_1 \leqslant G_0 = G,$$

such that G_j is a maximal subgroup of G_{j-1} for all $j = 1, \ldots i$.

3. Basic results

Polyakov has proved in [3] that a group G is supersoluble if every 2-maximal subgroup of G is permutable with every maximal subgroup. The first two of our theorems are local analogs of this result.

Theorem 3.1. *If G is a p-soluble group and every 2-maximal subgroup of G is permutable with all maximal subgroups M of G such that $|G : M|$ is a power of p, then G is p-supersoluble.*

Proof. Assume that the result is false and let G be a counterexample with minimal order. We now prove the theorem via the following steps.

(1) *G/N is a p-supersoluble groups for any non-identity normal subgroup N of G.*

Let M/N be a maximal subgroup of G/N such that $|G/N : M/N| = p^\alpha$ and M_2/N be an arbitrary 2-maximal subgroup of G/N. Then M is a maximal subgroup of G with $|G : M| = |G/N : M/N| = p^\alpha$ and M_2 is a 2-maximal subgroup of G. Therefore, by the hypothesis, the subgroup M permutes with M_2. Then we have

$$(M/N)(M_2/N) = MM_2/N = M_2M/N = (M_2/N)(M/N).$$

Thus, the condition of the theorem is inherited by G/N. But since $|G/N| < |G|$, G/N must be a p-supersoluble group, by the choice of the group G.

(2) $O_{p'}(G) = 1$.

Otherwise, $O_{p'}(G) \neq 1$. By (1), $G/O_{p'}(G)$ is p-supersoluble. Therefore G is p-supersoluble, a contradiction. So $O_{p'}(G) = 1$.

(3) *G has the only minimal normal subgroup N and $G = [N]M$, where M is a maximal subgroup of G with $M_G = 1$ and $|N| = p^\alpha$ where $\alpha \in \mathbb{N} \setminus \{1\}$.*

Let N be a minimal normal subgroup of G. Since by Lemma 2.1 the class of all p-supersoluble groups is a saturated formation, it follows that N is the only minimal normal subgroup of G and $N \nleq \Phi(G)$.

Since G is p-soluble, in view of (2), N is an elementary abelian p-group.

Let M be a maximal subgroup of G not containing N. Since N is abelian, $N \cap M \trianglelefteq G$ and so $N \cap M = 1$. Thus $G = [N]M$. Since $N = Soc(G)$, we see that $M_G = 1$. Besides, by (1) and by the choice of G, we have $|N| \neq p$.

(4) *The end of the proof.*

Let M_1 be a maximal subgroup of M. From (3), we know that $|G : M| = |G : M^x| = |N| = p^\alpha$ for some $\alpha \in \mathbb{N} \setminus \{1\}$. Then for every $x \in G$, we have $M_1 M^x = M^x M_1$. Thus, either $M^x M_1 = G$ or $M^x M_1 = M^x$.

Suppose that $M^xM_1 = G$. Then by Lemma 2.6, $G = MM_1 = M$, a contradiction. This shows that $M_1M^x = M^x$. Thus for every $x \in G$, we have $M_1 \leqslant M^x$ and $M_1 \leqslant M_G = 1$. Thus every maximal subgroup M_1 of M is trivial. Then $|M|$ is a prime and hence N is a maximal subgroup in G. In view of (3), we have $|N| \neq p$ and hence there is a non-identity maximal subgroup N_1 of N. It is clear that N_1 is a 2-maximal subgroup of G and N_1 permutes with M. This implies that $G = N_1M$ and $|G| = |N_1||M| = |N||M|$, a contradiction. This contradiction completes the proof.

Theorem 3.2. *Let G be a p-soluble group. Then the following statements hold:*

(a) If every maximal subgroup of G is permutable with every 2-maximal subgroup M of G such that $|G : M|$ is a power of p, then $l_p(G) \leqslant 1$.

(b) If every maximal subgroup of G is permutable with every 2-maximal subgroup M of G whose index is divisible by p, then G is p-supersoluble.

Proof. (a) Assume that this assertion is false and let G be a counterexample of minimal order. We divide the proof into the following steps.

(1) *G/N has the p-length $\leqslant 1$ for any non-identity normal subgroup N of G.*

Let M/N be a maximal subgroup of G/N. Let M_1/N be a 2-maximal subgroup of G/N such that $|G/N : M_1/N| = p^\alpha$ for some $\alpha \in \mathbf{N}$. Then M is a maximal subgroup of G and M_1 is a 2-maximal subgroup having the index $|G : M_1| = |G/N : M_1/N| = p^\alpha$ in G. Hence, by the hypothesis, M is permutable with M_1 and $MM_1 = M_1M$. Since

$$(M/N)(M_1/N) = MM_1/N = M_1M/N = (M_1/N)(M/N),$$

M/N is permutable with M_1/N. This shows that the hypothesis on G still holds on G/N. But $|G/N| < |G|$, and by our choice of G, we see that $l_p(G/N) \leqslant 1$.

(2) $O_{p'}(G) = 1$.

Indeed, assume that $O_{p'}(G) \neq 1$. In view of (1), $l_p(G/O_{p'}(G)) \leqslant 1$. This implies that $l_p(G) \leqslant 1$. This contradiction shows that $O_{p'}(G) = 1$.

(3) *G has the only minimal normal subgroup N and $G = [N]M$, where $N = F(G) = O_p(G)$ and M is a maximal subgroup of G such that $M_G = 1$ and $p||M|$.*

Let N be a minimal normal subgroup of G. Since by Lemma 2.4, the class of p-supersoluble groups with $l_p(G) \leqslant 1$ is a saturated formation, it follows that N is the only minimal normal subgroup of G and $N \nleq \Phi(G)$. Since, by the hypothesis, G is p-soluble, in view of (2), N is an elementary abelian p-group.

Let M be a maximal subgroup of G not containing N and $C = C_G(N)$. It is clear that $C \cap M \trianglelefteq G$. Since $N = Soc(G)$, we see that $M_G = 1$. This shows that $C \cap M = 1$ and $C = C \cap NM = N(C \cap M) = N$. By Lemma 2.5, we have that $O_p(G/C_G(N)) = O_p(G/N) = 1$. It is clear that $G = [N]M$ and, therefore, $O_p(M) = 1$.

We now demonstrate that p divides $|M|$. The quotient group $G/N \simeq M$. Consequently, if p does not divide $|M|$, then G/N is a p'-group. But this leads to $l_p(G) \leqslant 1$, which contradicts the choice of the group G.

(4) *Final contradiction.*

Let M_1 be a maximal subgroup of M such that $|M : M_1| = p^\beta$ for some $\beta \in \mathbf{N}$. It is clear that M_1 is a 2-maximal subgroup of G such that $|G : M_1|$ is a power of p. Then, by the hypothesis, $M_1M^x = M^xM_1$ for all $x \in G$. Therefore either $M^xM_1 = G$, or $M^xM_1 = M^x$. If $M^xM_1 = G$, then by Lemma 2.6, $G = MM_1 = M$, a contradiction. Hence $M_1M^x = M^x$. Thus, $M_1 \leqslant M^x$ for every $x \in G$ and $M_1 \leqslant M_G = 1$. This shows that a maximal subgroup M_1 of M such that $|M : M_1|$ is a power of p is trivial. Thus M is a p-group and therefore $l_p(G) \leqslant 1$. This contradiction completes the proof of our statement (a).

(b) Using the same arguments in the proof of Theorem 3.1, we can prove this statement.

Polyakov has proved in [3] that a group G is soluble if every 2-maximal subgroup of G is permutable with every 3-maximal subgroup. It was obtained that a group G is soluble if every maximal subgroup of G is permutable with every 3-maximal subgroup. The following theorems improve these results.

Theorem 3.3. *If G is a p-soluble group and every 2-maximal subgroup of G is permutable with all 3-maximal subgroups M of G such that $|G : M|$ is a power of p, then $l_p(G) \leqslant 2$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. We proceed the proof via the following steps.

(1) *G/N has the p-length $\leqslant 2$ for any non-identity normal subgroup N of G* (see the proof of Theorem 3.2).

(2) $O_{p'}(G) = 1$ (this fact follows directly from (1)).

(3) *G has the only minimal normal subgroup N and $G = [N]M$, where $N = F(G) = O_p(G)$ and M is a maximal subgroup of G with $M_G = 1$* (see the proof of Theorem 3.2).

(4) *The end of the proof.*

Let M_1 be an arbitrary maximal subgroup of M and M_2 be a 2-maximal subgroup of M such that $|M : M_2| = p^\beta$ for some $\beta \in \mathbb{N}$. Then M_1 is a 2-maximal subgroup of G and M_2 is a 3-maximal subgroup of G such that $|G : M_2| = |G : M||M : M_2| = |N||M : M_2|$ is a power of p. Thus, by the hypothesis, M_1 is permutable with M_2.

Since G is a p-soluble group, we see that M is a p-soluble group. By Theorem 3.2 (a), M has the p-length $\leqslant 1$. Since by (3), $M \simeq G/N$, we obtain that $l_p(G/N) \leqslant 1$. It is clear that $l_p(G) \leqslant 2$, this contradicts to the choice of G. Thus, the proof is completed.

Corollary 3.4. *If G is a p-soluble group and every 2-maximal subgroup of G is permutable with every 3-maximal subgroup of G whose index is divisible by p, then $l_p(G) \leqslant 2$.*

Corollary 3.5. *If every 2-maximal subgroup of G is permutable with all 3-maximal subgroups of G, then G is soluble and $l_p(G) \leqslant 2$ for all primes.*

Proof. The solubility of G follows from [3]. In view of Corollary 3.4, we have $l_p(G) \leqslant 2$ for all prime p.

Theorem 3.6. *If G is a p-soluble group and every 3-maximal subgroup of G is permutable with all 2-maximal subgroups M of G such that $|G : M|$ is a power of p, then $l_p(G) \leqslant 2$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. We proceed the proof as follows:

(1) *G/N has the p-length $\leqslant 2$ for any non-identity normal subgroup N of G* (see the proof of Theorem 3.2).

(2) $O_{p'}(G) = 1$ (this fact follows directly from (1)).

(3) *G has the only minimal normal subgroup N and $G = [N]M$, where $N = F(G) = O_p(G)$ and M is a maximal subgroup of G with $M_G = 1$* (see the proof of Theorem 3.2).

(4) *The end of the proof.*

Let M_2 be an arbitrary 2-maximal subgroup of M and M_1 be a maximal subgroup of M such that $|M : M_1|$ is a power of p. Then M_2 is a 3-maximal subgroup of G and M_1 is a 2-maximal subgroup of G. Moreover, $|G : M_1| = |G : M||M : M_1|$ is a power of p. Thus, by the hypothesis, M_1 is permutable with M_2.

Since G is p-soluble, we see that M is p-soluble. Thus, by Theorem 3.1, M is p-supersoluble, and $M/O_{p'}(M)$ is a p-supersoluble group. Since $O_{p'}(M/O_{p'}(M)) = 1$, by Lemma 2.3, we see that $M/O_{p'}(M)$ is supersoluble. Hence, by Lemma 2.2, $M/O_{p'}(M) = [P/O_{p'}(M)](H/O_{p'}(M))$, where $P/O_{p'}(M)$ is a Sylow p-subgroup and $H/O_{p'}(M)$ is a Hall p'-subgroup of $M/O_{p'}(M)$.

This shows that M has a chain of normal subgroups

$$1 < O_{p'}(M) < P < M$$

such that $O_{p'}(M)$ is a p'-subgroup, $P/O_{p'}(M)$ is a p-subgroup and M/P is a p'-subgroup.

Since $G = MN$, for all $g \in G$ we have $g = mn$, where $m \in M$ and $n \in N$. We have $(QN)^g = (QN)^{mn} = Q^nN = n^{-1}QnN = n^{-1}QN = n^{-1}NQ = QN$ for all normal subgroups Q of M and G has a chain of normal subgroups

$$1 < N < O_{p'}(M)N < PN < MN = G,$$

where $|NO_{p'}(M)/N| = |O_{p'}(M)|$ is a p'-number, $|PN/NO_{p'}(M)| = |P/O_{p'}(M)|$ is a p-number and $|G/PN| = |MN/PN| = |M/P|$ is a p'-number. This shows that $l_p(G) \leqslant 2$, contrary to the choice of G. Thus, the proof is completed.

Corollary 3.7. *If G is a p-soluble group and every 3-maximal subgroup of G is permutable with every 2-maximal subgroup of G whose index is divisible by p, then $l_p(G) \leqslant 2$.*

Theorem 3.8. *If G is a p-soluble group and every maximal subgroup of G is permutable with all 2-maximal subgroups M of G such that $|G : M|$ is a power of p, then $l_p(G) \leqslant 2$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Then:

(1) *G/N has the p-length $\leqslant 2$ for any non-identity normal subgroup N of G* (see the proof of Theorem 3.2).

(2) $O_{p'}(G) = 1$ (this fact follows directly from (1)).

(3) *G has the only minimal normal subgroup N and $G = [N]M$, where $N = F(G) = O_p(G)$ and M is a maximal subgroup of G such that $M_G = 1$ and $p||M|$* (see the proof of Theorem 3.2).

(4) *The end of the proof.*

Let M_1 be an arbitrary 2-maximal subgroup of M such that $|M : M_1| = p^\beta$ for some $\beta \in \mathbb{N}$. Clearly, M_1 is a 3-maximal subgroup of G such that $|G : M_1| = |G : M||M : M_1| = |N||M : M_1|$ is a power of p. As before (see the proof of Theorem 3.1), we can see that $M_1 \leqslant M_G = 1$. Thus, all 2-maximal subgroups M_1 of M such that $|M : M_1|$ is a power of p are trivial and therefore, M_1 is permutable with every maximal subgroup of M. In view of Theorem 3.2 (a), we have $l_p(M) \leqslant 1$. Since by (3), $M \simeq G/N$, we obtain that $l_p(G/N) \leqslant 1$. It is clear that $l_p(G) \leqslant 2$, this contradicts to the choice of G. Thus, the proof is completed.

Theorem 3.9. *If G is a p-soluble group and every maximal subgroup of G is permutable with every 3-maximal subgroup of G whose index is divisible by p, then $l_p(G) \leqslant 1$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. We now prove the theorem via the following steps.

(1) *G/N has the p-length $\leqslant 1$ for any non-identity normal subgroup N of G* (see the proof of Theorem 3.2).

(2) $O_{p'}(G) = 1$ (this fact follows directly from (1)).

(3) *G has the only minimal normal subgroup N and $G = [N]M$, where $N = F(G) = O_p(G)$ and M is a maximal subgroup of G such that $M_G = 1$ and $p||M|$* (see the proof of Theorem 3.2).

(4) *The end of the proof.*

Let M_1 be an arbitrary 2-maximal subgroup of M. As before (see the proof of Theorem 3.1), we can see that $M_1 \leqslant M_G = 1$. Thus, every 2-maximal subgroup of M is trivial, and therefore, all maximal subgroups of M have prime order. This shows that $|M|$ is either p^2 or pq. If $|M| = p^2$, then G is a p-group and $l_p(G) \leqslant 1$, contrary to our choice of G. Thus,

$|M| = pq$. Let P be a Sylow p-subgroup and Q be a Sylow q-subgroup of M. Then $M = PQ$. Since $|G : NQ| = p$, NQ is a maximal subgroup of G. Clearly, N is a maximal subgroup of NQ. Let N_1 be an arbitrary maximal subgroup of N. It is clear that N_1 is a 3-maximal subgroup of G and, by the hypothesis, N_1 permutes with M. Thus, either $N_1M = G$ or $N_1M = M$. Since $G = [N]M$, the first case is impossible. Thus, $N_1M = M$ and $N_1 \leqslant M$. Hence $N_1 \leqslant M \cap N = 1$. This shows that every maximal subgroup of N is trivial. Thus, $|N| = p$. Since $|NP| = p^2$, NP is abelian and $P \leqslant C_G(N) = N$. Hence $P \leqslant N \cap M = 1$, and $|M| = pq$. This contradiction completes the proof of Theorem.

Corollary 3.10. *If every maximal subgroup of G is permutable with every 3-maximal subgroup of G, then G is soluble and $l_p(G) \leqslant 1$ for all prime p.*

Proof. The solubility of G follows from [3]. In view of Theorem 3.9, we have $l_p(G) \leqslant 1$ for all prime p.

Using the same arguments in the proof of Theorem 3.9, we can also prove the following Theorem.

Theorem 3.11. *If G is a p-soluble group and every 3-maximal subgroup of G is permutable with all maximal subgroups M of G such that $|G : M|$ is a power of p, then $l_p(G) \leqslant 1$.*

Abstract. In 1966 L. Polyakov has obtained some criteria for the solubility and the supersolubility of groups by considering the conditions of permutability for maximal, 2-maximal and 3-maximal subgroups. The purpose of this article is obtaining local analogs of the results of Polyakov. Besides, we have extended the results of Polyakov, giving evaluation of the p-length of the considered groups.

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