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On $WCAP_p$-embedded and CAP_p-embedded subgroups of finite groups

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1. Introduction

Throughout this paper, all groups are assumed to be finite.

In a number of papers, several authors have studied groups some of whose distinguished families of subgroups have the cover and avoidance property. The results have included structure theorems (see [2]) and sometimes assertions that groups some but not all of whose relevant families of subgroups have the cover and avoidance property are soluble, supersoluble and nilpotent. For a fuller description of some of these results and other related ones, we refer the reader to the papers [5], [6], [8], [9] and [10].

Our object here is to take this programme of research a stage further by presenting common extensions of most of the results contained in the aforesaid articles.

We begin with the following definition:

Let $K \leq H \leq G$ and $A \leq G$ where K is normal in G. Then A *covers* the factor H/K if $(A \cap H)K = H$ and A *avoids* the factor H/K if $A \cap H \leq K$. We say following [4], Definition A.10.8, that *A has the cover and avoidance properties in G, A is a CAP-subgroup of G* for short, if it either covers or avoids every chief factor of G.

It is well known that in a soluble group every maximal subgroup and every Hall subgroup is a CAP-subgroup. The system normalisers of a soluble group are CAP-subgroups as well. Another interesting type of CAP-subgroups was found by Gaschütz in [7]: he proved that in every soluble group G there is the class of conjugate subgroups Σ such that every subgroup of Σ avoids the complemented chief factors of G and covers the rest.

Unfortunately, the cover and avoidance property is not hereditary in intermediate subgroups (see an example in [2]).

This nasty situation leads up to the introduction of a partial cover and avoidance property that although it is weaker than the "total " CAP-property, it is hereditary in intermediate subgroups (see [2], Lemma 2.4).

Definition 1.1. *Let A be a subgroup of a group G. Then, we say that A is a* partial CAP-subgroup *of G if there exists a chief series Γ_A of G such that A either covers or avoids each factor of Γ_A.*

This embedding property was first studied by Y. Fan, X. Guo and K. P. Shum in [6]. They named these subgroups *semi CAP-subgroups*. This is the name used in the subsequent papers [9] and [10]. However, we think that the name *partial CAP-subgroup* is more descriptive.

A local embedding property related to the partial CAP-property is the following:

Definition 1.2. *Let A be a subgroup of a group G and let p be a prime. We say that A is* a partial CAP_p-subgroup of G *(respectively* a CAP_p-subgroup of G*) if there exists a chief series Γ_A of G such that A either covers or avoids every factor of Γ_A (respectively every chief factor of G) whose order is divisible by p.*

Every subgroup of a p-supersoluble group G is a CAP_p-subgroup of G.

Assume now that G is a soluble group and D is a system normaliser of G. If G', the commutator subgroup of G, contains only central chief factors and T is the nilpotent residual of G, then $G = DT$ and $D \cap T = G'$ is a CAP-subgroup of G. This motivates the following:

Definition 1.3. *Let A be a subgroup of a group G and let p be a prime. Then we say that*

(1) A is (partially) CAP_p-embedded in G *if G has normal subgroups T and X such that $X = AT$ and $T \cap A$ is a (partial) CAP_p-subgroup of G.*

(2) A is well CAP_p-embedded in G *if G has a normal subgroup T such that $G = AT$ and $T \cap A$ is a CAP_p-subgroup of G.*

Note that every (partial) CAP_p-subgroup of G is (partially) CAP_p-embedded in G. However the converse does not hold in general.

A subgroup A of a group G is said to be c-normal in G if G has a normal subgroup T such that $G = AT$ and $T \cap A = A_G$. (See [17])

Example 1.4. Let A be any c-normal subgroup of a group G. Then A is well CAP_p-embedded in G for all primes p.

The first main result of the paper shows that p-supersolubility can be characterised by the CAP_p-embedding of some relevant families of subgroups.

Our second aim is to study another subgroup embedding property which is related to solubility. It is based on the fact that a soluble group can be characterised by the fact that every subgroup covers and avoids every composition factor of the group.

Definition 1.5. *Let A be a subgroup of a group G and let p be a prime. We say that A is* a partial $WCAP_p$-subgroup of G *(respectively* a $WCAP_p$-subgroup of G*) if there exists a composition series Γ_A of G such that A either covers or avoids every factor of Γ_A (respectively every composition factor of G) whose order is divisible by p.*

Every subgroup of a p-soluble group G is a $WCAP_p$-subgroup of G and every subnormal subgroup of a, non-necessarily p-soluble, group is a $WCAP_p$-subgroup.

The corresponding "composition"version of CAP_p-embedding is the following:

Definition 1.6. *Let A be a subgroup of a group G, p be a prime. Then we say that A* is (partially) $WCAP_p$-embedded in G *if G has subnormal subgroups T and X such that $X = AT$ and $T \cap A$ is a (partial) $WCAP_p$-subgroup of G.*

It is worth noting that every (partial) $WCAP_p$-subgroup of G is (partially) $WCAP_p$-embedded in G.

The second main result of the paper shows that p-soluble groups are to $WCAP_p$-embedding as p-supersoluble groups are to CAP_p-embedding.

2. General properties of $WCAP_p$-embedded and CAP_p-embedded subgroups

The purpose of the present section is to collect some properties of CAP_p and $WCAP_p$-embedded subgroups which are needed in the proofs of our main results.

Lemma 2.1. *Let $M \leq G$ and N be a normal subgroup of a group G. If M is a $WCAP_p$-subgroup (respectively, a CAP_p-subgroup) of G, then NM is a $WCAP_p$-subgroup (respectively, a CAP_p-subgroup) of G.*

Lemma 2.2. *Let A be a subgroup of a group G and suppose that G has subgroups K, L, T and H satisfying $K \leq L \leq T \leq H$, K is normal in H and L is normal in T. Then*

(1) If A covers H/K, then A covers T/L.

(2) If A avoids H/K, then A avoids T/L.

Lemma 2.3. *Let G be a group, let p be a prime and let A and D be subgroups of G such that A is contained in D. Then*

(1) If A is a partial $WCAP_p$-subgroup of G, then A is a partial $WCAP_p$-subgroup of D.

(2) If A is a partial CAP_p-subgroup of G, then A is a partial CAP_p-subgroup of D.

(3) If A is a $WCAP_p$-subgroup of G and D is subnormal in G, then A is a $WCAP_p$-subgroup of D.

Lemma 2.4. *Let G be a group and let p be a prime. For a subgroup A of G and a normal subgroup N of G, we have*

(1) If A is a partial $WCAP_p$-subgroup of G and either $N \leq A$ or $(|N|, |A|) = 1$, then AN is a partial $WCAP_p$-subgroup of G and AN/N is a partial $WCAP_p$-subgroup of G/N.

(2) If A is a $WCAP_p$-subgroup of G, then AN/N is a $WCAP_p$-subgroup of G/N.

(3) If A is a partial CAP_p-subgroup of G and either $N \leq A$ or $(|N|, |A|) = 1$, then AN is a partial CAP_p-subgroup of G and AN/N is a partial CAP_p-subgroup of G/N.

(4) If A is a CAP_p-subgroup of G, then AN/N is a CAP_p-subgroup of G/N.

We use $A^{\cdot\cdot G}$ to denote the subnormal closure of the subgroup A of a group G, that is, the intersection of all subnormal subgroups of G containing A (see Section 6.2 in [1] for the main properties of this subgroup).

Lemma 2.5. *Let A be a subgroup of a group G. Then*

(1) If A is (partially) $WCAP_p$-embedded in G, then G has a subnormal subgroup T such that $AT = A^{\cdot\cdot G}$ and $A \cap T$ is a (partial) $WCAP_p$-subgroup of G.

(2) If A is (partially) CAP_p-embedded in G, then G has a normal subgroup T such that $AT = A^G$ and $A \cap T$ is a (partial) CAP_p-subgroup of G.

Lemma 2.6. *Let G be a group and let p be a prime, Suppose that H and K are subgroups of G such that $H \leq K \leq G$. Then*

(1) If H is normal in G and K is $WCAP_p$-embedded in G, then K/H is $WCAP_p$-embedded in G/H.

(2) If H is normal in G and K is CAP_p-embedded in G, then K/H is CAP_p-embedded in G/H.

(3) If H is partially $WCAP_p$-embedded in G, then H is partially $WCAP_p$-embedded in K.

(4) If H is partially CAP_p-embedded in G, then H is partially CAP_p-embedded in K.

(5) If H is $WCAP_p$-embedded in G and K is subnormal in G, then H is $WCAP_p$-embedded in K.

(6) If H is normal in G and K is well CAP_p-embedded in G, then K/H is well CAP_p-embedded in G/H.

3. Groups with some families of CAP_p-embedded subgroups

In this section, p will denote a prime.

Lemma 3.1. *Let N be a non-trivial normal p-subgroup of a group G. If N is elementary and every maximal subgroup of N is well CAP_p-embedded in G, then some maximal subgroup of N is normal in G.*

It is obvious that a maximal subgroup M of a group G cannot be written as a proper intersection of subgroups of G. Proper subgroups of G with this property are called *primitive* (in [11]) or *meet-irreducible* (in [18] and [2]).

Theorem 3.2. *Let G be a group. Then any two of the following statement are equivalent:*

(1) G is p-supersoluble.

(2) Every subgroup of G is CAP_p-embedded in G.

(3) G is p-soluble and G has a normal subgroup E such that G/E is p-supersoluble and every maximal subgroup of every Sylow p-subgroup of E is well CAP_p-embedded in G.

(4) Every meet-irreducible subgroup of every maximal subgroup of G is CAP_p-embedded in G.

(5) Every meet-irreducible subgroup of G is CAP_p-embedded in G.

(6) G is p-soluble and every cyclic subgroup of G of prime order or order 4 is partially CAP_p-embedded in G.

(7) G is p-soluble and G has a normal subgroup E such that G/E is p-supersoluble and every maximal subgroup of every Sylow p-subgroup of $O_{p',p}(E)$ is well CAP_p-embedded in G.

We say that a subgroup A of a group G is *(partially, well) CAP-embedded in G* if A is (partially, well) CAP_p-embedded in G for all primes p dividing $|G|$.

Corollary 3.3. *For a group G the following statements are pairwise equivalent:*

(1) G is supersoluble.

(2) Every subgroup of G is CAP-embedded in G.

(3) G has a normal subgroup E such that G/E is supersoluble and every maximal subgroup of every Sylow subgroup of E is well CAP-embedded in G.

(4) Every meet-irreducible subgroup of G is CAP-embedded in G.

(5) Every meet-irreducible subgroup of every maximal subgroup of G is CAP-embedded in G.

(6) Every cyclic subgroup of G of prime order or order 4 is partially CAP-embedded in G.

(7) G is soluble and G has a normal subgroup E such that G/E is supersoluble and every maximal subgroup of every Sylow subgroup of $F(E)$ is well CAP-embedded in G.

Corollary 3.4 (Buckley [3]). *Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.*

Corollary 3.5 (Srinivasan [16]). *If the maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersoluble.*

The following two corollaries are the main resuls of the paper [17].

Corollary 3.6 (Wang [17]). A group G is supersoluble if G has a normal subgroup E such that G/E is supersoluble and every maximal subgroup of every Sylow subgroup of E is c-normal in G.

Corollary 3.7 (Wang [17]). A group G is supersoluble if every cyclic subgroup of prime order or order 4 is c-normal in G.

Corollary 3.8 (Ramadan [14]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

Corollary 3.9 (Li, Guo [12]). *Let G be a group and E be a soluble normal subgroup of G such that G/E is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 3.10 (Fan, Guo, Shum [6]). *If every cyclic subgroup of G of prime order or order 4 is a partially CAP-subgroup of, then G is supersoluble.*

4. Some new characterisations of p-soluble groups

In the sequel, p will be a prime.

Lemma 4.1. *For any Sylow subgroup P of a group G we have $P^{\cdot\cdot G} = P^G$.*

Theorem 4.2. *Let G be a group. Then the following statements are equivalent in pairs:*

(a) G is p-soluble.

(b) Every subgroup of G is $WCAP_p$-embedded in G.

(c) Every maximal subgroup of G is $WCAP_p$-embedded in G.

(d) Every 2-maximal subgroup of G is $WCAP_p$-embedded in G.

(e) Either G is cyclic or G has two p-soluble maximal subgroups M_1 and M_2 which are $WCAP_p$-embedded in G and $(|G : M_1|, |G : M_2|) = r^a q^b$ for some primes r, q, and some non-negative integers a, b.

(f) G is nilpotent or every non-supersoluble Schmidt subgroup of G is partially $WCAP_p$-embedded in G.

(g) Every Sylow p-subgroup of G is partially $WCAP_p$-embedded in G.

(h) p is the smallest prime dividing $|G|$ and every maximal subgroup of a Sylow p-subgroup of G is $WCAP_p$-embedded in G.

We say that a subgroup A of a group G is *(partially) $WCAP$-embedded in G* if A is (partially) $WCAP_p$-embedded in G for all primes p dividing $|G|$. We say that a subgroup A of a group G is *well CAP-embedded in G* if A is well $WCAP_p$-embedded in G for all primes p dividing $|G|$.

Corollary 4.3. *Given a group G, the following statements are pairwise equivalent:*

(a) G is soluble.

(b) Every subgroup of G is $WCAP$-embedded in G.

(c) Every maximal subgroup of G is $WCAP$-embedded in G.

(d) Every 2-maximal subgroup of G is $WCAP$-embedded in G.

(e) Either G is cyclic or G has two soluble maximal subgroups M_1 and M_2 which are $WCAP$-embedded in G and $(|G : M_1|, |G : M_2|) = r^a q^b$ for some primes r, q, and some non-negative integers a, b.

(f) G is nilpotent or every non-supersoluble Schmidt subgroup of G is partially $WCAP$-embedded in G.

(g) Every Sylow subgroup of G is partially $WCAP$-embedded in G.

(h) Every maximal subgroup of every Sylow subgroup of G is $WCAP$-embedded in G.

(j) G has a soluble maximal subgroup M such that M is well CAP-embedded in G.

Corollary 4.4 (Ore [13] or Wang [17]). *A group G is soluble if and only if every maximal subgroup of G is c-normal in G.*

Corollary 4.5 (Wang [17]). *A group G is soluble if and only if G has a maximal subgroup M such that M is c-normal in G.*

Corollary 4.6 (Guo, Shum [8]). *If every 2-maximal subgroup of G is a CAP-subgroup of G, then G is soluble.*

Abstract. Let G be a finite group and let p be a prime. Then a subgroup A of G is *$WCAP_p$-embedded in G*, if G has subnormal subgroups X and T such that $AT = X$ and $A \cap T$ either covers or avoids every composition factor H/K of G such that p divides $|H/K|$; A is said to be *CAP_p-embedded in G*, if G has normal subgroups X and T such that $AT = X$ and $A \cap T$ either covers or avoids every chief factor H/K of G such that p divides $|H/K|$. In this pa-per we study finite groups with some distinguished families of $WCAP_p$-embedded or CAP_p-embedded subgroups.

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