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On automorphisms for nilpotent subsemigroups of an order-decreasing transformation semigroup

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1. Introduction

Let $\mathcal{T}_n$ be a symmetric semigroup of all transformations of the set $N = \{1, .., n\}$. A transformation $\alpha \in \mathcal{T}_n$ is called an order-decreasing transformation if for all x of N it is true that $x\alpha \leqslant x$.

The set $\mathcal{D}_n$ of all order-decreasing transformations of $\mathcal{T}_n$ is a semigroup. This semigroup first appeared in Pin's monograph ([8]) and have been studied by various mathematicians afterwards (Howie [3], Higgins [2] and Umar [14], [15]).

A semigroup S with a zero 0 is called nilpotent if for some natural number l an equality $S^l = 0$ holds; a minimal number l satisfying this condition is called a nilpotency class of S. It is necessary to point that a transformation 0 which maps N into $\{1\}$ is the zero of the semigroup $\mathcal{D}_n$. An arbitrary subsemigroup S of $\mathcal{D}_n$ is nilpotent if and only if for all $x \in S$, $m \in N$ it is true that $x(m) < x$.

Let $Nil(n, k)$ stand for the set of the subsemigroups in $\mathcal{D}_n$ which are maximal among all nilpotent subsemigroups of nilpotency class k of T.

For any $m \in N$ and $A \subset N$ such that $m \notin A$ we define sets $Less(m, A) = \{x \in A | x < m\}$ and $Up(m, A) = \{x \in A | x > m\}$. Cardinalities of these sets we denote by $less(m, A)$ and $up(m, A)$ correspondingly.

For some fixed $k < n$ let $\Lambda(n, k)$ stand for the set of all ordered partitions $(Q_1, \cdots, Q_k)$ of $N \setminus \{1\}$ satisfying the following conditions for all l, $1 \leqslant l < k$:

$$1) \max_{i \in Q_l} i > \max_{i \in Q_{l+1}} i; \quad 2) \min_{i \in Q_l} i > \min_{i \in Q_{l+1}} i.$$

Under an ordered partition of some set A we mean an ordered chain of nonempty disjoint subsets (blocks) $Q_1, Q_2 \ldots, \subset A$ such that $A = Q_1 \cup Q_2 \ldots$.

For $\lambda \in \Lambda(n, k)$ with blocks $Q_1, \ldots, Q_k$ we define

$$T_\lambda = \{\varphi \in \mathcal{T}_n | (i \in Q_m) \Rightarrow (\varphi(i) \in Less(i, Q_{m+1} \cup \ldots \cup Q_k \cup \{1\}))\}.$$

Due to [9] T_λ is a subsemigroup in $Nil(n, k)$.

For any $A \subset N$ and $S \subset \mathcal{T}_n$ we define $S(A) = \{\varphi(a) | \varphi \in S, a \in A\}$. For an arbitrary semigroup $T \in Nil(n, k)$ let $Q_1, Q_2, \ldots, Q_p, \ldots$ be defined as follows :
$Q_1 = N \setminus T(N)$;
$Q_2 = (N \setminus Q_1) \setminus T(N \setminus Q_1)$;
$Q_3 = (N \setminus (Q_1 \cup Q_2)) \setminus T(N \setminus (Q_1 \cup Q_2)$;
$\ldots$;
$Q_p = (N \setminus (Q_1 \cup \cdots \cup Q_{p-1})) \setminus T(N \setminus (Q_1 \cup \cdots \cup Q_{p-1})$;
$\ldots$

As it is shown in [9], $Q_1, \ldots Q_p, \ldots$ form a partition of $N \setminus \{1\}$ and total number of blocks k. We shall refer to the ordered partition $N \setminus \{1\} = Q_1 \cup \cdots \cup Q_k$ as λ_T.

Theorem 1. *[9] Mappings $\varphi : \Lambda(n,k) \to Nil(n,k), \lambda \longmapsto T_\lambda$ and $\psi : Nil(n,k) \to \Lambda(n,k), T \longmapsto \lambda_T$ are reciprocal and determine one-to-one correspondence between $\Lambda(n,k)$ and $Nil(n,k)$.*

In [10] this theorem has been extended to the case of of order-decreasing transformations of a rooted tree.

While investigating semigroup it is naturally to consider also the automorphism groups of these semigroups. There are a lot of papers dedicated to the structure of automorphism groups of different semigroups (e.g. [1], [4], [6], [5], [7], [12], [11], [13]).

In our paper we investigate automorphism groups of semigroups in $Nil(n,k)$ and with the help of methods described in [1] we prove that each of these groups can be represented as a semidirect product of direct sums of symmetric groups.

Note that we perform transformations from left to right, i.e. $(\varphi \cdot \psi)(x) = \psi(\varphi(x))$.

2. Auxiliary propositions

Let $T \in Nil(n,k)$, $n \geqslant 2$ and $\lambda_T = (Q_1, \ldots, Q_k)$ be a corresponding partition of $N \setminus \{1\}$. For an arbitrary element s let $doms = \{m \in N | s(m) \neq 1\}$, $rans = s(N) \setminus \{1\}$. In the following we shall refer to $|rans|$ as $ranks$. One can construct an embedding ρ of $\mathcal{D}_n$ into the semigroup $\mathcal{PT}_{n-1}$ of all partial transformations of $\{2, \ldots, n\}$, putting $\rho(s)(m) = s(m)$ if and only if $s(m) \neq 1$. In particular, ρ maps the zero of the semigroup $\mathcal{D}_n$ to the zero of the semigroup $\mathcal{PT}_{n-1}$, i.e. to the completely undefined transformation. Thus $doms$, $rans$ and $ranks$ coincide with the domain, range and rank of the transformation $\rho(s)$ correspondingly.

For each indecomposable element $s \in T$ we consider a set $M_s = \{m \in Q_1 : s(m) \in$ $\in Q_k\}$ and an element s_*, where

$$s_*(m) = \begin{cases} 1, m \in M_s \\ s(m), m \notin M_s \end{cases}$$

Let $\sim$ be an equivalence relation, which coincides with equality relation on the set of all decomposable elements of T, and for indecomposable elements

$$a \sim b \Leftrightarrow a_* = b_*$$

Lemma 2. *The relation $\sim$ is a congruence on the semigroup T.*

Proof. It is easy to verify that for any a, b, c in T an inclusion $a \sim b$ implies $ac =$ $= bc$ and $ca = cb$; hence $ac \sim bc$ and $ca \sim bc$. Thus the relation $\sim$ is both left and right compatible.

Lemma 3. *If a and b are indecomposable elements of T, then*

$$a \sim b \Leftrightarrow ac = bc, \quad ca = cb \quad \forall c \in T$$

Proof. An implication $a \sim b \Rightarrow ac = bc, ca = cb \quad \forall c \in T$ follows from the proof of the lemma 2. Next, let $\forall c \in T$ $ac = bc, ca = cb$. For all $m \in (rana \setminus Q_k)$ we define a transformation c_m such that $domc_m = m$ and $ranc_m = \min\limits_{i \in Q_k} i$. Obviously, $c_m \in T$. Then $ac_m = bc_m$ implies an inclusion $m \in ranb \setminus Q_k$ and the equality $\{h \in N : a(h) = m\} = \{h \in$ $\in N : b(h) = m\}$. Analogously, for all $m \in doma \setminus Q_1$ we define a transformation d_m such that $rand_m = m$ and $domc_m = \max\limits_{i \in Q_1} i$. Again, $d_m \in T$ and $c_m a = c_m b$ implies $m \in domb \setminus Q_1$ and $a(m) = b(m)$. Hence $M_a = M_b$ and $a(m) = b(m)$ for all $m \notin M_a$ and so $a_* = b_*$.

Lemma 4. *The congruency $\sim$ is invariant under an arbitrary automorphism of the semigroup T.*

Proof. It implies from the lemma 3, for an automorphism preserves the decomposability and indecomposability of an element.

Let $T = \bigcup\limits_{i\geqslant 1} M_i$ be a decomposition of T into the union of equivalence classes of relation $\sim$, and $\oplus_{i\geqslant 1} S_{M_i}$ be a direct sum of symmetric groups S_{M_i} (as groups of permutations).

Lemma 5. *$\oplus_{i\geqslant 1} S_{M_i}$ is a normal subgroup of $Aut(T)$.*

Proof. Let π be in $\oplus_{i\geqslant 1} S_{M_i}$. As for any s_1, s_2 from T an element $s_1 s_2$ is decomposable, then it is true that $\pi(s_1 s_2)) = s_1 s_2$. Next, from the lemma 3 it follows that $s_1 s_2 = s_1 \pi(s_2) = \pi(s_1)\pi(s_2)$, hence $\pi(s_1 s_2) = \pi(s_1)\pi(s_2)$ and $\oplus_{i\geqslant 1} S_{M_i}$ is a subgroup of $Aut(T)$. Let γ be from $Aut(T)$. As $\pi(s) \sim s_1$, then from the lemma 4 it follows that $\gamma(\pi(s_1)) \sim \gamma(s_1)$. Thus there exists a permutation μ from $\oplus_{i\geqslant 1} S_{M_i}$ such that $\mu(\gamma(s_1)) = \gamma(\pi(s_1))$, in other words for any automorphism γ from $Aut(T)$

$$\gamma(\oplus_{i\geqslant 1} S_{M_i}) = (\oplus_{i\geqslant 1} S_{M_i})\gamma$$

Lemma 6. *Let s be from T, γ be from $Aut(T)$ and let a from Q_i such that $s(a) \in Q_{i+1}$ exist. Then there exists a' from Q_i such that $\gamma(s)(a') \in Q_{i+1}$.*

Proof. For some $s \in T$ the existence of $a \in Q_i$ and $b \in Q_{i+1}$, such that $s(a) \in Q_{i+1}$ is equipotential to the existence of $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{k-1}$ from T such that $s_1 \cdot s_2 \cdot \ldots \cdot s_{i-1} \cdot s \cdot s_{i+1} \cdot \ldots \cdot s_{k-1} \neq 0$. The statement of the lemma now implies from the fact that the latter inequality is equivalent to $\gamma(s_1) \cdot \gamma(s_2) \cdot \ldots \cdot \gamma(s_{i-1}) \cdot \gamma(s) \cdot \gamma(s_{i+1}) \cdot \ldots \cdot \gamma(s_{k-1}) \neq 0$.

The next lemma can be proved analogously.

Lemma 7. *Let s be from T, γ be from $Aut(T)$, $1 < i < k$. Then it is true that:*

1. *$doms \cap Q_i \neq \emptyset \Leftrightarrow dom\gamma(s) \cap Q_i \neq \emptyset$*
2. *$rans \cap Q_i \neq \emptyset \Leftrightarrow ran\gamma(s) \cap Q_i \neq \emptyset$*

For $i < k$ we define the set Φ_i as follows:

$$\Phi_i = \{s \in T : ranks_* = 1, doms_* \subset Q_i, rans_* \in Q_{i+1}, \text{ and } |doms_*| = 1 \text{ when } i > 1\}$$

Lemma 8. *For any $i < k$ the set Φ_i is invariant under an arbitrary γ from $Aut(T)$.*

Proof. Let A stand for the set of all transformations s from T, satisfying :
1)there exist such $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{k-1}$, that $s_1 \cdot s_2 \cdot \ldots \cdot s_{i-1} \cdot s \cdot s_{i-1} \cdot \ldots \cdot s_{k-1} \neq 0$
2)if $i \notin \{k-1, 1\}$, then for any $t \in sT \setminus \{0\}$ there exist such $t_1, \ldots, t_{i-1} \in T$, that $t_1 \cdot \ldots \cdot t_{i-1} \cdot t \neq 0$.
We observe that for all t from T it is true that $dom(st)_* \subset doms_*$. From the second clause of the definition of A it follows that for any t from T the intersection of the set $Q_i \cup Q_{i+1} \cup \cdots \cup Q_{k-1}$ and the domain of st is nonempty. Hence we conclude that for any $i \neq k-1$ and A it is true that $doms_* \in Q_i \cup Q_{i+1} \cup \cdots \cup Q_{k-1}$.

Let $i > 1$. We consider a set

$$\Psi_i^1 = \{t \in A : \text{ an ideal } Tt \text{ is a minimal left ideal}\}$$

It is quite clear that for any r from A an ideal Tr contains the elements of rank 1, whose domain is contained in $Up(\max\limits_{m\in Q_i} m, Q_1 \cup \cdots \cup Q_{i-1})$, and whose range is a number from $Q_{i+1} \cap ranr_*$. At the same time, if $r \in A$, $|domr_*| = 1$ and

$$domr_* \in \{b \in Q_1 : Up(b, Q_1 \cup \cdots \cup Q_{i-1}) = Up(\max_{m\in Q_i} m, Q_1 \cup \cdots \cup Q_{i-1})\},$$

then Tr consists of elements of rank 1 whose domain is a subset of the set $Up(\max_{m\in Q_i} m, Q_1 \cup \cup \cdots \cup Q_{i-1})$ and whose range is equal to $ranr_*$. Particulary, in such a case Tr is a minimal left ideal. Thus we can state that the set Ψ_i^1 is not empty. Next, the inclusion $\varphi \in \Psi_i^1$ implies that φ_* is a transformation of rank 1. Indeed, in such a case there exists $c \in Q_{i+1} \cap \cap ran\varphi_*$, and $T\varphi$ contains an ideal Tr_c, where $domr_c = \max_{m\in Q_i} m$, $ranr_c = c$. If there exists $d \in (Q_{i+1} \cup \cdots \cup Q_k) \cap ran\varphi_*$ and $c \neq d$, then $T\varphi$ contains a transformation r_d, defined as follows: $domr_d = \max_{m\in Q_1} m$, $ranr_d = d$. It is evident that $r_d \notin Tr_c$. So, we have come to the contradiction.

Next, let Ψ_i be a set of all r from A such that an ideal Tr contains exactly one ideal of type Tx, where $x \in \Psi_i^1$, and if for some t an ideal Tt is contained in Tr, then Tt contains the ideal Tx.

Now we show that if ψ belongs to Ψ_i, then $rank\psi_* = 1$. Let $c, d \in ran\psi_*$. Since $\psi \in A$ we can suppose that $c \in Q_{i+1}$ without loss of generality. Let ψ_c and ψ_d be transformations defined as follows: $ran\psi_c = c$, $ran\psi_d = d$, $dom\psi_c = \max_{m\in Q_i} m = dom\psi_d$. Clearly, $\psi_c \in \Psi_i^1$. An ideal $T\psi$ contains all the elements of rank 1, whose domain is a subset of $Up(\max_{m\in Q_i} m, Q_1 \cup \cup \cdots \cup Q_{i-1})$, and whose range is c or d. Then it means that $T\psi$ contains $T\psi_c$ and $T\psi_d$, while $T\psi_c$ is an ideal from $\{Ts : s \in \Psi_i^1\}$. Therefore the inclusion $T\psi_c \subset T\psi_d$ fulfills. Hence $d \in Q_{i+1}$ and $c = d$.

Next, let K_i be the set of all r from A satisfying the following term: for all ψ from Ψ_j a set $\{\tau \in T : \tau r \neq 0\}$ does not contain a set $\{\tau \in T : \tau\psi \neq 0\}$ strictly. We show that $|dom\phi_*| = 1$ for all $\phi \in K_i$. Let $c, d \in Q_i$, $c \in dom\phi_*$, $d \in dom\phi_*$ and ϕ_c, ϕ_d be transformations from T_n satisfying $dom\phi_c = c$, $dom\phi_d = d$, $ran\phi_c = ran\phi_d = ran\phi_*$. It is easy to verify that ϕ_c and ϕ_d belong to Ψ_i. Note that the set $\{s \in T : s\phi \neq 0\}$ contains elements t_1 such that $dom(t_1)* \subset Up(c, Q_1 \cup \cdots \cup Q_{i-1})$, $ran(t_1)_* = c$ and elements t_2 such that $dom(t_2)_* \subset Up(d, Q_1 \cup \cdots \cup Q_{i-1})$, $ran(t_2)_* = d$. Thus $\{s \in T : s\phi \neq 0\}$ contains sets $\{s \in T : s\phi_c \neq 0\}$ and $\{s \in T : s\phi_d \neq 0\}$, and so, $\phi \notin K_i$. At the same time, if $\phi \in \Psi_i$ and $|dom\phi_*| = 1$, then for each ρ from Ψ_i a set $\{s \in T : s\rho \neq 0\}$ either contains $\{s \in T : s\rho \neq 0\}$, or has empty intersection with it, and so ϕ belongs to K_i.

Let now $\psi \in T$, $|dom\psi_*| = 1$, $dom\psi_* \in Q_i$, $ran\psi_* \in Q_{i+1}$. Then an ideal $T\psi$ consists of elements whose domain is equal to a subset of $Up(dom\psi_*, Q_1 \cup \cdots \cup Q_{i-1})$, and whose range is equal to $ran\psi_*$. Thus $T\psi$ contains exactly one ideal from $\{Tx : x \in \Psi_i^1\}$, namely an ideal consisting of elements with range equal to $ran\psi_*$, and domain equal to a subset of the set $Up(\max_{m\in Q_i} m, Q_1 \cup \cdots \cup Q_{i-1})$. $\psi \in \Psi_i$, and since $|dom\psi_*| = 1$, we have $\psi \in K_i$. So, if $i > 1$, then $K_i = \Phi_i$. Invariance of the set Φ_i under automorphisms now follows from the construction of sets A, Ψ_i^1, Ψ_i and K_i, so, for $i > 1$ the lemma is proved.

Let now $i = 1$. We consider a set Ψ_1 of all transformations t from A satisfying the following conditions:

1) t belongs to the right annulator $Ann_R(T)$ of the semigroup T, in other words for all r from T an equality $rt = 0$ fulfills;

2) for each r from $Ann_R(T) \cap A$ a set $\{s \in T : rs \neq 0\}$ is not contained in the set $\{s \in \in T : ts \neq 0\}$ strictly.

It implies from the first condition that domains of all the transformations from Ψ_1 are contained in the first block of the partition λ. Next, we show that for any element t from Ψ_1 it is true that $rankt_* = 1$. Assume that there exist c, d of $(c, d \in rant_*) \cap (Q_2 \cup \cdots \cup \cup Q_{k-1})$ and $c \neq d$. Since t belongs to A, then without loss of generality one can suppose that $c \in Q_2$. Let r_c be a transformation from $\mathcal{T}_n$ such that $domr_c = \{a \in Q_1 : t(a) = c\}$, $ranr_c = c$. It is clear that r_c belongs to T, furthermore, r_c belongs to $Ann_R T \cap A$. A set $\{s \in T : ts \neq 0\}$ contains elements p such that $c \in domp$, and so $\{s \in T : ts \neq 0\}$ contains a set $\{s \in T : rs \neq 0\}$. The set $\{s \in T : ts \neq 0\}$ contains an element p_1 such that $domp_1 = d$, $ranp_1 = \min\limits_{m \in Q_3} m$. Obviously, p_1 does not belong to $\{s \in T : r_c s \neq 0\}$, so we have come to a contradiction with the assumption $t \in \Psi_1$.

Let now $t \in Ann_R(T) \cap A$ and $rankt_* = 1$. Then a set $\{s : \in T : ts \neq 0\}$ contains all the elements, whose domain contains $rant$; for any $r \in Ann_R(T) \cap A$ a set $\{s : \in T : ts \neq 0\}$ does not contain the set $\{s : \in T : rs \neq 0\}$. So, Ψ_1 is equal to the set of all transformations s from T, which map some subset of Q_1 into an element from Q_2 and $rans_* = 1$, so $\Psi_1 = \Phi_1$. From the construction of Ψ_1 it follows that each automorphism of the semigroup T maps an element from Ψ_1 into an element from Ψ_1, so lemma is proved for the case of $i = 1$.

Corollary 9. *Let $s_1, s_2 \in \bigcup\limits_{i \geqslant 1} \Phi_i$ and $\gamma \in Aut(T)$. Then it is true that:*

1. $dom(s_1)_* = dom(s_1)_* \Leftrightarrow dom(\gamma(s_1))_* = dom(\gamma(s_1))_*$
2. $ran(s_1)_* = ran(s_1)_* \Leftrightarrow ran(\gamma(s_1))_* = ran(\gamma(s_1))_*$

Proof. Since $(s_1)_*$ and $(s_2)_*$ are transformations of rank 1, an equality $dom(s_1)_* = dom(s_2)_*$ is equivalent to

$$(s_1T = s_2T) \vee (s_1T \subset s_2T) \vee (s_2T \subset s_1T).$$

The latter condition is equivalent to

$$(T\gamma(s_1) = T\gamma(s_2) \vee T\gamma(s_1) \subset T\gamma(s_2) \vee T\gamma(s_2) \subset T\gamma(s_1),$$

so $dom(\gamma(s_1))_* = dom(\gamma(s_1))_*$. Thus the first part of the corollary is proved. The other one can be proved analogously.

Corollary 10. *Let $s \in T$ and $ranks_* = 1$, $doms_* \subset Q_i$, $rans_* \in Q_j$, $i < j, j - i \neq k - 1$ and in case of $i > 1$ $|doms_*| = 1$. Then for any $\gamma \in Aut(T)$ it is true that $dom\gamma(s)_* \in Q_i$, $ran\gamma(s)_* \in Q_j$, $rank\gamma(s)_* = 1$ and $|dom\gamma(s)_*| = 1$ when $i > 1$.*

Proof. Note that for any s from T a condition $doms \in Q_i$, $i > 1$ is equivalent to the existence of r from $\bigcup\limits_{l \geqslant 1} \Phi_l(i-1)$ such that $rs \neq 0$; a condition $rans \in Q_i$, $i \neq k$ is equivalent to the existence of r from $\bigcup\limits_{l \geqslant 1} \Phi_l(i)$ such that $sr \neq 0$; an inclusion $doms \in Q_1$ is equivalent to $s \neq 0$ and $doms \bigcap (\bigcup\limits_{l=2}^{k} Q_l) = \emptyset$; an inclusion $rans \in Q_k$ is equivalent to $s \neq 0$ and $rans \bigcap (\bigcup\limits_{l=1}^{k-1} Q_l) = \emptyset$. Then a set A of all the mappings s from T satisfying $doms_* \subset Q_i$, $rans_* \subset Q_j$ is invariant under automorphisms of the semigroup T.

Let $i > 1$ and $s \in A$. $|doms_*| = 1$ implies that for any $\phi, \psi \in \Phi_{i-1}$ satisfying $\psi \cdot s \neq 0$, $\phi \cdot s \neq 0$, it is true that $ran\phi_* = ran\psi_*$. Then it means that for any $\phi, \psi \in \Phi_{i-1}$ such that $\psi \cdot \gamma(s) \neq 0$, $\phi \cdot \gamma(s) \neq 0$ it is true that $ran\phi_* = ran\psi_*$. As $\gamma(s) \in A$, then $|dom(\gamma(s))_*| = 1$.

Let now $i = 1$ and s belong to A. $|rans_*| = 1$ implies that for any $\phi, \psi \in \Phi_{i+1}$ such that $s \cdot \psi \neq 0$, $s \cdot \phi \neq 0$, it is true that $dom\phi_* = dom\psi_*$. Then for any $\phi, \psi \in \Phi_{i+1}$ satisfying $\gamma(s) \cdot \psi \neq 0$, $\gamma(s) \cdot \phi \neq 0$, an equality $dom\phi_* = dom\psi_*$ fulfills. Since $\gamma(s)$ belongs to A, we have $rank(\gamma(s))_* = 1$.

Corollary 11. *Let* $s_1 \in \bigcup_{j\geqslant 1} \Phi_j(i)$, $s_2 \in T$, $rank(s_2)_* = 1$ *and* $\gamma \in Aut(T)$. *Then*

1. $dom(s_1)_* = dom(s_2)_* \Leftrightarrow dom(\gamma(s_1))_* = dom(\gamma(s_2))_*$

2. $ran(s_1)_* = ran(s_2)_* \Leftrightarrow ran(\gamma(s_1))_* = ran(\gamma(s_2))_*$

Proof. Since for the present terms

$$dom(s_1)_* = dom(s_2)_* \Leftrightarrow s_1T \cap s_2T \neq \{0\},$$

$$ran(s_1)_* = ran(s_2)_* \Leftrightarrow Ts_1 \cap Ts_2 \neq \{0\};$$

it is enough to use corollary 10 .

Corollary 12. *Let* $s_1 \in \bigcup_{j\geqslant 1} \Phi_j(i)$, $s_2 \in T$, $\gamma \in Aut(T)$. *Then*

1. $ran(s_1)_* \in ran(s_2)_* \Leftrightarrow ran(\gamma(s_1))_* \in ran(\gamma(s_2))_*$,

2. *if* $i > 1$, *then* $dom(s_1)_* \in dom(s_2)_* \Leftrightarrow dom(\gamma(s_1))_* \subset dom(\gamma(s_2))_*$,

3. *if* $i = 1$, $dom(s_1)_* \subset dom(s_2)_*$ *and for all* $t_1, t_2 \in dom(s_1)_*$, $t_3 \in N \setminus dom(s_1)_*$ *it is true that* $(s_2)_*(t_1) = (s_2)_*(t_2), s_2(t_1) \neq s_2(t_3)$, *then* $dom(\gamma(s_1))_* \subset dom(\gamma(s_2))_*$ *and for all* $t_1, t_2 \in dom(\gamma(s_1))_*, t_3 \in N \setminus dom(\gamma(s_1))_*$ *it is true that* $\gamma(s_2)(t_1) = \gamma(s_2)(t_2)$, $\gamma(s_2)(t_1) \neq \gamma(s_2)(t_3)$

Proof. An inclusion $ran(s_1)_* \in ran(s_2)_*$ is equivalent to the existence of such $s \in T$, that $|doms| = 1$, $rans = ran(s_1)_*$ and $Ts \subset Ts_2$. The first part of the corollary easily implies from corollaries 10 and 11 .

For the proof of the second part of the corollary assertion it is enough to observe that $dom(s_1)_* \in dom(s_2)_*$ if and only if there exists an element $s \in \bigcup_{j\geqslant 1} \Phi_j(i)$, such that $ss_1 = ss_2 \neq 0$.

Let now $i = 1$. Then an inclusion $dom(s_1)_* \subset dom(s_2)_*$ is equivalent to the existence of $s \in T$ such that $ranks = 1$, $doms = dom(s_1)_*$ and $sT \subset s_2T$. Now using corollaries 10 and 11 is sufficient for the proof of the third part of the assertion.

For any of the blocks Q_i, $1 \leqslant i < k$ we define partition $Q_i = Q_i^1 \cup Q_i^2 \cup \cdots$, where blocks of the partition are defined by the following equivalence relation:

$$\text{if } 1 \leqslant i < k, \text{ then } (a \overset{i}{\sim} b) \Leftrightarrow (\forall m \in [1, k] \setminus \{i\} \quad less(a, Q_m) = less(b, Q_m));$$

$$(a \overset{k}{\sim} b) \Leftrightarrow (\forall m, 1 < m < k \quad less(a, Q_m) = less(b, Q_m)).$$

and the order of the blocks is defined by inequalities $\max\limits_{m\in Q_i^1} > \max\limits_{m\in Q_i^2} m > \cdots$.

Lemma 13. *Let* $s \in T$, $|doms_*| = 1$, $\gamma \in Aut(T)$. *Then for any* i, $1 < i < k$, *it is true that*

1. $dom(s)_* \subset Q_i^j \Leftrightarrow dom(\gamma(s))_* \subset Q_i^j$

2. $ran(s)_* \subset Q_{i+1}^l \Leftrightarrow ran(\gamma(s))_* \subset Q_{i+1}^l$

Proof. Let $dom(s)_* = \{a\} \subset Q_i^j$. For any l, $1 < l < i$, we consider a set of all the transformations r from T, such that $|dom(r)_*| = 1$, $dom(r)_* \subset Q_l$, $rs \neq 0$. It is clear that the cardinality of this set equals $up(a, Q_l) \cdot |Ann(T)|$, where $Ann(T)$ is the annulator of the semigroup T. Hence $up(dom(\gamma(s))_*, Q_l) \geqslant up(a, Q_l)$. From bijectivity γ it follows that $up(dom(\gamma(s))_*, Q_l) = up(a, Q_l)$, and so $less(dom(\gamma(s))_*, Q_l) = less(a, Q_i)$.

Now we consider a set of all r from T, such that $dom(r)_* \subset Q_1, |\{ran(r)_*\}| = 1$, $rs \neq = 0$. The cardinality of this set equals to $(2^{up(a,Q_1)} - 1) \cdot |Ann(T)|$. Analogously to mentioned above, $up(dom(\gamma(s))_*, Q_1) = up(a, Q_1)$ and thus $less(dom(\gamma(s))_*, Q_1) = less(a, Q_1)$.

Next, for each l, $i < l \leqslant k$ we consider a set of all r from T such that $|dom(r)_*| = 1$, $ran(r)_* \subset Q_l$ and $(rT = sT) \vee (rt \subset sT) \vee (sT \subset rT)$. It is immediate that s belongs to this set. Next, the cardinality of this set equals to $less(a, Q_l) \cdot \cdot |Ann(T)|$. Thus $less(dom(\gamma(s))_*, Q_l) \geqslant less(a, Q_l)$. From bijectivity of γ it follows that $less(dom(\gamma(s))_*, Q_l) = less(a, Q_l)$. So, for all $l \neq i$ it is true that $less(dom(\gamma(s))_*, Q_l) = = less(a, Q_l)$. Thus $dom(\gamma(s))_* \subset Q_i^j$. One can prove the conversed implication $(dom(\gamma(s))_* \subset Q_i^j) \Rightarrow (dom(s)_* \subset Q_i^j)$ using already proved assertion for the automorphism γ^{-1}.

The second part of lemma can be proved analogously.

Lemma 14. *Let $s \in T$, $rank(s)_* = 1$, $\gamma \in Aut(T)$.*

1. *If $ran(s)_* \subset Q_k$, $|dom(s)_*| = 1$, then for any i, $1 < i < k$, it is true that $up(ran(s)_*, Q_i) = up(ran(\gamma(s))_*, Q_i)$*
2. *if $dom(s)_* \subset Q_1$, then for any i, $1 < i < k$, it is true that $less(\min\limits_{a \in dom(s)_*} a, Q_i) = = less(\min\limits_{a \in dom(\gamma(s))_*} a, Q_i)$*

Proof. The proof follows from the fact that the cardinality of the set

$$\{r : domr_* \in Q_1,\ rankr_* = 1,\ ranr_* \in Q_i,\ (rT = sT) \vee (rT \subset sT) \vee (sT \subset rT)\}$$

equals to a number $less(\min\limits_{a \in dom(s)_*} a, Q_i)$; and the cardinality of the set

$$\{r : domr_* \in Q_i,\ |domr_*| = 1,\ ranr_* \in Q_k,\ (Tr = Ts) \vee (Tr \subset Ts) \vee (Ts \subset Tr)\}$$

equals to a number $up(ran(s)_*, Q_i)$.

Lemma 15. *If the nilpotency class k of the semigroup T is greater than 3, then for any d, $1 \leqslant d \leqslant |Q_1|$ a set*

$$\{s \in T, \quad rank(s)_* = 1, \quad doms \subset Q_1, \quad |doms_*| = d\}$$

is invariant under an arbitrary automorphism γ of the semigroup T.

Proof. For any $A \subset Q_1$, $A \neq \varnothing$ and $b \in Q_2 \cup \ldots \cup Q_{k-1}$ we consider a set $\Theta(A, b)$ of all the transformations s from T satisfying $rank(s)_* = 1$, $doms_* = A$, $rans_* = b$. Note that $|\Theta(A, b)| = 1$ if and only if $A = Q_1$. Next, for s_1, s_2 and T, such that

$$rank(s_1)_* = 1 = rank(s_2)_*,\ doms_1 \in Q_1,\ doms_2 \in Q_1$$

and equality $dom(s_1)_* \cap dom(s_1)_* = \emptyset$ is equivalent to the existence of transformation s_3 from T, such that

$$rank(s_3)_* = 2,\ dom(s_3) \subset Q_1,\ ran(s_3)_* = ran(s_1)_* \cup ran(s_2)_*,$$

$$\{a \in Q_1 : s_3(a) \in ran(s_1)_*\} = dom(s_1)_*; \ \{a \in Q_1 : s_3(a) \in (s_2)_*\} = dom(s_2)_*.$$

Then inequality $doms \cap domt \neq \varnothing$ holds for any $t \in T$ satisfying $rankt_* = 1, = domt \in Q_1$ if and only if $doms_* = Q_1$. Hence $|doms_*| = |Q_1| \Leftrightarrow |dom(\gamma(s))_*| = |Q_1|$.

We consider all the elements from T satisfying $rankt_* = 1$, $domt \in Q_1$ $doms \cap domt = \emptyset$. If it is possible to choose $|Q_1| - 1$ elements with pairwise nonequal domains among all such elements, but one can not choose $|Q_1|$ elements among all such elements though, then it will be if and only if $|doms_*| = |Q_{k-1}|$. Hence we conclude that $|doms_*| = |Q_1| - 1$ and $|dom(\gamma(s))_*| = |Q_1| - 1$ are equivalent.

A semigroup T contains an elements t satisfying $rankt_* = 1$, $domt \in Q_1$, $|domt_*| = |Q_1| - 1$, $doms \cap domt = \emptyset$ if and only if $|domt_*| = 1$. So, equalities $|doms_*| = 1$ and $|dom(\gamma(s))_*| = 1$ are equivalent.

It is possible to choose i element with pairwise nonequal domains among all the elements t satisfying $rankt_* = 1$, $domt \in Q_1$, $|domt| = 1$, $doms \cap domt \neq \emptyset$, while $i + 1$ elements among all the elements satisfying mentioned condition can not be chosen, if and only if $|domt_*| = i$. Thus equalities $|doms_*| = i$ and $|dom(\gamma(s))_*| = i$ are equivalent.

Corollary 16. *Let $s \in T$, $\gamma \in Aut(T)$. Then equalities $|doms_*| = 1$ and $|dom(\gamma(s))_*| = 1$ are equivalent.*

Corollary 17. *Let s_1, $s_2 \in T$, $doms_1 \subset Q_1$, $doms_2 \subset Q_1$, $|dom(s_1)_*| = 1$, $rank(s_2)_* = 1$, $\gamma \in Aut(T)$. Then $(dom(s_1)_* \subset dom(s_2)_*) \Leftrightarrow (dom(\gamma(s_1))_* \subset dom(\gamma(s_2))_*)$*

Corollary 18. *Let s_1, $s_2 \in T$, $doms_1 \subset Q_1$, $|dom(s_1)_*| = 1$, $\gamma \in Aut(T)$. Then $dom(s_1)_* \subset dom(s_2)_* \Leftrightarrow dom(\gamma(s_1))_* \subset dom(\gamma(s_2))_*$.*

Proof. The proof follows from corollaries 17 and 12.

Lemma 19. *Let $s \in T$, $|doms_*| = 1$, $\gamma \in Aut(T)$. Then*

$$dom(s)_* \subset Q_1^j \Leftrightarrow dom(\gamma(s))_* \subset Q_1^j.$$

Proof. We consider a set M of all such transformations t from T, such that

$$domt \subset Q_1, \ |domt_*| = |Q_1| - 1, \ rant_* = \{\min_{a \in Q_2} a\}, \ domt_* \cap doms_* = \emptyset.$$

It follows from lemma 15 and corollary 12 that M is invariant under automorphisms of the semigroup T. It is clear that for all t from M $domt_* = Q_1 \setminus doms_*$. Besides, the cardinality of the set M equals $less(doms_*, Q_k) + 1$. Next, with the use of lemma 15 and corollary 10, we obtain

$$less(dom(\gamma(s))_*, Q_k) + 1 = less(doms_*, Q_3) + 1,$$

and so $less(dom(\gamma(s))_*, Q_k) = less(doms_*, Q_3)$. An implication $(dom(s_1)_* \subset dom(s_2)_*) \Rightarrow (dom(\gamma(s_1))_* \subset dom(\gamma(s_2))_*)$ follows now from lemma 14. To prove the converse implication one can apply already proved assertion for the automorphism γ^{-1} and the element $\gamma(s)$.

Lemma 20. *Let s_1, $s_2 \in T$, $|doms_*| = 1$, element s_2 is decomposable and $\gamma \in Aut(T)$. Then*

$$dom(s_1)_* \subset (Q_1 \cap doms_2) \Leftrightarrow dom(\gamma(s_1))_* \subset (Q_1 \cap dom\gamma(s_2)),$$

$$ran(s_1)_* \subset (Q_k \cap rans_2) \Leftrightarrow ran(\gamma(s_1))_* \subset (Q_k \cap ran\gamma(s_2)).$$

Proof. Since the element s_2 is decomposable then there exist t_1, t_2 from T, such that

$$dom(t_1)_* = doms_2 \supset (Q_1 \cap doms_2),$$

$$ran(t_2)_* = rans_2 \supset (Q_1 \cap rans_2),$$

then the proof of lemma follows from the corollary 12.

We will say that t from T has an indecomposable arrow from a into b, if

- a belongs to Q_1, b belongs to Q_k;
- $t(a) = b$;
- $(t(a), a) \cap (\bigcup_{i=2}^{k-1} Q_i) = \emptyset$.

Let $S = \bigoplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}$ be a direct sum of symmetric groups of blocks Q_i^j. For any π from S let $\pi(1) = 1$ and $\overline{\pi} : t \mapsto t_\pi$ be transformation, defined as follows: for all a from N

$$t_\pi(a) = \begin{cases} t(\pi^{-1}(a)), & \text{if } a \text{ has an indecomposable arrow from } \pi^{-1}(a) \text{ into } t(\pi^{-1}(a)); \\ \pi(t(\pi^{-1}(a))), & \text{otherwise}. \end{cases}$$

Remark 21. *If t is decomposable, then $t_\pi(a) = \pi(t(\pi^{-1}(a)))$ for all a from N.*
Remark 22. *If a belongs to $dom(t)_*$, then $t_\pi(a) = \pi(t(\pi^{-1}(a)))$.*
Lemma 23. *For any permutation π from S following conditions are equivalent:*

- $\pi^{-1}(a) \in Q_1$, $t(\pi^{-1}(a)) \in Q_k$, $(t(\pi^{-1}(a)), \pi^{-1}(a)) \cap (\bigcup_{i=2}^{k-1} Q_i) = \emptyset$;
- $a \in Q_1$, $t_\pi(a) \in Q_k$, $(a, t_\pi(a)) \cap (\bigcup_{i=2}^{k-1} Q_i) = \emptyset$.

Proof. For the properties of partition $\cup Q_i^j$ it follows that a belongs to Q_1 if and only if $\pi^{-1}(a)$ belongs to Q_1, and that $t_\pi(a)$ belongs to Q_k if and only if $t(\pi^{-1}(a))$ belongs to Q_k. If $(t(\pi^{-1}(a)), \pi^{-1}(a)) \cap (\bigcup_{i=2}^{k-1} Q_i) = \emptyset$, then $(a, t_\pi(a)) \cap (\bigcup_{i=2}^{k-1} Q_i) = \emptyset$, as $\{c \in Q_2 \cup \dots \cup Q_{k-1} : c < a\} = \{c \in Q_2 \cup \dots \cup Q_{k-1} : c < \pi^{-1}(a)\}$. If there exists such b from $Q_2 \cup \dots \cup Q_{k-1}$, that $\pi^{-1}(a) > c > t(\pi^{-1}(a))$, then $a > c$ and $\pi(t(\pi^{-1}(a))) < c$, $a > c > t_\pi(a)$.

Lemma 24. *The mapping $\overline{\pi} : t \mapsto t_\pi$ is an embedding of the group $S = \bigoplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}$ into the group $AutT$.*

Proof. 1. We show that $\overline{\pi}$ maps T into T, i.e. for any $t \in T$ an inclusion $t_\pi \in T$ holds. We need to show that for any i, $1 \leqslant i \leqslant k$, a from Q_i conditions $t_\pi(a) \leqslant a$ and $t_\pi(a) \in N \setminus (Q_1 \cup \dots \cup Q_i)$ hold. Since for any i and j an inclusion $Q_i^j \subseteq Q_i$ holds, then it follows that $t_\pi(a) \in N \setminus (Q_1 \cup \dots \cup Q_i)$. It remains to show that t_π is an order-decreasing transformation. As t is order-decreasing, then $t(\pi^{-1}(a)) < \pi^{-1}(a)$.

Let $\pi^{-1}(a) \in Q_1$, $t(\pi^{-1}(a)) \in Q_k$ and $(t(\pi^{-1}(a)), \pi^{-1}(a)) \cap (\bigcup_{i=2}^{k-1} Q_i) = \emptyset$. Then a belongs to Q_1 and inequality $t_\pi(a) < a$ implies from $\{c \in Q_k : c < a\} = \{c \in Q_k : c < \pi^{-1}(a)\}$.

Let $\pi^{-1}(a) \in Q_1$ and $t(\pi^{-1}(a)) \in Q_k$ do not hold simultaneously. If $t(\pi^{-1}(a)) = 1$, then $t_\pi(a) = \pi(1) = 1$, that is $a \geqslant t_\pi(a)$. If $t(\pi^{-1}(a)) \neq 1$, then it implies from $a \in Q_{i_1}^{j_1}$ and $t(a) \in Q_{i_2}^{j_2}$ that $i_2 - i_1 < k - 1$. Then for the definition of the blocks Q_i^j we obtain that $\pi^{-1}(a) > \pi(t(a))$. An inequality $\pi^{-1}(a) > t(a)$ follows from the structure of Q_1^j.

Let $\pi^{-1}(a) \in Q_1$, $t(\pi^{-1}(a)) \in Q_k$ and there exists b from $Q_2 \cup \dots \cup Q_{k-1}$ such that $\pi^{-1}(a) > b > t(\pi^{-1}(a))$. Since numbers $\pi^{-1}(a)$ and a are contained in the same block of partition $\cup Q_i^j$, then $b < a$. Numbers $t(\pi^{-1}(a))$ and $\pi(t(\pi^{-1}(a)))$ are contained in the same block of partition $\cup Q_i^j$, thus $\pi(t(\pi^{-1}(a))) < b$. Hence $a > t_\pi(a)$.

So, t_π is an order-decreasing transformation.

2. Now we show that $\overline{\pi}$ is a homomorphism, i.e. $t_\pi \cdot s_\pi = (ts)_\pi$. Since ts is an indecomposable element, then for any a from N it is true that $ts_\pi(a) = \pi(ts(\pi^{-1}(a)))$. If $t(\pi^{-1}(a))$ belongs to Q_k, then $t_\pi(a)$ and $\pi^{-1}(t_\pi(a))$ belong to Q_k. Hence $ts_\pi(a) = \pi(s(t(\pi^{-1}(a))) = \pi(1)$ and $t_\pi s_\pi = \pi(s(\pi^{-1}(t_\pi(a))) = \pi(1)$, i.e. $ts_\pi(a) = t_\pi s_\pi(a)$.

If $t(\pi^{-1}(a))$ does not belong to Q_k, then $t_\pi(a) = \pi(t(\pi^{-1}(a)))$. As $t_\pi(a) \notin Q_1$, then $\pi^{-1}(t_\pi(a)) \notin Q_1$. We obtain that $(ts)_\pi(a) = \pi(ts(\pi^{-1}(a))) = \pi(s(t(\pi^{-1}(a)))) = s(t(\pi^{-1}(a))) = s(\pi^{-1}(\pi(t(\pi^{-1}(a))))) = t_\pi s_\pi$. Thus $\overline{\pi}$ is a homomorphism.

3. Now we show that $\overline{\pi}$ is injective. Let t, s belong to T, $s \neq t$. Then there exists a from N such that $s(a) \neq t(a)$. Assume that $t_\pi(\pi(a)) = s_\pi(\pi(a))$. Then, with the use of lemma 23 we obtain that following conditions are equivalent:

a). $a \in Q_1$, $t(a) \in Q_k$, $(t(a), a) \cap (\bigcup\limits_{i=2}^{k-1} Q_i) = \emptyset$;

b). $a \in Q_1$, $s(a) \in Q_k$, $(s(a), a) \cap (\bigcup\limits_{i=2}^{k-1} Q_i) = \emptyset$.

Then it implies from $t_\pi(\pi(a)) = s_\pi(\pi(a))$ that one of equalities holds: either $\pi(t(\pi^{-1}(\pi(a)))) = \pi(s(\pi^{-1}(\pi(a))))$, or $t(\pi^{-1}(\pi)(a))) = s(\pi^{-1}(\pi(a)))$. We have come to contradiction with statement $s(a) \neq t(a)$. Thus $t_\pi \neq s_\pi$, and so $\overline{\pi}$ is injective.

4. For any $\tau, \pi \in S = \bigoplus\limits_{i=1,m=1}^{k,m_i} S_{Q_i^m}$ we show that $\overline{\pi\tau} = \overline{\pi} \cdot \overline{\tau}$. It follows from the lemma 23 that t_π has an indecomposable arrow from $\tau^{-1}(a)$ into $t_\pi(\tau^{-1}(a))$ if and only if t has an indecomposable arrow from $\pi^{-1}(\tau^{-1}(a))$ into $t(\pi^{-1}(\tau^{-1}(a)))$.

Let for some a from N $t_{\pi\tau}(a) = t((\pi\tau)^{-1}(a)$. Then $(t_\pi)_\tau(a) = t_\pi(\tau^{-1}(a)) = t(\pi^{-1}(\tau^{-1}(a)))$, wherefrom $(t_\pi)_\tau = t_{\pi\tau}(a)$. In case when $t_{\pi\tau}(a) = \pi\tau(t((\pi\tau)^{-1}(a)))$, we have that $(t_\pi)_\tau(a) = \tau(t_\pi(\tau^{-1}(a))) = \tau(\pi(t(\pi^{-1}(\tau^{-1}(a)))))$, and hence $(t_\pi)_\tau = t_{\pi\tau}(a)$.

5. Now we show that $\overline{\pi}$ is surjective. From the proved above it follows that $(t_{(\pi)^{-1}})_\pi = t_{\pi\pi^{-1}}$. But $\pi\pi^{-1}$ — is an identical permutation, thus $t_{\pi\pi^{-1}} = t$, and so $\overline{\pi}$ is a surjection.

6. Now we show that if $\overline{\pi} = \overline{\tau}$, then $\pi = \tau$. For any $a \in N \setminus (Q_1 \cup \{1\})$ we denote by t_a a transformation from $\bigcup\limits_{i=1}^{k} \Phi_i$ such that $rant_a = \{a\}$. Then $\pi(a) = \tau(a)$ implies from $(t_a)_\pi = (t_a)_\tau$. For any $a \in Q_1$ we denote by t_a a transformation from T such that $domt_a = \{a\}$ and $t_a(a) \in Q_2$. Then $\pi(a) = \tau(a)$ implies from $(t_a)_\pi = (t_a)_\tau$. As $\pi(1) = \tau(1)$ then $\overline{\pi} = \overline{\tau}$.

In the following we will equate the group $S = \bigoplus\limits_{i=1,m=1}^{k,m_i} S_{Q_i^m}$ to the image of embedding into the group $Aut(T)$.

Lemma 25. $Aut(T)/\oplus_{i\geqslant 1} S_{M_i} = \oplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}$

Proof. For any a from N we define transformation t_a on the set N as follows:

- $domt_a = \{a\}$, $rant_a = \{\min\limits_{m \in Q_{i+1}} m\}$, if a belongs to Q_i, $1 \leqslant i < k$;
- $rant_a = \{a\}$, $domt_a = \{\min\limits_{m \in Q_{i+1}, m > a} m\}$, if a belongs to Q_k;
- t=0, if $a = 1$.

It easy to see that for any a transformation t_a belongs to T. Let ς belong to T. We define a transformation π_ς on the set N as follows:

$$\pi_\varsigma(a) \in \begin{cases} domt_a, & \text{if } a \text{ belong to } Q_i \text{ and } 1 \leqslant i < k; \\ rant_a, & \text{if } a \text{ belong to } Q_k; \\ 1, & \text{if } a = 1; \end{cases}$$

It follows from lemmas 13, 14, 19 that π_ς belongs to $\oplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}$.

Let t be decomposable element of T. Then for the remark 21 for all a from N $t_{\pi_\varsigma}(a) = \pi_\varsigma(t(\pi_\varsigma^{-1}(a)))$ (t_{π_ς} we define as in previous lemma). Then for the lemma 20 we obtain that $dom\varsigma(t) = \bigcup\limits_{a \in domt} dom(\varsigma(t_a))_* = domt_{\pi_\varsigma}$ and for all a of $dom\varsigma(t)$ it is true that $\varsigma(t) = t_{\pi_\varsigma}$. Let t be indecomposable. Then for all a of $dom(t_{\pi_\varsigma})_*$ we have that $t_{\pi_\varsigma}(a) = \pi_\varsigma(t(\pi_\varsigma^{-1}(a)))$. From corollaries — 12, 16 — 18 it implies that $dom(\varsigma(t))_* = \bigcup\limits_{a \in domt} dom(\varsigma(t_a))_* = dom(t_{\pi_\varsigma})_*$ and for all a from $dom(\varsigma(t))_*$ it is true that $\varsigma(t) = t_{\pi_\varsigma}$. Then $\varsigma(t) \sim t_{\pi_\varsigma}$. Note that if π_ς is identical mapping, then for all t from T we have that $t_{\pi_\varsigma}(a) = t(a)$ if a belongs to $dom(t)_*$ and $t_{\pi_\varsigma}(a) = t(a)$ if a belongs to $domt$ and t is decomposable. Thus $t_{\pi_\varsigma} \sim t$ and $\varsigma(t) \sim t$.

If $\varsigma(t) \sim t$ for all $t \in T$, then $t_{\pi_\varsigma} \sim t$ and $t_{\pi_\varsigma}(a) = t(a)$ if a belongs to $domt_*$ and $t_{\pi_\varsigma}(a) = t(a)$ if a belongs to $domt$, if t is identical. Thus t_{π_ς} is identical.

We denote a mapping I from th set $Aut(T)$ into the set $\oplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}$ as follows: $I : \varsigma \mapsto \pi_\varsigma$. It is easy to verify that I is a homomorphism. $\oplus_{i \geqslant 1} S_{M_i}$ is a kernel of mapping I, so lemma is proved.

Proof. From lemmas 20, 3 and corollaries 9 — 12, 16 — 18 it follows that if t is decomposable and $a \in N$ or if t is indecomposable and $a \in domt_*$, then $\gamma(t) = t_{\pi_\gamma}(a)$. Therefore $\gamma(t) \sim t_{\pi_\gamma}$. If π_γ is an identical mapping, then for all t from T an equality $t_{\pi_\gamma}(a) = t(a)$ fulfills for any a from $domt_*$; if t is decomposable, then $t_{\pi_\gamma}(a) = t(a)$ for any a from $domt$. Thus $t_{\pi_\gamma} \sim t$ and $\gamma(t) \sim (t)$. At the same time, if $\gamma(t) \sim (t)$ for all $t \in T$, then $t_{\pi_\gamma} \sim t$ and $t_{\pi_\gamma}(a) = t(a)$ for all a from $domt_*$ and if t is decomposable, then $t_{\pi_\gamma}(a) = t(a)$ for all a from $domt$. Thus π_γ is an identical mapping. Next we consider a mapping $I : \gamma \mapsto \pi_\gamma$ from $Aut(T)$ into $\oplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}$. It is easy to verify that I is a homomorphism, while the kernel of I equals to $\oplus_{i \geqslant 1} S_{M_i}$, wherefrom the assertion of lemma easily implies.

3. The main theorem

Theorem 26. *Let T be a nilpotent subsemigroup from $\mathcal{D}_n$ which is maximal among all the nilpotent subsemigroups of nilpotency class k.*

1. *If $k = 2$, then $Aut(T) \simeq S_{T \setminus \{0\}}$.*

2. *If $k > 3$, the group $Aut(T)$ may be represented as a semidirect product of direct sums of the symmetric groups*

$$Aut(T) = \oplus_{i \geqslant 1} S_{M_i} \leftthreetimes \oplus_{i=1,m=1}^{k,m_i} S_{Q_i^m}.$$

Proof. 1. Since a nilpotent semigroup of nilpotency class 2 is a semigroup where the product of two arbitrary elements is zero then any permutation of any element is an automorphism.

2. It follows from the lemmas 2, 5 and 25.

Corollary 27. *If T is the maximal nilpotent subsemigroup from $\mathcal{D}_n$, then* $Aut(T) = \underbrace{C_2 \oplus \ldots \oplus C_2}_{r}$, *where r equals the number of the indecomposable elements in the maximal nilpotent subsemigroup of $\mathcal{D}_{n-1}$.*

Proof. It implies from fact that all the blocks of the partition λ are one-element and each equivalence class for the $\sim$ contains either one or two elements, while s is contained in two-element class if and only if s_* is an indecomposable transformation and $n \notin doms_*$.

Abstract. The paper deals with the automorphism groups of maximal nilpotent subsemigroups of $\mathcal{D}_n$ of nilpotency class k. It is proved that each of these groups can be represented as a semidirect product of direct sums of symmetric groups.

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