# Higher Order Structural Relativistic Effects for $\boldsymbol{s}$-States of Hydrogen-Like Systems 

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#### Abstract

A calculation of structural relativistic corrections of higher orders for the ground states of hydro-gen-like systems, such as a hydrogen atom and a muonic hydrogen atom, has been performed. The dependence of the corrections on the parametrization of proton form factors is investigated. The numerical estimates show that the corrections are within the limits of sensitivity of modern experiments to measure the energy characteristics of hydrogen-like systems.


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## INTRODUCTION

Studying the characteristics of bound systems is the most important source of information on the properties of interactions of elementary particles. In this regard, to explain the experimental data within the framework of quantum field models, it is necessary to take into account both relativistic effects and effects of higher orders in the interaction constant. Such calculations require techniques that allow numerical calculations of these effects with high accuracy.

Calculating various corrections for energy characteristics of $s$-states of hydrogen-like systems is relevant for several reasons. First, experimental measurements of $s$-states of the hydrogen atom are carried out with high accuracy $\left(\delta \sim 10^{-15}\right)$ [1]. Second, there is still no final solution to the problem of the difference between the values of the proton charge radius obtained in the experiment with muonic hydrogen [2] and an ordinary hydrogen atom. This situation has stimulated numerous theoretical studies of various corrections that can improve the accuracy of theoretical calculations [3-6].

This work is devoted to calculating higher order structural relativistic effects for $s$-states of hydrogenlike systems. For the calculations, a calculation technique is used based on the impulse representation of the interaction potential and the exact calculation of the radial kernel of the equation of state (without expansion in powers of the velocities) carried out in [7, 8]. Using this technique, higher order relativistic contributions are calculated for the muonic and ordinary hydrogen atoms, the bound states of which are described by the gauge-invariant Poincaré-covariant model. This model is based on the point form of Poin-caré-invariant quantum mechanics (PIQM).

## 1. DESCRIPTION <br> OF A BOUND TWO-PARTICLE SYSTEM IN THE POINCARÉ COVARIANT MODEL

The main requirement of PICM is the condition for the preservation of Poincaré invariance both for systems without interaction and for interacting particles. In the case of a system of two particles with masses $m_{1}$ and $m_{2}$ and, accordingly, with 4-pulses $p_{1}=\left(\omega_{m_{1}}\left(p_{1}\right), \mathbf{p}_{1}\right)$ and $p_{2}=\left(\omega_{m_{2}}\left(p_{2}\right), \mathbf{p}_{2}\right)$, this requirement, within the framework of the instantaneous and point forms of the PIQM, leads to the equation for the bound state with the wave function $\Phi_{\ell, S}^{J}(k)$ and mass $M$ :

$$
\begin{gather*}
\sum_{\ell, S^{\prime}} \int_{0}^{\infty} V_{\ell, S ; \ell^{\prime}, S^{\prime}}^{J}\left(k, k^{\prime}\right) \Phi_{\ell^{\prime}, S^{\prime}}^{J \mu}\left(k^{\prime}\right) k^{\prime 2} d k^{\prime}  \tag{1}\\
=\left(M-M_{0}\right) \Phi_{\ell, S}^{J}(k),
\end{gather*}
$$

where $M_{0}=\omega_{m_{1}}(k)+\omega_{m_{2}}(k)$ is the effective mass of a system of noninteracting particles with an impulse of relative motion $\mathbf{k}\left(\mathrm{k}=|\mathbf{k}|, \omega_{m}(k)=\sqrt{k^{2}+m^{2}}\right)$.

The nucleus of the fermion-fermion system in the $\ell-S$ basis for an arbitrary total angular momentum $J$, after the exact calculation of the spinor part by the method of basis spinors [11], can be written in the form

$$
\begin{align*}
& V_{\ell^{\prime}, S^{\prime} ; \ell, S}^{J}\left(k^{\prime}, k\right)=-\frac{\sqrt{(2 \ell+1)\left(2 \ell^{\prime}+1\right)}}{2 J+1} \\
& \times \sum_{\lambda_{k_{1}, 2}, \lambda_{p_{1}, 2}=-1}^{1} \mathbf{C}\left\{\begin{array}{ccc}
1 / 2 & 1 / 2 & S \\
\lambda_{k_{1}} / 2, & -\lambda_{k_{2}} / 2, & \lambda
\end{array}\right\} \mathbf{C}\left\{\begin{array}{ccc}
\ell & S & J \\
0, & \lambda, & \lambda
\end{array}\right\}  \tag{2}\\
& \left.\times \mathbf{C}^{\ell^{\prime}} \begin{array}{ccc}
S^{\prime} & J^{\prime} \\
0, & \lambda^{\prime}, & \lambda^{\prime}
\end{array}\right\} \mathbf{C}\left\{\begin{array}{ccc}
1 / 2 & 1 / 2 & S \\
\lambda_{p_{1}} / 2, & -\lambda_{p_{2}} / 2, & \lambda
\end{array}\right\} \\
& \times \frac{Z \alpha}{4 \pi} \sum_{l=I-V} V_{\lambda_{1},}^{l} \lambda_{k_{2}}, \lambda_{p}, \lambda_{p_{2}},
\end{align*}
$$

where

$$
\begin{align*}
& V_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{I}=2 \sum_{\sigma, \rho=-1}^{1} W_{-\sigma \lambda_{k_{1}},-\rho \lambda_{k_{2}}}(k) W_{-\sigma \lambda_{p_{1}},-\rho \lambda_{p_{2}}}\left(k^{\prime}\right) \\
& \times\left[\delta_{\lambda_{k_{1}}, \lambda_{k_{2}}} \rho \sigma G_{-\lambda_{k_{1}}, \lambda_{k},}^{J, \lambda / 2,1 / 2}, \lambda_{p_{1}}, \lambda_{p_{2}}\left[\tilde{R}_{\ell}^{(\mathrm{I})}\left(k, k^{\prime}\right)\right]\right.  \tag{3}\\
& \left.+\delta_{\rho \lambda_{k_{1}}, \sigma \lambda_{k_{2}}} G_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2}\left[\tilde{R}_{\ell}^{(\mathrm{I})}\left(k, k^{\prime}\right)\right]\right], \\
& V_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{\text {II }}=-\frac{1}{2 m_{1}} \sum_{\sigma, \rho=-1}^{1} W_{-\sigma \lambda_{k_{1}},-\rho \lambda_{k_{2}}}(k) \\
& \times W_{\sigma \lambda_{p_{1}},-\rho \lambda_{p_{2}}}\left(k^{\prime}\right)\left[k ^ { \prime } \rho \lambda _ { k _ { 2 } } \left(3 G_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2}\right.\right. \\
& \left.\times\left[\tilde{Z}^{(\mathrm{II})}\left(k^{\prime}, k\right)\right]-2 G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,2 / 2}\left[\tilde{R}^{(\mathrm{II})}\left(k^{\prime}, k\right)\right]\right)  \tag{4}\\
& +G_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,2}\left[\tilde{R}^{(\mathrm{II})}\left(k^{\prime}, k\right)\right] \\
& \left.\times\left\{\rho\left(3 \lambda_{k_{2}} k-2 \lambda_{p_{2}} k^{\prime}\right)-3\left(\omega_{m_{1}}(k)+\omega_{m_{1}}\left(k^{\prime}\right)\right)\right\}\right], \\
& V_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{\text {II }}=-\frac{1}{2 m_{2}} \sum_{\sigma, \rho=-1}^{1} W_{-\sigma \lambda_{k_{1}},-\rho \lambda_{k_{2}}}(k) \\
& \times W_{-\sigma \lambda_{p_{1}}, \rho \lambda_{p_{2}}}\left(k^{\prime}\right)\left[k ^ { \prime } \sigma \lambda _ { k _ { 1 } } \left(3 G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2}\left[\tilde{Z}^{(\mathrm{III})}\left(k^{\prime}, k\right)\right]\right.\right. \\
& \left.-2 G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 3 / 2,1 / 2}\left[\tilde{R}^{(\text {III })}\left(k^{\prime}, k\right)\right]\right)+G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2}  \tag{5}\\
& \times\left[\tilde{R}^{(\text {III })}\left(k^{\prime}, k\right)\right] \\
& \left.\times\left\{\sigma\left(3 \lambda_{k_{1}} k-2 \lambda_{p_{1}} k^{\prime}\right)-3\left(\omega_{m_{2}}(k)+\omega_{m_{2}}\left(k^{\prime}\right)\right)\right\}\right], \\
& V_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{\mathrm{IV}}=\frac{1}{4 m_{1} m_{2}} \sum_{\sigma, \rho=-1}^{1} W_{-\sigma \lambda_{k_{1}},-\rho \lambda_{k_{2}}} \\
& \times(k) W_{\sigma \lambda_{p_{1}}, \rho \lambda_{p_{2}}}\left(k^{\prime}\right)  \tag{6}\\
& \times\left[\left(k^{\prime 2}+k^{2}+\left(\omega_{m_{1}}(k)+\omega_{m_{1}}\left(k^{\prime}\right)\right)\left(\omega_{m_{2}}(k)+\omega_{m_{2}}\left(k^{\prime}\right)\right)\right)\right. \\
& \times G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2}\left[\tilde{R}^{(\mathrm{IV})}\left(k^{\prime}, k\right)\right] \\
& \left.+2 k k^{\prime} G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2}\left[\tilde{Z}^{(\mathrm{IV})}\left(k^{\prime}, k\right)\right]\right] .
\end{align*}
$$

To shorten the record, additional functions have been introduced:

$$
\begin{gather*}
G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1},}, \lambda_{p_{2}}}^{J, s_{2}}[\Phi(x)]=\sum_{s=\left|s_{1}-s_{2}\right|}^{s_{1}+s_{2}} \sum_{\ell=|J-s|}^{J+s} \frac{(2 \ell+1)}{(2 J+1)} \\
\times \mathbf{C}\left\{\begin{array}{cc}
s_{1} & s_{2} \\
\lambda_{k_{1}} / 2, & -\lambda_{k_{2}} / 2, \\
\hline
\end{array}\right\} \mathbf{C}\left\{\begin{array}{cc}
\ell & s \\
0, & \lambda, \\
\hline
\end{array}\right\}  \tag{7}\\
\times \mathbf{C}\left\{\begin{array}{cc}
s_{1} & s_{2} \\
\lambda_{p_{1}} / 2, & -\lambda_{p_{2}} / 2, \\
\lambda^{\prime}
\end{array}\right\} \mathbf{C}\left\{\begin{array}{cc}
\ell & S \\
0, & \lambda^{\prime}, \\
\lambda^{\prime}
\end{array}\right\} \Phi_{\ell}(x), \\
\tilde{Z}_{\ell}\left(k^{\prime}, k\right) \frac{1}{2 \ell+1}\left[(\ell+1) \tilde{R}_{\ell+1}\left(k^{\prime}, k\right)+\ell \tilde{R}_{\ell-1}\left(k^{\prime}, k\right)\right], \\
W_{\lambda, \rho}(k)=\sqrt{1+\lambda v_{k_{1}}} \sqrt{1+\rho v_{k_{2}}}, \quad W_{\lambda, \rho}\left(k^{\prime}\right)  \tag{8}\\
=\sqrt{1+\lambda v_{p_{1}}} \sqrt{1+\rho v_{p_{2}}},
\end{gather*}
$$

with

$$
\begin{equation*}
v_{k_{1}} \frac{k}{\omega_{m_{1}}(k)}, v_{p_{1}} \frac{k^{\prime}}{\omega_{m_{1}}\left(k^{\prime}\right)}, v_{k_{2}} \frac{k}{\omega_{m_{2}}(k)}, v_{p_{2}} \frac{k^{\prime}}{\omega_{m_{2}}\left(k^{\prime}\right)} \tag{9}
\end{equation*}
$$

An analytical expression for the last part of the potential is written in the form

$$
\begin{gather*}
V_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{\mathrm{V}}=\sum_{\sigma, \rho=-1}^{1} W_{-\sigma \lambda_{k_{1}},-\rho \lambda_{k_{2}}}(k) \\
\times\left[G_{\lambda_{k_{1}}, \lambda_{k_{2}} ; \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,1 / 2} \tilde{U}^{(\mathrm{I})}\left(k^{\prime}, k\right)\right]  \tag{10}\\
\times W_{-\sigma \lambda_{p_{1}},-\rho \lambda_{p_{2}}}\left(k^{\prime}\right)-\frac{1}{2 m_{2}} G_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}^{J, 1 / 2,2}\left[\tilde{U}^{(\mathrm{II})}\left(k^{\prime}, k\right)\right] \\
\left.\times W_{-\sigma \lambda_{p_{1}}, \rho \lambda_{p_{2}}}\left(k^{\prime}\right)\left(\omega_{m_{2}}\left(k^{\prime}\right)+\omega_{m_{2}}(k)\right)\right]
\end{gather*}
$$

For the Clebsch-Gordan coefficients of the group $S U(2)$, we use the notation of the form $\mathbf{C}\left\{\begin{array}{ccc}s_{1} & s_{2} & S \\ \lambda_{p_{1}}, & \lambda_{p_{2}}, & \lambda\end{array}\right\}$.

Components $V_{\lambda_{k_{1}}, \lambda_{k_{2}}, \lambda_{p_{1}}, \lambda_{p_{2}}}\left(k, k^{\prime}\right)$ are linear combinations of functions

$$
\begin{align*}
& K^{(\mathrm{I}, \mathrm{II})}\left(q^{2}\right)=\Pi\left(\alpha, q^{2}\right) G_{M}^{p}\left(q^{2}\right)\left\{G_{M}^{e}\left(\alpha, q^{2}\right), F_{2}^{e}\left(\alpha, q^{2}\right)\right\},  \tag{11}\\
& K^{(\mathrm{III}, \mathrm{IV})}\left(q^{2}\right)=\Pi\left(\alpha, q^{2}\right) F_{2}^{p}\left(q^{2}\right)\left\{G_{M}^{e}\left(q^{2}\right), F_{2}^{e}\left(\alpha, q^{2}\right)\right\}, \tag{12}
\end{align*}
$$

where the form factor $\Pi\left(\alpha, q^{2}\right)$ is the result of vacuum polarization and $G_{M}^{e, p}\left(q^{2}\right)$ and $F_{2}^{e, p}\left(q^{2}\right)$ are fermion form factors.

In Eqs. (3)-(10), the functions $\tilde{R}_{\ell}\left(k^{\prime}, k\right)$ and $\tilde{U}_{\ell}\left(k^{\prime}, k\right)$ are defined by integrals of the form

$$
\begin{gather*}
\tilde{R}_{\ell}\left(k^{\prime}, k\right) \int_{-1}^{1} \frac{K\left(q^{2}\right) P_{\ell}(x)}{q^{2}} d x  \tag{13}\\
\tilde{U}_{\ell}\left(k^{\prime}, k\right)=\varrho_{12}\left(k^{\prime}, k\right) \int_{-1}^{1} \frac{K\left(q^{2}\right) P_{\ell}(x)}{q^{4}} d x \tag{14}
\end{gather*}
$$

where
$\varrho_{12}\left(k^{\prime}, k\right)=\left(\omega_{m_{1}}\left(k^{\prime}\right)-\omega_{m_{1}}(k)\right)\left(\omega_{m_{2}}(k)-\omega_{m_{2}}\left(k^{\prime}\right)\right),(15)$
and

$$
\begin{gather*}
q^{2}=-2 k k^{\prime}(y-x)  \tag{16}\\
y=\frac{k^{2}+k^{\prime 2}}{2 k k^{\prime}} \tag{17}
\end{gather*}
$$

The radial part of the potential was obtained on the basis of the scattering amplitude of fermions, which took into account the diagrams of one-boson exchange, the structure of the proton, the diagrams associated with the polarization of the vacuum, and the electromagnetic corrections for the electron line [7, 8]. In this work, we investigate the corrections associated only with the structure of the proton.

## 2. ESTIMATING <br> THE RELATIVISTIC CONTRIBUTIONS

As a rule, when calculating the energy corrections of hydrogen-like systems, the potential expansion is used $\sim k^{2} / m^{2}$ (see survey [9]). Including terms of higher order than $\mathbb{O} \sim k^{2} / m^{2}$ in the expansion in terms of fermion velocities leads to complications. Thus, when calculating relativistic corrections, divergent integrals appear due to the appearance of high degrees $(k / m)^{n}$. Therefore, to study relativistic contributions of a higher order, we use the exact expression for the kernel of integral Eq. (1). The potential kernel (2) was obtained in [7] without any assumptions about the fermion velocities and the quantity $q^{2}$, so it is an adequate way to analyze relativistic contributions of a higher order than $k^{2} / m^{2}$.

For numerical calculations, we use the values of fundamental physical constants taken from [10]. To find the energy corrections for a relativistic hydrogenlike system with $J=S$, we use the expression

$$
\begin{align*}
\Delta E & =\int_{0}^{\infty} \int_{0}^{\infty} R_{n \ell=0}^{C}(k) \Delta V^{J=S}\left(k, k^{\prime}\right)  \tag{18}\\
& \times R_{n \ell=0}^{C}\left(k^{\prime}\right) k^{\prime 2} k^{2} d k^{\prime} d k
\end{align*}
$$

where $\Delta V^{J=S}\left(k, k^{\prime}\right)$ is an addition to the potential with point fermions and $R_{n \ell}^{C}(k)$ is Coulomb wave functions

$$
\begin{gather*}
R_{n \ell}^{C}(k)=\sqrt{\frac{2(n-\ell-1)!}{\pi(n+\ell)!}} \\
\times \frac{n^{2} 2^{2(\ell+1)} \ell!n^{\ell}(k / \beta)^{\ell}}{\beta^{3 / 2}\left(n^{2}(k / \beta)^{2}+1\right)^{\ell+2}} G_{n-\ell-1}^{\ell-1}\left(\frac{\left.n^{2}(k / \beta)^{2}+1\right)}{n^{2}(k / \beta)^{2}+1}\right) \tag{19}
\end{gather*}
$$

with Gegenbauer polynomials $\mathscr{G}_{n}^{\ell}(x)$.

## 3. ESTIMATING

## RELATIVISTIC CONTRIBUTIONS

Let us estimate the higher order relativistic corrections associated with the motion of fermions, as well as the effects of high momentum transfers for structural contributions to $1 s$ - and $2 s$-states of hydrogen-like systems. To isolate the terms associated with the finite size of the proton, we represent the Saxon form factors of the proton in the form of sums:

$$
G_{M}^{p}\left(q^{2}\right)=\mu_{p}+\Delta G_{M}^{p}\left(q^{2}\right), \quad G_{E}^{p}\left(q^{2}\right)=1+\Delta G_{E}^{p}\left(q^{2}\right)
$$

where $\Delta G_{M, E}^{p}\left(q^{2}\right) \sim\left\langle r_{p}^{2}\right\rangle$ at small $q^{2}$.
Next, we will calculate three types of corrections: $\Delta E_{\mathrm{NR}}, \Delta E_{\mathrm{LC}}$ and $\Delta E_{\mathrm{rel}}$ using various approximations:
nonrelativistic approximation $\Delta E_{\mathrm{NR}} \quad\left(k^{2} / m_{1, p}^{2}\right.$, $k^{\prime 2} / m_{1, p}^{2} \ll 1$ and $q^{2} \ll m_{p}^{2}$ );
leading contribution $\Delta E_{\mathrm{LC}}$ (only the approximation is used $k^{2} / m_{1, p}^{2}, k^{\prime 2} / m_{1, p}^{2} \ll 1$ );
precise calculation $\Delta E_{\text {rel }}$ (without decomposition in terms of parameters $k^{2}, k^{\prime 2} / m_{1, p}^{2}$ and $q^{2} / m_{p}^{2}$ ). Amendment $\Delta E_{\mathrm{Rel}}$ with potential (2) will give the result taking into account the relativistic motion of the system fermions. To estimate the higher order relativistic contributions, we use the quantity

$$
\Delta_{\mathrm{HO}}=\Delta E_{\mathrm{Rel}}-\Delta E_{\mathrm{LC}}
$$

The explicit form of the terms of the potential, which contain integrals of the form (functions (13) and (14))

$$
\begin{gathered}
\tilde{G}_{\ell}^{p}\left(k^{\prime}, k\right)=\int_{-1}^{1} \frac{\Delta G^{p}\left(q^{2}\right)}{q^{2}} P_{\ell}(x) d x \\
\tilde{F}_{\ell}^{p}\left(k^{\prime}, k\right)=\int_{-1}^{1} \frac{\Delta G^{p}\left(q^{2}\right)}{q^{2}\left(1-q^{2} /\left(4 m_{p}^{2}\right)\right)} P_{\ell}(x) d x
\end{gathered}
$$

will depend on the explicit form of the proton form factors.

To calculate the corrections, we use various parameterizations of the proton form factors:
-standard dipole parameterization

$$
\begin{gather*}
G_{E}^{p}\left(Q^{2}\right)=G_{D}\left(Q^{2}\right), \quad G_{M}^{p}\left(Q^{2}\right)=\mu_{p} G_{D}\left(Q^{2}\right), \\
G_{D}\left(Q^{2}\right)=\left(1+\frac{Q^{2}}{m_{D}^{2}}\right)^{-2}, \quad Q^{2}=-q^{2}, \tag{20}
\end{gather*}
$$

where $m_{D}^{2}=0.71 \mathrm{GeV}^{2}$. This variant of parameterization will be referred to as fit I;
parameterization variant [12] will be called fit II:

$$
\begin{equation*}
G_{E, M}\left(Q^{2}\right)=\left(1+\frac{Q^{2}}{a_{E, M}}\right)^{-2},\left\langle r_{E, M}^{2}\right\rangle=\frac{12}{a_{E, M}} \tag{21}
\end{equation*}
$$

where

$$
\begin{aligned}
& \left\langle r_{M}^{2}\right\rangle^{1 / 2}=\left(0.777 \pm 0.013_{\text {stat }} \pm 0.009_{\text {syst }}\right. \\
& \left. \pm 0.005_{\text {model }} \pm 0.002_{\text {group }}\right) \mathrm{Fm}, \\
& \left\langle r_{E}^{2}\right\rangle^{1 / 2}=\left(0.879 \pm 0.005_{\text {stat }} \pm 0.004_{\text {syst }}\right. \\
& \left.\quad \pm 0.002_{\text {model }} \pm 0.004_{\text {group }}\right) \mathrm{Fm} ;
\end{aligned}
$$

Table 1. Corrections for the fit II variant associated with the internal structure of the proton and contributions $\Delta E_{\mathrm{NR}}, \Delta E_{\mathrm{LC}}$ and high-order relativistic corrections $\Delta E_{\mathrm{Rel}}$ for the hydrogen atom (in kHz )

| $n$ | $\Delta E_{\mathrm{NR}}$ | $\Delta E_{\mathrm{LC}}$ | $\Delta E_{\mathrm{Rel}}$ | $\Delta_{\mathrm{HO}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1208.31 | 1208.30 | 1202.52 | -5.78 |
| 2 | 151.04 | 151.04 | 150.31 | 0.72 |

parameterization [13]

$$
\begin{gathered}
G_{M}^{p}\left(Q^{2}\right)=\mu_{p} \frac{1+a_{p, \tau}^{M} \tau_{p}}{1+b_{p, 1}^{M} \tau_{p}+b_{p, 2}^{M} \tau_{p}^{2}+b_{p, 3}^{M} \tau_{p}^{3}} \\
\tau_{p}=Q^{2} /\left(4 m_{p}^{2}\right), \\
G_{E}^{p}\left(Q^{2}\right)=\frac{G_{M}^{p}\left(Q^{2}\right)}{\mu_{p}}\left(c_{0}+c_{1} Q^{2}\right)
\end{gathered}
$$

was applied in [14] to describe the behavior of proton form factors with parameters:

$$
\begin{gathered}
a_{p, 1}^{M}=1.53 \pm 0.01, \quad b_{p, 1}^{M}=12.87 \pm 0.07 \\
c_{0}=1.02 \pm 0.01 \\
b_{p, 2}^{M}=29.16 \pm 0.25, \quad b_{p, 3}^{M}=41.40 \pm 0.33 \\
c_{1}=-0.13 \pm 0.01
\end{gathered}
$$

(1) This variant of parameterization is denoted by fit III. Table 1 shows the results of calculations for the hydrogen atom.

As follows from the data in Table 1, accounting for high $q^{2}$ through parametrization (21) does not significantly change in comparison with linear behavior. However, the corrections associated with the relativistic motion of fermions in this situation give a visible, albeit relatively small in percentage terms ( $\approx 0.48 \%$ ), effect.

Calculations using the standard dipole parameterization (20) (fit I) and parameterization fit II differ due to the different behavior of functions (20) and (21) at small $q^{2}$. So much for $1 s$ - and $2 s$-states for the parameterization of fit $I$ we have that $\Delta E_{\mathrm{Rel}}=1024.33 \mathrm{kHz}$ and $\Delta E_{\mathrm{Rel}}=150.32 \mathrm{kHz}$, respectively.

For a more complete understanding of the dependence of the structural corrections for hydrogen-like atoms on the type of parametrization, let us consider a numerical calculation for fit III.

Higher order relativistic corrections $\Delta_{\text {но }}:-5.67$ and 0.71 kHz make up $\approx 0.47 \%$ for $1 s$ - and $2 s$-states and practically do not differ from the parameterization of fit II, while the contributions themselves differ due to different behavior at small $q^{2}$. However, the interval estimates of both fits are practically the same.

In this approach, it is possible to calculate contribution $\Delta_{\text {Jlab }}$ associated with the Jlab effect [15] (the difference between the two cases $\mathscr{R}\left(q^{2}\right)=1$ and $\left.\mathscr{R}\left(q^{2}\right)=c_{0}+c_{1} q^{2}, \quad c_{1}=0.13 \mathrm{GeV}^{2}\right)$ : with kHz$]$ $\Delta_{\text {Jlab }}\{-23.42 \mathrm{kHz},-2.93 \mathrm{kHz}\}$ for $1 s$ - and $2 s$-states.

Let us carry out calculations similar to the above for the muonic hydrogen atom. Interest in this system is associated with unusual consequences arising from the experiment [2].

The calculations for the muonic hydrogen atom for the fit II situation are presented in Table 3.

From Table 3 it follows that, in contrast to the hydrogen atom, the $\mu-p$ system is more sensitive to the behavior of the proton form factor and less sensitive to high-order relativistic effects. This is explained by the fact that the first Bohr orbit of this system is closer to the nucleus than in the hydrogen atom. Therefore, in calculating the structural contributions of the muonic hydrogen atom, an important role is played by the behavior of the form factors from the committed momentum. In this case, the dependence on the parametrization is stronger than for the hydrogen atom.

Numerical estimates show that relativistic effects $(\approx 0.38 \%)$ are more than the required accuracy $\sim 10^{-4} \mathrm{meV}$ and therefore they also need to be considered.

As in the case of the hydrogen atom, the calculations using the standard dipole parameterization (20) (fit I) differ from the data in Table 3. This effect becomes more significant in comparison with the hydrogen atom; the relative deviation of the calculations is almost $14.7 \%$.

Calculations for the "fit III" option are presented in Table 4.

The data from Tables 3 and 4 show that, in contrast to the ordinary hydrogen atom, in the muonic hydrogen atom, the effects of high $q^{2}$ are partially compensated by relativistic effects associated with the motion of fermions.

Table 2. Corrections for the option of parameterization fit III (in kHz )

| $n$ | $\Delta E_{\mathrm{NR}}$ | $\Delta E_{\mathrm{LC}}$ | $\Delta E_{\mathrm{Rel}}$ | $\Delta_{\mathrm{HO}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $1223.15 \pm 22.36$ | $1200.10 \pm 34.86$ | $1194.43 \pm 33.67$ | -5.67 |
| 2 | $152.89 \pm 2.39$ | $150.01 \pm 4.23$ | $149.30 \pm 4.21$ | 0.71 |

Table 3. Corrections for the fit II variant associated with the internal structure of the proton $\Delta E_{\mathrm{NR}}$ and $\Delta E_{\mathrm{LC}}$ and taking into account high-order relativistic contributions
$\Delta E_{\mathrm{Rel}}$ for the muonic hydrogen atom ( $\mu^{-} p$ system) (in meV)

| $n$ | $\Delta E_{\mathrm{NR}}$ | $\Delta E_{\mathrm{LC}}$ | $\Delta E_{\mathrm{Rel}}$ | $\Delta_{\mathrm{HO}}, \mathrm{meV}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 32.1261 | 31.9498 | 32.0591 | 0.1093 |
| 2 | 4.0158 | 3.9937 | 4.0065 | 0.0128 |

Table 4. Corrections for the "fit III" variant associated with the internal structure of the proton $\Delta E_{\mathrm{NR}}$ and $\Delta E_{\mathrm{LC}}$ and taking into account high-order relativistic contributions $\Delta E_{\mathrm{Rel}}$ for the muonic hydrogen atom ( $\mu^{-} p$ system) (in meV)

| $n$ | $\Delta E_{\mathrm{NR}}$ | $\Delta E_{\mathrm{LC}}$ | $\Delta E_{\mathrm{Rel}}$ | $\Delta_{\mathrm{HO}}, \mathrm{meV}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 32.5205 | 31.7717 | 31.8929 | 0.1212 |
| 2 | 4.0651 | 3.9715 | 3.9866 | 0.0152 |

Let us also estimate the additional contribution associated with the effect found on Jlab [15]. Since it modifies the behavior of the proton charge form factor, one should expect a greater sensitivity of the muonic hydrogen atom to these effects. Indeed, for $1 s$ and 2 s states of the hydrogen atom, this effect gives additional corrections equal to -0.6158 and -0.0770 meV respectively.

## CONCLUSIONS

As follows from the calculations, the proposed calculation method, which uses the representation of the interaction potential of fermions in form (2), is a tool that makes it possible to estimate the relativistic contributions higher than $k^{2} / m^{2}$ terms. It was used to investigate the dependence of the corrections on the parametrization of proton form factors and calculate higher order relativistic contributions. Calculations show that such corrections are within the limits of sensitivity of modern experiments to measure the energy characteristics of such systems.

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