Ricerche di Matematica
https://doi.org/10.1007/s11587-021-00627-8

New characterizations of σ-nilpotent finite groups

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Received: 17 February 2021 / Revised: 11 June 2021 / Accepted: 26 July 2021

Abstract
Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition of the set of all primes. We characterize the class of all σ-nilpotent groups as a hereditary formation $\mathfrak{F}$ that contains every group G all whose Sylow subgroups are K-$\mathfrak{F}$-subnormal in their product with the generalized Fitting subgroup $\mathrm{F}^*(G)$.

Keywords Finite group · Generalized Fitting subgroup · Hereditary formation · K-$\mathfrak{F}$-subnormal subgroup · σ-nilpotent group

Mathematics Subject Classification Primary 20D25; Secondary 20F17 · 20F19

1 Introduction

Throughout this paper all groups are finite. Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes. Following Skiba (see [18] or [19]), a group is called σ-nilpotent if it is the direct product of all its Hall π_i-subgroups for $\pi_i \in \sigma$. The class of all σ-nilpotent groups is denoted by $\mathfrak{N}_\sigma$. It is a hereditary saturated formation. Note that the classes of all nilpotent groups and of all π-decomposable groups coincide with $\mathfrak{N}_\sigma$ for $\sigma = \{\{p\} \mid p \in \mathbb{P}\}$ and $\sigma = \{\pi, \pi'\}$ respectively. The class of all σ-nilpotent groups is of the great interest in the theory of classes of groups because it has many properties of the class of all nilpotent groups.

This document is the results of the research project funded by Belarusian Republican Foundation for Fundamental Research, Project No. Φ20P-291.

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Published online: 09 August 2021

There are many interesting characterizations of the class of all σ-nilpotent groups in the universe of all soluble groups. For example every hereditary saturated lattice formation [2] or formation closed under taking products of abnormal subgroups [20] is the class of all σ-nilpotent groups for some σ.

In the universe of all groups the class $\mathfrak{N}_\sigma$ appeared when Shemetkov [16] studied the generalizations of systems normalizers. He called groups from this class σ-decomposable. Kazarin, Martínez-Pastor, and Pérez-Ramos [9] proved that a group is σ-nilpotent if and only if all normalizers of its Sylow subgroups are σ-nilpotent. From [10] it follows that if $G = AB = AC = BC$ where A, B and C are σ-nilpotent, then G is σ-nilpotent. Skiba extended the theory of S-permutable subgroups for such classes [18] and [19]. For more results about these groups see also [1,4] and [7].

The aim of this paper is to find new properties of the class $\mathfrak{N}_\sigma$ that distinguish it from all other hereditary formations.

Let $\mathfrak{F}$ be a formation. Recall [3, Definition 6.1.4] that a subgroup H of G is called *K-$\mathfrak{F}$-subnormal* in G if there is a chain $H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ with $H_{i-1} \trianglelefteq H_i$ or $H_i/\mathrm{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for all $i = 1, \ldots, n$. Denoted by H K-$\mathfrak{F}$-sn G. If $\mathfrak{F} = \mathfrak{N}$ is the formation of all nilpotent groups, then the notions of K-$\mathfrak{F}$-subnormal and subnormal subgroups coincide.

Groups with different systems of K-$\mathfrak{F}$-subnormal subgroups are the main object of many papers (for example, see [1,14,18] and [23]). In this paper we consider K-$\mathfrak{F}$-subnormality of a subgroup not in the whole group but in some subgroup containing it in the sense of the following definition:

Definition 1 Let $\mathfrak{F}$ be a formation and R be a subgroup of a group G. We shall call a subgroup H of G *R-K-$\mathfrak{F}$-subnormal* if H is K-$\mathfrak{F}$-subnormal in $\langle H, R\rangle$. If $\mathfrak{F} = \mathfrak{N}$, then we just obtain the notion of R-subnormal subgroup.

Recall [8, X, Definition 13.9] that *the generalized Fitting subgroup* $\mathrm{F}^*(G)$ is the set of all elements of G which induce an inner automorphism on every chief factor of G. One of the important properties of $\mathrm{F}^*(G)$ is $C_G(\mathrm{F}^*(G)) \subseteq \mathrm{F}^*(G)$.

In [11,12,15] and [22] the products of R-subnormal subgroups were studied for $R \in \{\mathrm{F}(G), \mathrm{F}^*(G)\}$. It was shown that if G is the product of two nilpotent (resp. quasinilpotent) $\mathrm{F}(G)$-subnormal (resp. $\mathrm{F}^*(G)$-subnormal) subgroups, then it is nilpotent (resp. quasinilpotent).

It is well known that a group is nilpotent if and only if all its Sylow subgroups are subnormal. Formations $\mathfrak{F}$ of all groups whose Sylow subgroups are K-$\mathfrak{F}$-subnormal are studied for example in [23]. This is a rather wide family of formations. This property have formations of σ-nilpotent groups, p-nilpotent groups, w-supersoluble groups and other.

Theorem 1 *Let $\mathfrak{F}$ be a hereditary formation. The following statements are equivalent:*

(1) *$\mathfrak{F}$ contains every group G all whose cyclic primary subgroups are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal.*
(2) *$\mathfrak{F}$ contains every group G all whose Sylow subgroups are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal.*
(3) *There is a partition σ of $\mathbb{P}$ such that $\mathfrak{F}$ is the class of all σ-nilpotent groups.*

Remark 1 In the proof of Theorem 1 we use [24, Theorem 5.4]. The proof of the last result is based on the deep results mod CFSG of [9].

Corollary 1 *A group G is nilpotent if and only if all its Sylow subgroups are* $\mathrm{F}^*(G)$-*subnormal.*

Corollary 2 *A group G is σ-nilpotent if and only if every π_i-element of* $\mathrm{F}^*(G)$ *permutes with every π_i'-element of G for every $\pi_i \in \sigma$.*

The proof of the next result is based on the previous theorem.

Theorem 2 *Let $\mathfrak{F}$ be a hereditary formation. The following statements are equivalent:*

(1) *$\mathfrak{F}$ contains every group $G = AB$ where all cyclic primary subgroups of A and B are* $\mathrm{F}^*(G)$-*K-$\mathfrak{F}$-subnormal.*
(2) *$\mathfrak{F}$ contains every group $G = AB$ where all Sylow subgroups of A and B are* $\mathrm{F}^*(G)$-*K-$\mathfrak{F}$-subnormal.*
(3) *There is a partition σ of $\mathbb{P}$ such that $\mathfrak{F}$ is the class of all σ-nilpotent groups.*

Since in every σ-nilpotent group all Sylow subgroups are K-$\mathfrak{N}_\sigma$-subnormal, the following holds.

Corollary 3 *Let A and B be a σ-nilpotent* $\mathrm{F}^*(G)$-*K-$\mathfrak{N}_\sigma$-subnormal* ($\mathrm{F}^*(G)$-*subnormal*) *subgroups of a group G. If $G = AB$, then G is σ-nilpotent.*

Corollary 4 ([18]) *If A and B are normal σ-nilpotent subgroups of a group G, then AB is σ-nilpotent.*

Corollary 5 *A group $G = AB$ is σ-nilpotent if and only if every π_i-element of* $\mathrm{F}^*(G)$ *permutes with every π_i'-element of $A \cup B$ for every $\pi_i \in \sigma$.*

Recall that a subgroup H of G is called R-conjugate-permutable [13] if $H^r H = HH^r$ for all $r \in R$. If $R = G$, then we obtain the notion of conjugate-permutable subgroup [6]. From (1) of [13, Lemma 2.2] it follows that an $\mathrm{F}^*(G)$-conjugate-permutable subgroup is $\mathrm{F}^*(G)$-K-$\mathfrak{N}$-subnormal. Hence the main result of [25] follows from Theorem 2.

Corollary 6 ([25, Theorem 3.1]) *Let A and B be subgroups of a group G and $G = AB$. If every Sylow subgroup of A is $B\mathrm{F}^*(G)$-conjugate-permutable and every Sylow subgroup of B is $A\mathrm{F}^*(G)$-conjugate-permutable, then G is nilpotent.*

2 Preliminaries

The notation and terminology agree with [3] and [5]. We refer the reader to these books for the results about formations.

Recall that a *formation* is a class of groups which is closed under taking epimorphic images and subdirect products. A formation $\mathfrak{F}$ is called *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$. The following two lemmas follow from [3, Lemmas 6.1.6 and 6.1.7].

Lemma 1 *Let $\mathfrak{F}$ be a formation, H and R be subgroups of G and $N \trianglelefteq G$.*

(1) *If H K-$\mathfrak{F}$-sn G, then HN/N K-$\mathfrak{F}$-sn G/N.*

(2) *If* H/N K-$\mathfrak{F}$-*sn* G/N, *then* H K-$\mathfrak{F}$-*sn* G.
(3) *If* H K-$\mathfrak{F}$-*sn* R *and* R K-$\mathfrak{F}$-*sn* G, *then* H K-$\mathfrak{F}$-*sn* G.

Lemma 2 *Let* $\mathfrak{F}$ *be a hereditary formation,* H *and* R *be subgroups of* G.

(1) *If* H K-$\mathfrak{F}$-*sn* G, *then* $H \cap R$ K-$\mathfrak{F}$-*sn* R.
(2) *If* H K-$\mathfrak{F}$-*sn* G *and* R K-$\mathfrak{F}$-*sn* G, *then* $H \cap R$ K-$\mathfrak{F}$-*sn* G.

The following lemma directly follows from Lemma 1.

Lemma 3 *Let* $\mathfrak{F}$ *be a formation,* H *and* R *be subgroups of* G *and* $N \trianglelefteq G$. *If* H K-$\mathfrak{F}$-*sn* R, *then* HN K-$\mathfrak{F}$-*sn* RN.

The following result directly follows from [5, B, Theorem 10.3].

Lemma 4 *If* $\mathrm{O}_p(G) = 1$ *and* G *has a unique minimal normal subgroup, then there exists a faithful irreducible* $\mathbb{F}_pG$*-module.*

Recall [3, Chapter 6.3] or [21] that a formation $\mathfrak{F}$ has *the lattice property for* K-$\mathfrak{F}$-*subnormal subgroups* if the set of all K-$\mathfrak{F}$-subnormal subgroups is a sublattice of the lattice of all subgroups in every group.

Lemma 5 (see [21], [18, Lemma 2.6(3)]) *Let* σ *be a partition of* $\mathbb{P}$. $\mathfrak{N}_\sigma$ *has the lattice property for* K-$\mathfrak{N}_\sigma$*-subnormal subgroups.*

Recall [24] that a Schmidt (p, q)-group is a Schmidt group with a normal Sylow p-subgroup. An *N-critical graph* $\Gamma_{Nc}(G)$ of a group G [24, Definition 1.3] is a directed graph on the vertex set $\pi(G)$ of all prime divisors of $|G|$ and (p, q) is an edge of $\Gamma_{Nc}(G)$ iff G has a Schmidt (p, q)-subgroup. An *N-critical graph* $\Gamma_{Nc}(\mathfrak{X})$ of a class of groups $\mathfrak{X}$ [24, Definition 3.1] is a directed graph on the vertex set $\pi(\mathfrak{X}) = \cup_{G\in\mathfrak{X}}\pi(G)$ such that $\Gamma_{Nc}(\mathfrak{X}) = \cup_{G\in\mathfrak{X}}\Gamma_{Nc}(G)$.

Lemma 6 ([24, Theorem 5.4]) *Let* $\sigma = \{\pi_i \mid i \in I\}$ *be a partition of the vertex set* $V(\Gamma_{Nc}(\mathfrak{X}))$ *such that for* $i \neq j$ *there are no edges between* π_i *and* π_j. *Then every* $\mathfrak{X}$*-group is the direct product of its Hall* π_k*-subgroups, where* $k \in \{i \in I \mid \pi(G) \cap \pi_k \neq \emptyset\}$.

Let $\mathfrak{F}$ be a hereditary formation. In [14] and [23] the classes of groups $\overline{w}\mathfrak{F}$ and $v^*\mathfrak{F}$ all whose Sylow and cyclic primary subgroups respectively are K-$\mathfrak{F}$-subnormal were studied. According to these papers the following result holds.

Lemma 7 *If* $\mathfrak{F}$ *is a hereditary formation, then* $\mathfrak{N} \cup \mathfrak{F} \subseteq \overline{w}\mathfrak{F} \subseteq v^*\mathfrak{F}$.

Lemma 8 *Let* $\mathfrak{F}$ *be a hereditary formation. Then there is a largest by inclusion subgroup* $S_\mathfrak{F}(G)$ *among normal subgroups* N *of* G *with* P K-$\mathfrak{F}$-*sn* PN *for every Sylow subgroup* P *of* G.

Proof Let $N_i \trianglelefteq G$ with P K-$\mathfrak{F}$-*sn* PN_i for every Sylow subgroup P of G and $i = 1, 2$. Note that PN_2 K-$\mathfrak{F}$-*sn* $(PN_1)N_2$ by P K-$\mathfrak{F}$-*sn* PN_2 and Lemma 3. Hence P K-$\mathfrak{F}$-*sn* PN_1N_2 by (3) of Lemma 1. Let S be a product of all normal subgroups N of G with P K-$\mathfrak{F}$-*sn* PN. Now P K-$\mathfrak{F}$-*sn* PS. It means that $S = S_\mathfrak{F}(G)$. □

3 Proofs of theorems

Proof of Theorem 1 (1) $\Rightarrow$ (2). Note that every cyclic primary subgroup is subnormal in some Sylow subgroup. Hence if all Sylow subgroups of G are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal, then all cyclic primary subgroups of G are also $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. Thus $G \in \mathfrak{F}$.

(2) $\Rightarrow$ (3). (a) $\mathfrak{N} \subseteq \mathfrak{F}$.

Assume that $\mathfrak{F}$ contains every group G all whose Sylow subgroups are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. Now $\mathfrak{F}$ contains every group G all whose Sylow subgroups are K-$\mathfrak{F}$-subnormal. Hence $\mathfrak{F} = \overline{w}\mathfrak{F}$. Now $\mathfrak{N} \subseteq \mathfrak{F}$ by Lemma 7.

(b) *Assume that L is a faithful irreducible $\mathbb{F}_pG$-module, $T = L \rtimes G$ and $L \leq S_{\mathfrak{F}}(T)$. Then $G \in \mathfrak{F}$.*

In this case $L = \mathrm{F}^*(T) \leq S_{\mathfrak{F}}(T)$. Now $T \in \mathfrak{F}$ by (2). Thus $G \in \mathfrak{F}$ as a quotient group of T, the contradiction.

(c) *Let $\pi(p) = \{q \in \mathbb{P} \mid (p, q) \in \Gamma_{Nc}(\mathfrak{F})\} \cup \{p\}$. Then $\mathfrak{F}$ contains every q-closed $\{p, q\}$-group for every $q \in \pi(p)$.*

Assume the contrary. Let G be a minimal order counterexample. Since $\mathfrak{F}$ and the class of all q-closed groups are hereditary formations, we see that G is an $\mathfrak{F}$-critical group, G has a unique minimal normal subgroup N and $G/N \in \mathfrak{F}$. Let P be a Sylow p-subgroup of G. If $NP < G$, then $NP \in \mathfrak{F}$. Hence P K-$\mathfrak{F}$-sn PN and PN/N K-$\mathfrak{F}$-sn G/N. From Lemma 1 it follows that P K-$\mathfrak{F}$-sn G. Since G is a q-closed $\{p, q\}$-group, we see that every Sylow subgroup of G is K-$\mathfrak{F}$-subnormal. Hence $G \in \mathfrak{F}$, a contradiction.

Now N is a Sylow q-subgroup and $\mathrm{O}_p(G) = 1$. By Lemma 4 there exists a faithful irreducible $\mathbb{F}_pG$-module L. Let $T = L \rtimes G$. Assume that $NL \notin \mathfrak{F}$. Then it has an $\mathfrak{F}$-critical subgroup H with normal Sylow p-subgroup K and the elementary abelian Sylow q-subgroup Q. From Maschke's theorem it follows that K is the direct product of minimal normal subgroups of H. Note that each of this subgroups has a complement in H. It means that K is the unique minimal normal subgroup of H. Hence K is a faithful irreducible $\mathbb{F}_pQ$-module. From [5, B, Theorem 10.3] it follows that Q is a group of order p. Now H is a Schmidt (p, q)-group with the trivial Frattini subgroup. From $(p, q) \in \Gamma_{Nc}(\mathfrak{F})$ it follows that $\mathfrak{F}$ contains a Schmidt (p, q)-group with trivial Frattini subgroup. According to [26] all such Schmidt groups are isomorphic. Hence $H \in \mathfrak{F}$, a contradiction. Therefore $NL \in \mathfrak{F}$. Note that $L \leq \mathrm{O}_p(T)$. Hence $L \leq S_{\mathfrak{F}}(T)$ by Lemma 8. Thus $G \in \mathfrak{F}$ by (b), a contradiction.

From (c) it follows that

(d) *$\Gamma_{Nc}(\mathfrak{F})$ is undirected, i.e $(p, q) \in \Gamma_{Nc}(\mathfrak{F})$ iff $(q, p) \in \Gamma_{Nc}(\mathfrak{F})$.*

(e) *Let p, q and r be different primes. If $(p, r), (q, r) \in \Gamma_{Nc}(\mathfrak{F})$, then $(p, q) \in \Gamma_{Nc}(\mathfrak{F})$.*

There exists a faithful irreducible $\mathbb{F}_pZ_q$-module P by Lemma 4. Let $G = P \rtimes Z_q$. Then there exists a faithful irreducible $\mathbb{F}_rG$-module R by Lemma 4. Let $T = R \rtimes G$. From (c) it follows that $\mathfrak{F}$ contains all r-closed $\{p, r\}$-groups and $\{q, r\}$-groups. Hence $R \leq S_{\mathfrak{F}}(T)$ by Lemma 8. Thus $G \in \mathfrak{F}$ by (b). Note that G is a Schmidt (p, q)-group.

(f) *$\mathfrak{F} = \mathfrak{N}_\sigma$ for some partition σ of $\mathbb{P}$.*

From (d) and (e) it follows that $\Gamma_{Nc}(\mathfrak{F})$ is a disjoint union of complete (directed) graphs Γ_i, $i \in I$. Let $\pi_i = V(\Gamma_i)$. Then $\sigma = \{\pi_i \mid i \in I\}$ is a partition of $\mathbb{P}$. From

Lemma 6 it follows that every $\mathfrak{F}$-group G has a normal Hall π_i-subgroups for every $i \in I$ with $\pi_i \cap \pi(G) \neq \emptyset$. Now G is σ-nilpotent. Hence $\mathfrak{F} \subseteq \mathfrak{N}_\sigma$.

Let show that the class $\mathfrak{G}_{\pi_i}$ of all π_i-groups is a subset of $\mathfrak{F}$ for every $i \in I$. It is true if $|\pi_i| = 1$. Assume now $|\pi_i| > 1$. Suppose the contrary and let a group G be a minimal order group from $\mathfrak{G}_{\pi_i} \setminus \mathfrak{F}$. Then G has a unique minimal normal subgroup, $\pi(G) \subseteq \pi_i$ and $|\pi(G)| > 1$. Note that $\mathrm{O}_q(G) = 1$ for some $q \in \pi(G)$. Hence there exists a faithful irreducible $\mathbb{F}_q G$-module N by Lemma 4. Let $T = N \rtimes G$. Hence $NP \in \mathfrak{F}$ for every Sylow subgroup P of T by (c). Now $N \leq S_{\mathfrak{F}}(T)$ by Lemma 8. Hence $G \in \mathfrak{F}$ by (b), the contradiction.

Since a formation is closed under taking direct products, we see that $\mathfrak{N}_\sigma \subseteq \mathfrak{F}$. Thus $\mathfrak{F} = \mathfrak{N}_\sigma$.

(3) $\Rightarrow$ (1). Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition of $\mathbb{P}$. Then $\mathfrak{N}_\sigma$ has the lattice property for K-$\mathfrak{N}_\sigma$-subnormal subgroups by Lemma 5. According to [14, Theorem B and Corollary E.2] $v^*\mathfrak{F} = \mathfrak{F}$.

Assume that all cyclic primary subgroups of G are $\mathrm{F}^*(G)$-K-$\mathfrak{N}_\sigma$-subnormal. Note that every cyclic primary subgroup of $\mathrm{F}^*(G)$ is K-$\mathfrak{N}_\sigma$-subnormal in it. Hence $\mathrm{F}^*(G) \in \mathfrak{N}_\sigma$. Now $\mathrm{F}^*(G)$ is a direct product of all its normal Hall π_i-subgroup F_{π_i} where $\pi_i \in \sigma$ and $\pi_i \cap \pi(\mathrm{F}^*(G)) \neq \emptyset$.

Let C be a cyclic primary subgroup of G. Then C K-$\mathfrak{N}_\sigma$-sn $C\mathrm{F}^*(G)$ and $C \in \mathfrak{N}_\sigma$. Let $C = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = C\mathrm{F}^*(G)$ be a chain with $C_{i-1} \trianglelefteq C_i$ or $C_i/\mathrm{Core}_{C_i}(C_{i-1}) \in \mathfrak{N}_\sigma$ for all $i = 1, \dots, n$. Note that if $C_{i-1} \trianglelefteq C_i$, then $C_i/\mathrm{Core}_{C_i}(C_{i-1}) = C_i/C_{i-1}$ is isomorphic to a section of $\mathrm{F}^*(G) \in \mathfrak{N}_\sigma$. Hence $C_i/C_{i-1} \in \mathfrak{N}_\sigma$. Now $C\mathrm{F}^*(G) \in \mathfrak{N}_\sigma$ by [3, Proposition 6.1.11].

It means that if C is a π_i'-group, then $C \leq C_G(F_{\pi_i})$. Now $(H/K) \rtimes G/C_G(H/K)$ is a π_i-group for some $\pi_i \in \sigma$ and every chief factor H/K of G below $\mathrm{F}^*(G)$. Also note that $G^{\mathfrak{N}_\sigma} \leq \mathrm{O}^{\pi_i}(G) \leq C_G(F_{\pi_i})$. Hence $G^{\mathfrak{N}_\sigma} \leq C_G(\mathrm{F}^*(G)) \leq \mathrm{F}^*(G)$. From this it follows that $(H/K) \rtimes G/C_G(H/K)$ is a π_i-group for some $\pi_i \in \sigma$ and for every chief factor H/K of G. Now G is σ-nilpotent by [18]. □

Proof of Theorem 2 (1) $\Rightarrow$ (2). Assume that $G = AB$ where all Sylow subgroups of A and B are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. Since every cyclic primary subgroup C is subnormal in some Sylow subgroup P of A, we see that $C \trianglelefteq\trianglelefteq P$ K-$\mathfrak{F}$-sn $P\mathrm{F}^*(G)$. Now C K-$\mathfrak{F}$-sn $C\mathrm{F}^*(G)$ by Lemma 1. Hence C is $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. Thus all cyclic primary subgroups of A are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. We can prove the same statement for B. Now $G \in \mathfrak{F}$ by (1).

(2) $\Rightarrow$ (3). From $G = GG$ and (2) it follows that $\mathfrak{F}$ contains every group G all whose Sylow subgroups are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. Thus there is a partition σ of $\mathbb{P}$ such that $\mathfrak{F} = \mathfrak{N}_\sigma$ by Theorem 1.

(3) $\Rightarrow$ (1). Let $G = AB$ where all cyclic primary subgroups of A and B are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. By [17, Lemma 11.6] there are Sylow p-subgroups P_1, P_2 and P of A, B and G respectively with $P_1P_2 = P$.

Let $C \leq P_1$ be a cyclic primary subgroup. Since C K-$\mathfrak{F}$-sn P_1, we see that $C\mathrm{F}^*(G)$ K-$\mathfrak{F}$-sn $P_1\mathrm{F}^*(G)$ by Lemma 3. From C K-$\mathfrak{F}$-sn $C\mathrm{F}^*(G)$ it follows that C K-$\mathfrak{F}$-sn $P_1\mathrm{F}^*(G)$ by (3) of Lemma 1.

Since $\mathfrak{F}$ has the lattice property for K-$\mathfrak{F}$-subnormal subgroups by Lemma 5 and P_1 is generated by all its cyclic primary subgroups, we see that P_1 K-$\mathfrak{F}$-sn $P_1\mathrm{F}^*(G)$.

From P_1 K-$\mathfrak{F}$-sn P it follows that $P_1\mathrm{F}^*(G)$ K-$\mathfrak{F}$-sn $P\mathrm{F}^*(G)$ by Lemma 3. Since P_1 K-$\mathfrak{F}$-sn $P_1\mathrm{F}^*(G)$, we see that P_1 K-$\mathfrak{F}$-sn $P\mathrm{F}^*(G)$ by (3) of Lemma 1. The same argument shows that P_2 K-$\mathfrak{F}$-sn $P\mathrm{F}^*(G)$. Thus P K-$\mathfrak{F}$-sn $P\mathrm{F}^*(G)$ by the lattice property.

Since all Sylow p-subgroups of G are conjugate, they all are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. By analogy one can show that all Sylow subgroups of G are $\mathrm{F}^*(G)$-K-$\mathfrak{F}$-subnormal. Now $G \in \mathfrak{F}$ by Theorem 1. □

Proof of Corollaries 2 and 5 Let prove that if H is a π_i-subgroup for some $\pi_i \in \sigma$ and every element of H permutes with every π_i'-element of $\mathrm{F}^*(G)$, then H K-$\mathfrak{N}_\sigma$-sn $H\mathrm{F}^*(G)$. Note that $\mathrm{O}^{\pi_i}(\mathrm{F}^*(G)) \trianglelefteq G$ and $\mathrm{O}^{\pi_i}(\mathrm{F}^*(G)) \leq C_G(H)$. Now $H \trianglelefteq H\mathrm{O}^{\pi_i}(\mathrm{F}^*(G))$. Since $H\mathrm{F}^*(G)/\mathrm{O}^{\pi_i}(\mathrm{F}^*(G))$ is a π_i-group, we see that H K-$\mathfrak{N}_\sigma$-sn $H\mathrm{F}^*(G)$.

Now Corollaries 2 and 5 directly follows from Theorems 1 and 2 respectively. □

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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