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On finite factorized groups with permutable subgroups of factors

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Abstract. Two subgroups A and B of a group G are called msp-permutable if the following statements hold: AB is a subgroup of G; the subgroups P and Q are mutually permutable, where P is an arbitrary Sylow p-subgroup of A and Q is an arbitrary Sylow q-subgroup of B, $p \neq q$. In the present paper, we investigate groups that are factorized by two msp-permutable subgroups. In particular, the supersolubility of the product of two supersoluble msp-permutable subgroups is proved.

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1. Introduction. Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [3]. The notation $Y \leq X$ means that Y is a subgroup of a group X.

Two subgroups A and B of a group G are called *mutually (totally) permutable* if $UB = BU$ and $AV = VA$ (respectively, $UV = VU$) for all $U \leq A$ and $V \leq B$.

The idea of totally and mutually permutable subgroups was first initiated by Asaad and Shaalan in [1]. This direction has since been subject of an in-depth study of many authors. An exhaustive report on this matter appears in [3, chapters 4–5].

It is quite natural to consider a factorized group $G = AB$ in which certain subgroups of the factors A and B are mutually (totally) permutable. In this direction, Monakhov [7] obtained the solubility of a group $G = AB$ under the assumption that the subgroups A and B are soluble and the Carter subgroups (Sylow subgroups) of A and of B are permutable.

We introduce the following

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Definition. Two subgroups A and B of a group G are called msp-permutable if the following statements hold:

(1) AB is a subgroup of G;
(2) the subgroups P and Q are mutually permutable, where P is an arbitrary Sylow p-subgroup of A and Q is an arbitrary Sylow q-subgroup of B, $p \neq q$.

In the present paper, we investigate groups that are factorized by two msp-permutable subgroups. In particular, the supersolubility of the product of two supersoluble msp-permutable subgroups is proved.

2. Preliminaries. In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called *supersoluble*. Recall that a *p-closed* group is a group with a normal Sylow p-subgroup and a *p-nilpotent* group is a group with a normal Hall p'-subgroup.

Denote by G', $Z(G)$, $F(G)$, and $\Phi(G)$ the derived subgroup, centre, Fitting, and Frattini subgroups of G, respectively; $\mathbb{P}$ the set of all primes. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$.

The monographs [2,5] contain the necessary information of the theory of formations. The formations of all nilpotent, p-groups, and supersoluble groups are denoted by $\mathfrak{N}$, $\mathfrak{N}_p$, and $\mathfrak{U}$, respectively. A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. A *formation function* is a function f defined on $\mathbb{P}$ such that $f(p)$ is a, possibly empty, formation. A formation $\mathfrak{F}$ is said to be *local* if there exists a formation function f such that $\mathfrak{F} = \{G \mid G/F_p(G) \in f(p)\}$. Here $F_p(G)$ is the greatest normal p-nilpotent subgroup of G. We write $\mathfrak{F} = LF(f)$ and f is a local definition of $\mathfrak{F}$. By [5, Theorem IV.3.7], among all possible local definitions of a local formation $\mathfrak{F}$, there exists a unique f such that f is integrated (i.e., $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full (i.e., $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$). Such a local definition f is said to be the *canonical local definition* of $\mathfrak{F}$. By [5, Theorem IV.4.6], a formation is saturated if and only if it is local.

A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G, see [12], if either $H = G$, or there is a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G, \ |H_i : H_{i-1}| \in \mathbb{P} \ \forall i.$$

A group G is called w-*supersoluble* (widely supersoluble) if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G. Denote by w$\mathfrak{U}$ the class of all w-supersoluble groups, see [12]. In [12, Theorem 2.7, Proposition 2.8], it is proved that w$\mathfrak{U}$ is a subgroup-closed saturated formation and every group from w$\mathfrak{U}$ has an ordered Sylow tower of supersoluble type. By [9, Theorem B], [8, Theorem 2.6], [12, Theorem 2.13], $G \in$ w$\mathfrak{U}$ if and only if G has an ordered Sylow tower of supersoluble type and every metanilpotent (biprimary) subgroup of G is supersoluble.

Denote by v$\mathfrak{U}$ the class of groups all of whose primary cyclic subgroups are $\mathbb{P}$-subnormal. In [9, Theorem B], it is proved that v$\mathfrak{U}$ is a subgroup-closed saturated formation and $G \in \mathrm{v}\mathfrak{U}$ if and only if G has an ordered Sylow tower of supersoluble type and every biprimary subgroup of G with a cyclic Sylow subgroup is supersoluble. It is easy to verify that $\mathfrak{U} \subseteq \mathrm{w}\mathfrak{U} \subseteq \mathrm{v}\mathfrak{U} \subseteq \mathcal{D}$. Here $\mathcal{D}$ is the formation of all groups which have an ordered Sylow tower of supersoluble type.

If H is a subgroup of G, then $H_G = \bigcap_{x \in G} H^x$ is called *the core* of H in G. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*. A simple check proves the following lemma.

Lemma 2.1. *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Lemma 2.2 ([5, Theorem 15.6]). *Let G be a soluble primitive group and M be a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$*;*
(2) $F(G) = C_G(F(G)) = O_p(G)$ *and* $F(G)$ *is an elementary abelian subgroup of order* p^n *for some prime* p *and some positive integer* n*;*
(3) G *contains a unique minimal normal subgroup* N *and moreover,* $N = F(G)$*;*
(4) $G = F(G) \rtimes M$ *and* $O_p(M) = 1$*.*

Lemma 2.3 ([10, Lemma 2.16]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.4 *Let $\mathfrak{F}$ be a formation, G a group, A and B subgroups of G such that A and B belong to $\mathfrak{F}$. If $[A, B] = 1$, then $AB \in \mathfrak{F}$.*

Proof. Since

$$[A, B] = \langle [a, b] \mid a \in A,\ b \in B \rangle = 1,$$

it follows that $ab = ba$ for all $a \in A$, $b \in B$. Let

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\},$$
$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2) \ \ \forall a_1, a_2 \in A,\ b_1, b_2 \in B$$

be the external direct product of the groups A and B. Since $A \in \mathfrak{F}$, $B \in \mathfrak{F}$ and $\mathfrak{F}$ is a formation, we have $A \times B \in \mathfrak{F}$. Let $\varphi : A \times B \to AB$ be a function with $\varphi((a, b)) = ab$. It is clear that φ is a surjection. Because

$$\begin{aligned}\varphi((a_1, b_1)(a_2, b_2)) &= \varphi((a_1a_2, b_1b_2)) = a_1a_2b_1b_2 \\ &= a_1b_1a_2b_2 = \varphi((a_1, b_1))\varphi((a_2, b_2)),\end{aligned}$$

it follows that φ is an epimorphism. The core Ker φ contains all elements (a, b) such that $ab = 1$. In this case, $a = b^{-1} \in A \cap B \leq Z(G)$. By the fundamental homomorphism theorem,

$$A \times B/\mathrm{Ker}\ \varphi \cong AB.$$

Since $A \times B \in \mathfrak{F}$ and $\mathfrak{F}$ is a formation, $A \times B/\mathrm{Ker}\ \varphi \in \mathfrak{F}$. Hence $AB \in \mathfrak{F}$. □

Lemma 2.5 ([4]). *Let a group $G = HK$ be the product of subgroups H and K. If L is normal in H and $L \leq K$, then $L \leq K_G$.*

Lemma 2.6. *Let $G = P \rtimes M$ be a primitive soluble group, where M is a primitivator of G and P is a Sylow p-subgroup of G. Let A and B be subgroups of M and $M = AB$. If $B \leq N_G(X)$ for every subgroup X of P, then the following statements hold:*

(1) B is a cyclic group of order dividing $p - 1$;
(2) $[A, B] = 1$.

Proof. We fix an element $b \in B$. If $x \in P$, then $x^b \in \langle x \rangle$ since $B \leq N_G(\langle x \rangle)$ by hypothesis. Hence $x^b = x^{m_x}$, where m_x is a positive integer and $1 \leq m_x \leq p$. If $y \in P \setminus \{x\}$, then

$$(xy)^b = (xy)^{m_{xy}} = x^{m_{xy}} y^{m_{xy}}, \ (xy)^b = x^b y^b = x^{m_x} y^{m_y},$$
$$x^{m_{xy}} y^{m_{xy}} = x^{m_x} y^{m_y}, \ x^{m_{xy} - m_x} = y^{m_y - m_{xy}} = 1, \ m_{xy} = m_x = m_y.$$

Therefore we can assume that $x^b = x^{n_b}$ for all $x \in P$, where $1 \leq n_b \leq p$ and n_b is a positive integer.

Assume that there exist $d \in B$ and $y \in P \setminus \{1\}$ such that $y^d = y$. Then $n_d = 1$ and $x^d = x$ for all $x \in P$, i.e., $d \in C_G(P) = P$ and $d = 1$. Consequently B is an automorphism group of a group of order p. Hence B is cyclic of order dividing $p - 1$.

Now we show that $[A, B] = 1$. We fix an element $[b^{-1}, a^{-1}] \in [A, B]$. Since P is normal in G, it follows that $x^a \in P$ for any $a \in A$ and any $x \in P$. Hence

$$\begin{aligned} x^{[b^{-1}, a^{-1}]} &= x^{bab^{-1}a^{-1}} = (x^b)^{ab^{-1}a^{-1}} \\ &= ((x^{n_b})^a)^{b^{-1}a^{-1}} = ((x^a)^{n_b})^{b^{-1}a^{-1}} = ((x^a)^b)^{b^{-1}a^{-1}} = (x)^{abb^{-1}a^{-1}} = x. \end{aligned}$$

Therefore $[b^{-1}, a^{-1}] \in C_G(P) = P$. Since $[A, B] \leq M$, we have $[b^{-1}, a^{-1}] \in M \cap P = 1$ and $[A, B] = 1$. □

3. Properties of msp**-permutable subgroups.** We will say that a group G satisfies the property:

E_π if G has at least one Hall π-subgroup;

C_π if G satisfies E_π and any two Hall π-subgroups of G are conjugate in G;

D_π if G satisfies C_π and every π-subgroup of G is contained in some Hall π-subgroup of G.

Such a group is also called an E_π-group, C_π-group, and D_π-group, respectively.

Lemma 3.1. *Let A and B be* msp*-permutable subgroups of G and $G = AB$.*

(1) If N is a normal subgroup of G, then $G/N = (AN/N)(BN/N)$ is the msp*-permutable product of the subgroups AN/N and BN/N.*
(2) If $A \leq H \leq G$, then H is the msp*-permutable product of the subgroups A and $H \cap B$.*
(3) If $G \in D_\pi$, then there exist Hall π-subgroups G_π, A_π, and B_π of G, of A, and of B, respectively, such that $G_\pi = A_\pi B_\pi$ is the msp*-permutable product of the subgroups A_π and B_π.*

Proof. (1) Let $p \in \pi(AN/N)$, X/N be a Sylow p-subgroup of AN/N, and P be a Sylow p-subgroup of A. Then $PN/N = X/N$. Similarly, if $q \in \pi(BN/N)$ such that $q \neq p$, Y/N is a Sylow q-subgroup of BN/N and Q is a Sylow q-subgroup of B. Then $QN/N = Y/N$. By hypothesis, P and Q are mutually permutable. Hence X/N and Y/N are mutually permutable.

(2) By Dedekind's identity, $H = A(H \cap B)$. Let A_q be a Sylow q-subgroup of A, R be a Sylow r-subgroup of $H \cap B$, where $q \neq r$, and B_r be a Sylow r-subgroup of B containing R. Since $(H \cap B_r)$ is a Sylow r-subgroup of $H \cap B$ and $R \leq H \cap B_r$, it follows that $R = H \cap B_r$.

Because A_q and B_r are mutually permutable, we have $A_qU \leq G$ for every subgroup U of R.

Let V be an arbitrary subgroup of A_q. Since A_q and B_r are mutually permutable,

$$VB_r \leq G, \quad H \cap VB_r = V(H \cap B_r) = VR \leq G.$$

Hence A_q and R are mutually permutable.

(3) By [3, Theorem 1.1.19], there are Hall π-subgroups G_π, A_π, and B_π of G, of A, and of B, respectively, such that $G_\pi = A_\pi B_\pi$. Since A and B are msp-permutable, it obviously follows that A_π and B_π are msp-permutable. □

Lemma 3.2. *Let A and B be* msp-*permutable subgroups of G and $G = AB$. Let $p, r \in \pi(G)$, p be the greatest prime in $\pi(G)$, and r be the smallest prime in $\pi(G)$. Then the following statements hold:*

(1) if A and B are p-closed, then G is p-closed;

(2) if A and B are r-nilpotent, then G is r-nilpotent;

(3) if A and B have an ordered Sylow tower of supersoluble type, then G has an ordered Sylow tower of supersoluble type.

Proof. (1) By [3, Theorem 1.1.19], there are Sylow p-subgroups P, P_1, and P_2 of G, of A, and of B, respectively, such that $P = P_1P_2$. By hypothesis, P_1 is normal in A and P_2 is normal in B. Let H_1 and H_2 be Hall p'-subgroups of A and of B, respectively, and Q be a Sylow q-subgroup of H_1, where $q \in \pi(H_1)$. Choose a chain of subgroups

$$1 = Q_0 < Q_1 < \cdots < Q_{t-1} < Q_t = Q, \quad |Q_{i+1} : Q_i| = q.$$

Since A and B are msp-permutable, we have that P_2Q_i is a subgroup of G for every i. Since $|P_2Q_1 : P_2| = q$ and $p > q$, it follows that P_2 is normal in P_2Q_1. Then by induction, we have that P_2 is normal in P_2Q. Because q is an arbitrary prime in $\pi(H_1)$, it follows that P_2 is normal in P_2H_1 and $\langle H_1, H_2\rangle \leq N_G(P_2)$. Similarly, $\langle H_1, H_2\rangle \leq N_G(P_1)$. Hence $P = P_1P_2$ is normal in G.

(2) Let R, R_1, and R_2 be Sylow r-subgroups of G, of A, and of B, respectively, such that $R = R_1R_2$. Let K_1 and K_2 be Hall r'-subgroups of A and of B. Let $q \in \pi(G) \setminus \{r\}$, Q, Q_1, and Q_2 be Sylow q-subgroups of G, of A, and of B, respectively, such that $Q = Q_1Q_2$. Choose a chain of subgroups

$$1 = V_0 < V_1 < \cdots < V_{t-1} < V_t = R_1, \quad |V_{i+1} : V_i| = r.$$

Since A and B are msp-permutable, V_iQ_2 is a subgroup of G for every i. Since $|V_1Q_2 : Q_2| = r$ and $q > r$, it follows that Q_2 is normal in V_1Q_2. Then by induction, we have that $R_1 \leq N_G(Q_2)$. By hypothesis, A is r-nilpotent, hence $R_1 \leq N_G(Q_1)$ and $R_1 \leq N_G(Q)$. Similarly, $R_2 \leq N_G(Q)$ and G has a r-nilpotent Hall $\{r, q\}$-subgroup RQ. Since q is an arbitrary prime in $\pi(G) \setminus \{r\}$, it follows that G is soluble and r-nilpotent by [11, Corollary].

(3) By (1), we have that a Sylow p-subgroup P is normal in G for the greatest $p \in \pi(G)$. By Lemma 3.1 (1), G/P is the product of the msp-permutable subgroups AP/P and BP/P. By induction, G/P has an ordered Sylow tower of supersoluble type, hence G has an ordered Sylow tower of supersoluble type. □

Theorem 3.3. *Let A and B be* msp-*permutable subgroups of G and $G = AB$. If A and B are soluble, then G is soluble.*

Proof. We use induction on the order of G and the method of the proof from [7, Theorem 2]. Let $N \neq 1$ be a soluble normal subgroup of G. By Lemma 3.1 (1), G/N is the product of the soluble msp-permutable subgroups AN/N and BN/N. By induction, G/N is soluble, hence G is soluble. In what follows, we assume that G contains no non-trivial soluble normal subgroups.

Since A is soluble, $U = O_s(A) \neq 1$ for some $s \in \pi(A)$. If B is an s-subgroup of G, then $G = AG_s$, $U \leq G_s$, and $U^G \leq (G_s)_G$ by Lemma 2.5, a contradiction. Hence B is not an s-subgroup of G and let Q be an arbitrary Sylow q-subgroup of B, where $q \in \pi(B) \setminus \{s\}$. Since A and B are msp-permutable,

$$UQ^x = UQ^{ba} = U^a(Q^b)^a = (UQ^b)^a = (Q^bU)^a = Q^xU$$

for every $x = ba \in G$, where $b \in B$ and $a \in A$. By [6, Theorem 7.2.5], $D = U^Q \cap Q^U$ is subnormal in G. Since $U^Q \leq UQ$ and UQ is soluble, it follows that D is a soluble subnormal subgroup of G and $D = 1$. Hence

$$[U, Q] \leq [U^Q, Q^U] \leq D = 1.$$

This is true for any Sylow q-subgroup of B, therefore $[U, Q^B] = 1$.

Let $H = N_G(U)$. By Dedekind's identity, $H = A(H \cap B)$. By Lemma 3.1 (2), H is the product of the soluble msp-permutable subgroups A and $H \cap B$. By induction, H is soluble. Since $[U, Q^B] = 1$, we have $Q^B \leq N_G(U) = H$. Because $G = AB = HB$, Q^B is normal in B, and $Q^B \leq H$, it follows that $Q^B \leq H_G = 1$ by Lemma 2.5, a contradiction. □

Lemma 3.4. *Let $G = G_1G_2$ be the product of* msp-*permutable subgroups G_1 and G_2. If a Sylow p-subgroup P of G is normal in G and abelian, then $P \cap G_i$ is normal in G for every $i \in \{1, 2\}$.*

Proof. Assume that $i, j \in \{1, 2\}$ and $i \neq j$. It is clear that $P \cap G_i$ is a Sylow p-subgroup of G_i and $P \cap G_i = (G_i)_p$ is normal in G_i. Hence G_i has a Hall p'-subgroup $(G_i)_{p'}$. Since G_i and G_j are msp-permutable, it follows that $(G_i)_p(G_j)_{p'}$ is a subgroup of G and $(G_j)_{p'} \leq N_G((G_i)_p)$ because every subgroup of G is p-closed. By hypothesis, P is abelian, therefore $(G_i)_p$ is normal in P and

$$G_j = (G_j)_p(G_j)_{p'} = (P \cap G_j)(G_j)_{p'} \le N_G((G_i)_p).$$

Hence $(G_i)_p$ is normal in $G = G_iG_j = G_1G_2$ for every $i \in \{1, 2\}$. □

4. Proof of the main theorem.

Theorem 4.1. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq \mathcal{D}$. Let $G = G_1G_2$ be the product of* msp*-permutable subgroups G_1 and G_2. If $G_1, G_2 \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. By Lemma 3.2 (3), G has an ordered Sylow tower of supersoluble type. Let P be a Sylow p-subgroup of G, where p is the greatest prime in $\pi(G)$. Then P is normal in G.

Assume that $G \notin \mathfrak{F}$. Let N be a non-trivial normal subgroup of G. Hence

$$G/N = (G_1N/N)(G_2N/N),$$
$$G_1N/N \cong G_1/G_1 \cap N \in \mathfrak{F},\ G_2N/N \cong G_2/G_2 \cap N \in \mathfrak{F}.$$

By Lemma 3.1 (1), G_1N/N and G_2N/N are msp-permutable. Consequently, G/N satisfies the hypothesis of the theorem, and by induction, $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, G is primitive by Lemma 2.1. Hence $\Phi(G) = 1$, $G = N \rtimes M$, where $N = C_G(N) = F(G) = O_p(G) = P$ is the unique minimal normal subgroup of G by Lemma 2.2. Therefore M is a Hall p'-subgroup of G and $M = (G_1)_{p'}(G_2)_{p'}$ for some Hall p'-subgroups $(G_1)_{p'}$ and $(G_2)_{p'}$ of G_1 and of G_2, respectively.

Suppose that p divides $|G_1|$ and $|G_2|$. By Lemma 3.4, $P \le G_1 \cap G_2$. Let $P_1 \le P$ and $|P_1| = p$. Since $P \le G_1$ and Q permutes with P_1 for every Sylow subgroup Q of $(G_2)_{p'}$, we have $P_1(G_2)_{p'} \le G$ and $(G_2)_{p'} \le N_G(P_1)$. Similarly, since $P \le G_2$ and R permutes with P_1 for every Sylow subgroup R of $(G_1)_{p'}$, it follows that $P_1(G_1)_{p'} \le G$ and $(G_1)_{p'} \le N_G(P_1)$. Hence $M = (G_1)_{p'}(G_2)_{p'} \le N_G(P_1)$ and P_1 is normal in G. By Lemma 2.3, $G \in \mathfrak{F}$, a contradiction.

Thus $P \le G_1$ and G_2 is a p'-subgroup of G. By Lemma 2.6 (1), G_2 is a cyclic group of order dividing $p-1$. Hence $G_2 \in g(p)$, where g is the canonical local definition of the saturated formation $\mathfrak{U}$. Since $\mathfrak{U} \subseteq \mathfrak{F}$, we have by [5, Proposition IV.3.11], $g(p) \subseteq f(p)$, where f is the canonical local definition of the saturated formation $\mathfrak{F}$. Hence $G_2 \in f(p)$. Since $P \le G_1$, it follows that $G_1 = P \rtimes (G_1)_{p'}$. Because $G_1 \in \mathfrak{F}$ and $F_p(G_1) = P$, we have $G_1/F_p(G_1) = G_1/P \cong (G_1)_{p'} \in f(p)$. By Lemma 2.6 (2), $[(G_1)_{p'}, (G_2)_{p'}] = 1$. Since $(G_1)_{p'} \in f(p)$, $(G_2)_{p'} \in f(p)$, and $f(p)$ is a formation, it follows by Lemma 2.4 that $G/P \cong M = (G_1)_{p'}(G_2)_{p'} \in f(p)$. Because $P \in \mathfrak{N}_p$, we have $G \in \mathfrak{F}$, a contradiction. The theorem is proved. □

Corollary 4.2. *Let $G = G_1G_2$ be the product of* msp*-permutable subgroups G_1 and G_2.*

1. *If $G_1, G_2 \in \mathfrak{U}$, then $G \in \mathfrak{U}$.*
2. *If $G_1, G_2 \in \mathrm{w}\mathfrak{U}$, then $G \in \mathrm{w}\mathfrak{U}$.*
3. *If $G_1, G_2 \in \mathrm{v}\mathfrak{U}$, then $G \in \mathrm{v}\mathfrak{U}$.*

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