Vector properties of Bessel light beams

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ABSTRACT

In the present report the new strict solutions of Maxwell equations and vector wave equation for different modes of an electromagnetic field of nodiffraction beams of a light are obtained. It is shown, that the set of solutions is multiparameter and consequently there is an arbitrariness in choice of main types of modes. The expressions for the polarization characteristics of different TE and TM modes of Bessel beams are obtained. The principle of reciprocity permitting to realize transition from E-modes to M-modes and back is justified. It is has been calculated their average density of energy and Poynting's vector. The main legitimacies of TE and TM modes are established. Founed results may be used at the description of transformation of Bessel light beams of different types.

1. INTRODUCTION

The learning of legitimacies of distribution of electromagnetic waves, as acutely directional, transversally restricted beams, has major importance in connection with necessity of usage in systems of processing and transmission of information. It will be used more often Gaussian [1] light beams. Recently Durnin and other [2,3] have paid attention that there are precise solutions of a scalar wave equation of the Helmholtz describing the new type of light beams. The transversal distribution of amplitude of such beams is featured not by the function of the Gauss. Such beams were called Bessel or nodiffraction and they have begun intensive theoretical and experimental researches [4-14.]

Recently major attention is attracted with so-called Bessel-Gaussian beams [15-17], for which the transversal allocation of amplitude of a field is featured by product of cylindrical functionses and Gauss ones. Such nodiffraction beams, as well as Gaussian, have limited energy. They are more realistic model, than the simple Bessel beams, but are least studied.

It is necessary to mark however, that the polarization properties of electromagnetic waves are featured not scalar, but vector characteristics. Therefore, it is necessary to study vector solution, instead of scalar wave equation. Besides at learning Bessel light beams frequently are restricted to parabolic approximation of a wave equation, and the solutions of the latter search in paraxial approximation.

The learning of vector properties of Bessel beams on the basis of strict solution of Maxwell equations was conducted in [7-9]. It was shown, that the solution vector, instead of scalar equation of the Helmholtz leads to modes of Bessel beams of a light have a longitudinal component. The number of solutions is obtained for TE, TM of modes, for plain polarized modes and their properties are considered. So, in [8] is shown, that the discrepancy between the strict vector and approximate scalar theory of Bessel beams of a light becomes essential, if the size of cut of a bundle is comparable from a long light wave. Density of energy is distinct from zero in any point, the central peak is much less, than in the scalar theory. At the same time, in these operations the separate solutions from many, modes, describing possible types, of nodiffraction fields are retrieved only.

Thus, the fulfilled till now theoretical and experimental operations on learning nodiffraction beams have demonstrated high prospects of their further researches.

However usage of paraxial approximation, ignoring of vector character of an electromagnetic field and illconditioned registration of a state of polarization of waves generating a Bessel bundle, reduce to certainly approximate, and in a number of cases - erratic outcomes. As it was already pointed above, only in some operations the attempts were done to leave for frameworks of scalar approximation and to take into account vector character of a light field. Thus the first steps are made only and before the contributors serious problems still stand.

The main purpose of the given report is obtaining correct solutions of a vector wave equation circumscribing TE - mode of Bessel beams of different types and learning of their energy characteristics.

2. THE VECTOR EQUATION OF THE HELMHOLTZ AND VECTOR BESSEL LIGHT BEAMS. A PRINCIPLE OF RECIPROCITY

Maxwell equations

$$\begin{cases} rot\ \overline{E} = -\dfrac{1}{c}\dfrac{\partial \overline{B}}{\partial t},\ rot\ \overline{H} = \dfrac{1}{c}\dfrac{\partial \overline{D}}{\partial t}\ ; \\ div\ \overline{D} = 0,\ div\ \overline{B} = 0 \end{cases} \tag{1}$$

for simple harmonic waves of sort $\overline{E}(\bar{r}, t) = E(\bar{r})e^{-i\omega t}$ become simpler

$$\begin{cases} rot\ \overline{E} = ik_o\overline{B},\ rot\ \overline{H} = -ik_o\overline{D},\ k_o = \omega / c. \\ div\ \overline{D} = 0,\ div\ \overline{B} = 0. \end{cases} \tag{2}$$

For material equations

$$\overline{D} = \varepsilon\overline{E}, \overline{B} = \mu\overline{H} \tag{3}$$

the relations (2) are reduced to expressions

$$\begin{cases} rot\ \overline{E} = ik_o\mu\overline{H}, \qquad div\ \varepsilon\overline{E} = 0, & (4a) \\ rot\ \overline{H} = -ik_o\varepsilon\overline{E}, \qquad div\ \mu\overline{H} = 0. & (4b) \end{cases}$$

For homogeneous isotropic media a wave equation (vector equation of the Helmholtz) has form

$$\left(\Delta + \frac{\varepsilon\mu\omega^2}{c^2}\right)\overline{E}(\bar{r}) = 0\,. \tag{5}$$

It is fair also for the medium with an ε -anisotropy and μ -scalar.

For an isotropic medium $\varepsilon\mu\omega^2 / c^2 = \bar{k}^2$ and the equation of the Helmholtz becomes simpler a little

$$\left(\Delta + k^2\right)\overline{E}(r) = 0\,. \tag{6}$$

For nodiffraction beams we shall search for solutions of the equation (6) by the way

$$\overline{E}(\bar{r}) = \overline{E}(\bar{r}_\perp)\exp(ik_{II}z), \tag{7}$$

Where $\bar{r} = \bar{r}_\perp + \bar{r}_{II},,\ \bar{r}_\perp = (x, y),\quad \bar{r}_{II} = \bar{z}$. Then $\left|\overline{E}(\bar{r}, t)\right|^2 = const$, as well as it is required for nodiffraction beams. In the total we obtain the equation of the Helmholtz for nodiffraction fields

$$\left(\overline{\nabla}_\perp^2 + k_\perp^2\right)\overline{E}(\bar{r}_\perp) = 0\,. \tag{8}$$

The denotations here are entered

$$\bar{k} = \bar{k}_{II}^2 + \bar{k}_\perp^2,\quad k_{II} = k_z\,. \tag{9}$$

It is in most cases expedient to pass to a cylindrical coordinate system. Now equation of the Helmholtz (8) accepts the following form:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\rho^2} + k_\perp^2\right)\overline{E}(\bar{r}_\perp) = 0\,. \tag{10}$$

For Ez of components this is written, for example, in [18]. However as we have shown, it is fair also for each component of a field.

For solutions of the equation (10) for each component we search by a separation of variables of a field by the way

$$E_i(\bar{r}_\perp) = E_i(\rho, \varphi) = E_i(\rho)\exp(i\mu\varphi). \tag{11}$$

Let's suppose, that these solutions are periodic on an index n, therefore n = 0, ± 1, $\pm$ 2, ± 3, …. Having substituted (11) in (10) we obtain the system of three equations of the Bessel [19]

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + 1 - \frac{v^2}{u^2}\right)E_i(u) = 0\,. \tag{12}$$

For simplification a new variable $u = k_\perp\rho$ here is entered. As it is known [19,20], solutions of the equations (12) are, in particular, the cylindrical functions of 1 sorts $J_m(u)$ about *m* - order.

If to pass to a concrete coordinate system $\left(\bar{E}(u)=\{E_\rho, E_\varphi, E_z\}\right.$, or $\{\bar{E}_\perp, E_z\}$, or to cartesian $\left.\bar{E}(u)=\{E_x, E_u, E_z\}\right)$, it is possible to take solution of the equations (12) in the form of linear combinations of cylindrical functions of the Bessel of 1 sort with different indexes n.

$$E_{\sigma_v}(\bar{r},t)=E_{\sigma_v} J_v(u)\exp\left[i(k_{II} z-\omega t+v\varphi)\right], \tag{13}$$

Where $\sigma = x, y, z, \rho, \varphi$, v = 0, ± 1, ± 2, … . Coefficients E_{σ_v} are arbitrary constants defined from boundary conditions.

Thus, the expressions (13) represent the multiparameter set of solutions of the vector equation of the Helmholtz (5) in cylindrical functions, since each component of a field is linear combination of cylindrical functions of the first sort.

Let's mark, that, in order to prevent tangle, we mean $k_z = k_{II}$, $k_\perp = \sqrt{k^2 - k_2^2}$. The writers of reports [4, 5, 9, 15] a transversal component $k_\perp$ of a wave vector mean through β, and longitudinal component k_{II} - through α. Other writers, for example [2, 3, 14], use the return denotations.

The similar reasonings are fair, certainly, and for vectors of a magnetic field $\bar{H}$ light waves. In the total for components of a vector $\bar{H}$ it is possible also to write solutions of sort of wave functions with transversal Bessel allocation of amplitudes.

All six solutions of scalar equations for components of two vectors $\bar{E}(u)$ and $\bar{H}(u)$ are not independent, and obey to the second pair of Maxwell equations, which for an isotropic medium becomes

$$div\ \bar{E}(\bar{r})=0, \quad div\ \bar{H}(\bar{r})=0. \tag{14}$$

Therefore it is possible to select from 6 unknowns of components any 2 independent ones, then remaining are defined uniquely.

Two components of an electromagnetic field uniquely (within a constant factor) define a field. In the total we obtain a mode. Thus, the modes of nodiffraction light fields are defined ambiguously and depend on many parameters.

The Maxwell equations (4a), (4b) together with material equations (3) have a considerable symmetry. At replacements

$$\bar{E} \vartriangle \bar{H},\ \varepsilon \vartriangle -\mu,\ \bar{B} \vartriangle -\bar{D} \tag{15}$$

the expressions (4a), (4b), (3) are conversed to themselves. Therefore relations (15) as a matter of fact represent a principle of reciprocity.

The relations (15) allow, on the basis of expressions for the polarization characteristics of an electric field, at once to write similar expressions for the polarization characteristics of a magnetic field of an electromagnetic wave. Further, in the present report, we shall apply a principle of reciprocity (15) to Bessel light fields. It is clear, however, that the principle of transition (15) can be used for Gaussian, Bessel-Gaussian and other sorts of light beams.

The offered principle of reciprocity (15) is distinct from two principles of a dualism justified by F.I. Phedorov [21] for plane harmonic waves. It is fair for different sorts of monochromatic light (plane, spherical, cylindrical and other) fields in homogeneous anisotropics media.

3. THE COMMON ENERGY CHARACTERISTICS TE - MODES OF BESSEL FIELDS

A time average electromagnetic energy density w and vector of an energy flux density (Poynting's vector) $\bar{S}$ disregarding influences of a frequency dispersion are determined by common expressions [22]:

$$\bar{S}=\frac{c}{8\pi}\mathrm{Re}\left[\bar{E}^*\bar{H}\right]; \tag{16}$$

$$w=\frac{1}{16\pi}\left(\varepsilon\left|\bar{E}\right|^2+\mu\left|\bar{H}\right|^2\right). \tag{17}$$

In a cylindrical coordinate system

$$\bar{S}=\frac{c}{8\pi}\mathrm{Re}\left\{\bar{e}_\rho\left(E_\varphi H_z^* - E_z H_\varphi^*\right)+\bar{e}_\varphi\left(E_z H_\rho^* - E_\rho H_z^*\right)+\bar{e}_z\left(E_\rho H_\varphi^* - E_\varphi H_\rho^*\right)\right\}. \tag{18}$$

as it was already mentioned above, there is a considerable arbitrariness at choice of vector amplitudes of nodiffraction fields. In particular, always one of six components can be put equal to zero.

At $E_z = 0$ we obtain TE - modes. By this, the equations (4a) become simpler and in a coordinate system become

$$\begin{cases} -k_0 \mu H_\rho = k_{II} E_\varphi, & (19) \\ k_0 \mu H_\varphi = k_{II} E_\rho, & (20) \\ k_0 \mu H_z = -i \dfrac{\partial (E_\varphi \rho)}{\partial \rho} - i \dfrac{\partial E_\rho}{\partial \varphi}; & (21) \\ \dfrac{\partial}{\partial \rho}(\rho E_\rho) + \dfrac{\partial E_\varphi}{\partial \varphi} = 0. & (22) \end{cases}$$

Their solutions will be different Bessel TE - modes. Let's consider their common properties. From (19, 20) it is visible, that for any TE - modes

$$\overline{H}_\rho \text{II}\ \overline{\mathrm{E}}_\varphi,\ \overline{H}_\varphi \text{II}\ \overline{\mathrm{E}}_\rho. \tag{23}$$

According to (21) longitudinal component of vector magnetic field H_z is expressed through transversal components of an electric field.

Similarly, the longitudinal component of energy flux also depends only on $\overline{E}_\perp$:

$$\overline{S}_z = \frac{c}{8\pi} \frac{k_{II}}{k_o \mu} \left|\overline{E}_\perp\right|^2 \overline{e}_z . \tag{24}$$

For transversal components of a Poynting's vector it is easy to receive expression

$$\overline{S}_\perp = \frac{c k_0 \mu}{8\pi k_\perp^2} \operatorname{Im}\left(H_z^* \left(\overline{\nabla}_\perp \overline{H}_z\right)\right). \tag{25}$$

The expression (17) for an electromagnetic energy density of Bessel TE - modes also becomes simpler a little

$$w = \frac{\mu}{16\pi} \left(\frac{k^2 + k_{II}^2}{k_0^2 \mu^2} \left|\overline{E}_\perp\right|^2 + \left|\overline{H}_z\right|^2 \right). \tag{26}$$

4. BESSEL TE-MODES ($E_z = 0, E_\varphi = J_m(u)\exp(im\varphi)$)

Let's take a TE - mode, in which the azimuth component of an electric field of a light wave is featured by one cylindrical functions of the first sort about m, depending from normalized transversal coordinate $u = k_\perp \rho$, i.e.

$$E_z = 0, \quad E_\varphi = J_m(u) \exp(im\varphi). \tag{27}$$

Then from (19)

$$E_\rho = \frac{-im \exp(im\varphi)}{u} F_m(u). \tag{28}$$

Here $F_m(u)$ represents a particular integral from a cylindrical functions about m

$$F_m(u) = \int_0^u J_m(u) du . \tag{29}$$

At $m = 2n + 1$ integral $F_m(u)$ is expressed by a linear combination of cylindrical functions of the even orders [23]

$$\int_0^u J_{2n+1}(u) du = 1 - J_0(u) - 2\sum_{k=1}^{n} J_{2k}(u). \tag{30}$$

At $m = 2n$

$$\int_0^u J_{2n}(u) du = F_0(u) - 2\sum_{k=1}^{n-1} J_{2k+1}(u). \tag{31}$$

Thus, the integrals from cylindrical functionses of the first sort are reduced to one integral $F_0(u)$ and simple superposition of cylindrical functions. Function $F_0(u)$ represents the restricted damping oscillating function, and at $u \to 0$ function $F_0(u) \to u$, and at $u \to \infty$ $F_0(u) \to 1$.

The expressions for E_ρ and H_φ are obtained from (19, 20).

The energy characteristics of given TE - modes are described by expressions (21), (24-26).

5. BESSEL TE - MODES ($E_z = 0,\ E_\rho = J_m(u)\exp(im\varphi)$)

From (22) at $m \neq 0$ it is expressed an azimuth component of an electric field

$$E_\varphi = -\frac{1}{im}\frac{\partial}{\partial u}\left(uE_\rho\right). \tag{32}$$

At $m = 0$ given modes can not.

Further remaining components of an electromagnetic field easily express. Let's mark, that for the given type of nodiffraction TE - modes of a radial component S_ρ the Poynting`s vector $\overline{S}$ has no.

6. BESSEL TE - MODES ($E_z = 0,\ H_z = J_m(u)\exp(im\varphi)$)

It is in this case expedient to express all values of a field through E_z и H_z. Then from (4a), (4в) it is possible to express $\overline{E}_\perp$ and $\overline{H}_\perp$ only through H_z, as $E_z = 0$:

$$\overline{E}_\perp = ik_0\mu\left[\overline{\nabla}_\perp H_z, \overline{e}_z\right]/k_\perp^2; \tag{33}$$

$$\overline{H}_\perp = ik_{II}\overline{\nabla}_\perp H_z / k_\perp^2. \tag{34}$$

We shall take the component H_z in the form

$$H_z = J_m(u)\exp(im\varphi) \tag{35}$$

After some conversions it is discovered, that for the given type of TE - modes transversal components of vectors $\overline{E}$ and $\overline{H}$ the fields are equal:

$$\overline{H}_\perp = \frac{k_{II}}{k_\perp}\left(iJ'_m\,\overline{e}_\rho - \frac{m}{u}J_m\,\overline{e}_\varphi\right)\exp\,(im\varphi); \tag{36}$$

$$\overline{E}_\perp = -\frac{k_0\mu}{k_\perp}\left(iJ'_m\,\overline{e}_\varphi + \frac{m}{u}J_m\,\overline{e}_\rho\right)\exp\,(im\varphi). \tag{37}$$

Hereinafter stroke means a rate of change of a function on a variable u.

Let's mark, that such Bessel TE - modes were considered also in [9], however energy characteristics there were not considered.

The electromagnetic energy density of the given mode is determined by expression

$$w = \frac{\mu}{16\pi}\left(\frac{k^2 + k_{II}^2}{k_\perp^2}\left(J_m'^2 + \frac{m^2}{u^2}J_m^2\right) + J_m'^2\right). \tag{38}$$

An energy flux density is equal to

$$\overline{S} = \frac{ck_0\mu}{8\pi k_\perp^2}\left\{k_{II}\left(J_m'^2 + \frac{m^2}{u^2}J_m^2\right)\overline{e}_z + \frac{m}{u}J_m^2\,\overline{e}_\varphi\right\}. \tag{39}$$

The analysis of the obtained outcomes allows to make the following outputs: a) transversal components of vectors $\overline{E}$ and $\overline{H}$ are orthogonally related; в) there is no radial energy, component of vector stream, and its lines represent composite spirals with an axis along an axis z.

The case, when m = 0 is most interesting. Then we obtain

$$E_z = E_\rho = H_\varphi = 0, \quad E_\varphi = ik_0\mu\ J_1 / k_\perp . \qquad (40)$$

It is azimuthal polarized beam of the Bessel type. Thus $J_0' = J_1$ and the stream of energy is directed strictly along an axis Z and is proportional to J_1^2. As immediately on an axis of a bundle $\bar{S} = 0$, such bundle is tubular J_1 bundle.

Recently it was shown theoretically [24] and experimentally [25], that some types of semiconducting lasers emit infrared circularly symmetric azimuthally polarized light J_1 bundle. Therefore further learning of azimuthally polarized Bessel beams is represented rather perspective.

7. OTHER TYPES OF BESSEL MODES

Recently precise solutions of Maxwell equations for linearly polarized on an axis X Bessel modes were obtained for the cases of cartesian [9] and cylindrical [7] coordinate systems. It is shown, that the trajectories of stream of energy of a nodiffraction field represent composite spiral curves with generator along an axis Z.

We shall mark, that for obtaining appropriate expressions for TM-modes of Bessel fields it is enough to take a principle of reciprocity (15). Thus $\bar{S} \to -\bar{S}$, $w \to -w$.

Other types of nondiffraction beams are possible which will be considered by us hereinafter.

CONCLUSIONS

In the present report on the basis of Maxwell equations for monochromatic radiation in an isotropic medium the strict vector equation of the Helmholtz is obtained. It is shown, that each component of an electric field of nodiffraction light fields can be expressed as a linear combination of determined wave functions. Each such function represents streaming along an axis Z a wave with transversal allocation of amplitude circumscribed by a cylindrical Bessel functions of the first sort. Certainly, all told are fairly also concerning components of a vector of a magnetic field of a bundle.

The correlations between different six components of an electromagnetic field of a bundle are provided with Maxwell equations. In the total it is obtained the multiparameter set of solutions circumscribing a nodiffraction Bessel field.

The common strict expressions for the analytical characteristics of TE- modes of Bessel fields are obtained. The vectors of a field $\bar{E}, \bar{H}$, density of an electromagnetic energy w and vector Poynting's $\bar{S}$ for different types of TE - modes are obtained, for which one of components of a field $\left(E_\varphi, E_\rho \quad , or\ H_z\right)$ are described by one simple cylindrical functions of the first sort.

The principle of reciprocity permitting to transfer from TE - modes to TM-modes and back is grounded. On the basis of the latter the similar results for TM-modes follow.

Other types of nodiffraction modes are considered also.

At reflection and refraction of Bessel beams of a light on the boundary of medium, and also at passing them through different optical elements it will be a transformation of one types of modes in other, therefore further learning of different types of modes of nodiffraction fields is obviously necessary also perspective.

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