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КРИТЕРИЙ ПРИНАДЛЕЖНОСТИ КОНЕЧНОЙ ГРУППЫ НАСЫЩЕННОЙ ФОРМАЦИИ

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A CRITERION FOR A FINITE GROUP TO BELONG A SATURATED FORMATION

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Доказывается следующий результат: *пусть* $\mathcal{F}$ *– такая наследственная насыщенная формация p-разрешимых групп, содержащая все p-сверхразрешимые группы, что* $\mathcal{F}=\mathcal{G}_{p'}\mathcal{F}$. *Пусть* $G=AT$, *где A – холлова* π*-подгруппа из G,* $p\notin\pi$ *и T – p-сверхразрешимая подгруппа из G. Предположим, что для силовской p-подгруппы P из T мы имеем* $|P|>p$. *Если A перестановочна с холловой* p'*-подгруппой из T и со всеми такими максимальными подгруппами V из P, что* $G^{\mathcal{F}}\cap P\nleq V$, *то* $G\in\mathcal{F}$.

Ключевые слова: *конечная группа, насыщенная формация, p-разрешимая группа, p-сверхразрешимая группа, холлова подгруппа.*

We prove the following result: *Let* $\mathcal{F}$ *be a hereditary saturated formation of p-soluble groups containing all p-supersoluble groups such that* $\mathcal{F}=\mathcal{G}_{p'}\mathcal{F}$. *Let* $G=AT$, *where A is a Hall* π*-subgroup of G,* $p\notin\pi$ *and T is a p-supersoluble subgroup of G. Suppose that for a Sylow p-subgroup P of T we have* $|P|>p$. *If A permutes with a Hall* p'*-subgroup of T and with all maximal subgroups V of P such that* $G^{\mathcal{F}}\cap P\nleq V$, *then* $G\in\mathcal{F}$.

Keywords: *finite group, saturated formation, p-soluble group, p-supersoluble group, Hall subgroup.*

Introduction

Throughout this paper, all groups are finite and *G* always denotes a finite group. Moreover, *p* is always supposed to be a prime and π is a non-empty subset of the set $\mathbb{P}$ of all primes; p' denotes the set of all primes $q\neq p$. A subgroup *H* of *G* is said to permute with a subgroup *K* of *G* if $HK=KH$.

By the well known Hall theorem [1], *G* is soluble if every Sylow subgroup *P* of *G* has a complement *T* in *G*, that is, a subgroup of *G* such that $PT=G$ and $P\cap T=1$. The example of the alternating group A_5 shows that such a result is incorrect in general if we consider only the Sylow *p*-subgroups for some fixed *p*. Nevertheless, B. Huppert [2] proved that if a Sylow *p*-subgroup *P* of *G* has a complement *T* in *G*, $|P|>p$ and *T* permutes with every maximal subgroup of *P*, then *G* is *p*-soluble. This result was improved in some directions. V. Sergienko [3] on the base of this result proved that if a Sylow *p*-subgroup *P* of *G* has a complement *T* in *G*, there is a number p^k such that $1<p^k<|P|$ and *T* permutes with all subgroups of *P* of order p^k and *P* is abelian in the case $p^k=2$, then *G* is *p*-so-luble and the *p*-length of *G* is equal to 1. Further, M. Borovikov [4] proved that under these conditions, *G* is even *p*-supersoluble. In [5] W. Guo, K.P. Shum and A.N. Skiba proved that if $G=AT$, where *A* is a Hall π-subgroup of *G*, *T* is nilpotent, and *A* permutes with all Sylow subgroups of *T* and with all maximal subgroups of any Sylow subgroup of *T*, then *G* is *p*-supersoluble, for each prime $p\notin\pi$ such that $|T_p|>p$ for a Sylow *p*-subgroup T_p of *T*. See also papers [6], [7].

In this paper we prove the following result in this line researches.

Theorem. *Let* $\mathcal{F}$ *be a hereditary saturated formation of p-soluble groups containing all p-supersoluble groups such that* $\mathcal{F}=\mathcal{G}_{p'}\mathcal{F}$. *Let* $G=AT$, *where A is a Hall* π*-subgroup of G,* $p\notin\pi$ *and T is a p-supersoluble subgroup of G. Suppose that for a Sylow p-subgroup P of T we have* $|P|>p$. *If A permutes with a Hall* p'*-subgroup of T and with all maximal subgroups V of P such that* $G^{\mathcal{F}}\cap P\nleq V$, *then* $G\in\mathcal{F}$.

All unexplained notation and terminology are standard. The reader is referred to [8]–[10] or [11] if necessary.

1 Preliminaries

***Lemma* 1.1.** *Let $\mathcal{F}$ be a hereditary formation. Let $H \le E \le G$ and $E_p \le G_p$, where E_p and G_p are Sylow p-subgroups of E and G, respectively. Suppose also that $H \le E_p$.*

(1) *If N is a normal subgroup of G and $(G/N)^{\mathcal{F}} \cap (PN/N) \nleq HN/N$, then $G^{\mathcal{F}} \cap P \nleq H$.*

(2) *If $E^{\mathcal{F}} \cap E_p \nleq H$, then $G^{\mathcal{F}} \cap G_p \nleq H$.*

Proof. (1) Assume that $G^{\mathcal{F}} \cap P \le H$. Then $N(G^{\mathcal{F}} \cap P) \le NH$, so

$$(G/N)^{\mathcal{F}} \cap (PN/N) =$$
$$= (G^{\mathcal{F}}N/N) \cap (PN/N) = N(G^{\mathcal{F}}N \cap P)/N =$$
$$= N(G^{\mathcal{F}} \cap P)(N \cap P)/N =$$
$$= N(G^{\mathcal{F}} \cap P)/N \le NH/N,$$

a contradiction. Hence we have $G^{\mathcal{F}} \cap P \nleq H$.

(2) Since the formation $\mathcal{F}$ is hereditary, $E/E \cap G^{\mathcal{F}} \simeq EG^{\mathcal{F}}/G^{\mathcal{F}} \in \mathcal{F}$. Hence this assertion directly follows from the inclusion $E^{\mathcal{F}} \cap E_p \le G^{\mathcal{F}} \cap G_p$. □

***Lemma* 1.2.** *If G is p-supersoluble and $O_{p'}(G) = 1$, then G is supersoluble and $F(G) = O_p(G)$ is normal Sylow p-subgroup of G, where p is the largest prime dividing $|G|$.*

***Lemma* 1.3.** *Let $\mathcal{F}$ be a saturated formation containing supersoluble groups and E a minimal normal subgroup of G such that $G/E \in \mathcal{F}$. If E is abelian and $\mathcal{F}$-central, then $G \in \mathcal{F}$.*

Proof. Clearly, we can suppose that $E \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = E \rtimes M$ and let $C = C_G(E)$. Then $M_G = C \cap M$ and so

$$G/M_G = (EM_G/M_G) \rtimes (M/M_G) \in \mathcal{F}$$

since $M/M_G \simeq G/C \in \mathcal{F}$. Thus

$$G \simeq G/E \cap M_G \in \mathcal{F}. \qquad \square$$

***Lemma* 1.4** (O.H. Kegel [12]). *Let A and B be subgroups of G such that $G \ne AB$ and $AB^x = B^xA$ for all $x \in G$. Then G has a proper normal subgroup N such that either $A \le N$ or $B \le N$.*

***Lemma* 1.5** (V.N. Knyagina and V.S. Monakhov [13]). *Let H, K and N be subgroups of G. If H is a Hall subgroup of G and H permutes with K, then*

$$N \cap HK = (N \cap H)(N \cap K).$$

2 Proof of Theorem

Assume that this theorem is false and let G be a counterexample of minimal order. Then $G^{\mathcal{F}} \ne 1$. We proceed our proof by proving the following claims:

(1) $O_{p'}(G) = 1$.

In view of Lemma 1.1(1), the hypothesis still holds for G/D and so $G/D \in \mathcal{F}$ by the choice of G. But then $G \in \mathcal{F}$ since $\mathcal{F} = \mathcal{G}_{p'}\mathcal{F}$, a contradiction. Thus we have (1).

(2) $G^{\mathcal{F}} \cap P \ne 1$.

Indeed, if $G^{\mathcal{F}} \cap P = 1$, then $G^{\mathcal{F}}$ is a p'-group. Hence $G^{\mathcal{F}} = 1$ by Claim (1), a contradiction.

(3) *T is supersoluble, $O_{p'}(T) = 1$ and P is normal in T.*

Since $O_{p'}(T)$ is normal in T,

$$(O_{p'}(T))^G = (O_{p'})^{AT} = (O_{p'})^A \le AT_{p'} = T_{p'}A,$$

where $T_{p'}$ is a Hall p'-subgroup of T. Hence $(O_{p'}(T))^G \le O_{p'}(G) = 1$, so $O_{p'}(T) = 1$. Hence, since T is p-supersoluble by hypothesis, T is supersoluble and P is normal in T by Lemma 1.2.

(4) *G is not p-soluble. Hence $G^{\mathcal{F}}$ is not p-soluble.*

Assume that G is p-soluble. Let L be a minimal normal subgroup of G. Then by Claim (1), L is a p-group and so $L \le P$. Next note that $G/L \in F$. Indeed, if $|P/L| \le p$, then the assertion follows from Lemma 1.3. On the other hand, if $|P/L| > p$, the hypothesis is true for G/L by Lemma 1.1 (1). Hence $G/L \in \mathcal{F}$ by the choice of G. Therefore $L \nleq \Phi(G)$. Hence $|L| > p$ and $L \nleq \Phi(T)$. Let M be a maximal subgroup of T such that $LM = T$. Then every Hall p'-subgroup of M is a Hall p'-subgroup of T. Since T is soluble, any two Hall p'-subgroups are conjugate in T. Hence without loss of generality we may suppose that $M = M_pM_{p'}$, where M_p is a Sylow p-subgroup of M and $M_{p'}$ is a Hall p'-subgroup of M such that $M_{p'}A = AM_{p'}$. Since T is supersoluble, $|T : M| = p$, so M_p is a maximal subgroup of P. Note also that $L \le G^{\mathcal{F}}$. Indeed, if $L \nleq G^{\mathcal{F}}$, then from the G-isomorphism

$$G^{\mathcal{F}}L/G^{\mathcal{F}} \simeq L/L \cap G^{\mathcal{F}}$$

we deduce that L is $\mathcal{F}$-central in G and hence $G \in \mathcal{F}$ by Lemma 1.3, contrary to the choice of G. Hence A permutes with M_p. Therefore

$$MA = M_pM_{p'}A = AM = M_pM_{p'}$$

is a subgroup of G with $|G : MA| = p$ and with $L \nleq MA$. But then $|L| = p$, a contradiction. Thus we have (4).

(5) *If H is a minimal normal subgroup of G and $|H| = p$, then $|P| = p^2$.*

Indeed, if $|P| > p^2$, the hypothesis is still true for G/H and so $G/H \in \mathcal{F}$ by the choice of G. Hence $G \in \mathcal{F}$ by Lemma 1.3, contrary to the choice of G.

(6) *If* H *is a normal subgroup of* G *and* $H \cap A \neq A$, *then* H *is p-soluble.*

It is clear that $H = (A \cap H)(T \cap H)$. Let $E = (H \cap A)T$. Let V be a maximal subgroup of P. Suppose that $E^{\mathcal{F}} \cap P \nleq V$. Then, by Lemma 1.1 (2), $G^{\mathcal{F}} \cap P \nleq V$. Hence $AV = VA$ is a subgroup of G. Therefore

$$AV \cap (A \cap H)P = $$
$$= (A \cap H)(AV \cap P) = (A \cap H)V(A \cap T) =$$
$$= (A \cap H)V = V(A \cap H).$$

Thus the hypothesis is still true for E. If $E = G$, then

$$A = A \cap (H \cap A)P = (H \cap A)(A \cap T) = H \cap A,$$

a contradiction. Hence, $E \neq G$ and so $E \in \mathcal{F}$ by the choice of G. Since every group in $\mathcal{F}$ is p-soluble by hypothesis, we conclude that $H \leq E$ is p-soluble.

(7) $O^{p'}(G) = G$.

Suppose that $O^{p'}(G) \neq G$. Since the hypothesis holds for $O_{p'}(G)$ by Lemma 1.1 (2), $O_{p'}(G) \in \mathcal{F}$ by the choice of G. But then G is p-soluble, contrary to Claim (4).

(8) *If* H *is a p-soluble minimal normal subgroup of* G, *then* $|H| = p$ *and* $H \leq Z(G)$.

First note that if $|H| = p$ and $C = C_G(H)$, then G/C, as a group of automorphisms of H, is a cyclic group of order dividing $p-1$. Hence in this case we have $H \leq Z(G)$ by Claim (6). Therefore we need only show that $|H| = p$. Clearly, H is either p'-group or p-group. But the former case is impossible by Claim (1), so $|H| = p^a$ for some natural a. If either $H = P$ or $|P/H| > p$, then G is clearly p-soluble, contrary to Claim (4). Hence H is a maximal subgroup of P. Suppose that $a > 1$. Then P is not cyclic. Therefore for some maximal subgroup V of P we have $P = HV$. Suppose that $G^{\mathcal{F}} \cap P \leq V$. Then $G^{\mathcal{F}} \neq G$ and $H \nleq G^{\mathcal{F}}$. Thus, in view of Claims (1) and (6), $G = G^{\mathcal{F}}H$. Since $G/G^{\mathcal{F}}$ is p-soluble and $O^{p'}(G) = G$, there is a normal maximal subgroup of G such that $G^{\mathcal{F}} \leq M$ and $|G : M| = p$. Since $|H| > p$, it follows that $H \leq M$. Hence $G = G^{\mathcal{F}}H \leq M$, a contradiction. Then $G^{\mathcal{F}} \cap P \nleq V$, which implies that A permutes with V. Now, as in the proof of Claim (4), it may be proved that there is a subgroup W of G such that $|G : W| = p$ and $H \nleq W$. But then $|H| = p$, a contradiction. Hence we have (8).

(9) P *is not cyclic.*

Suppose on the contrary that P is cyclic. First we show that in this case G does not have a proper normal subgroup E with $EP = G$. Indeed, if $EP = G$, where E is normal in G and $E \neq G$, then for any Sylow q-subgroup Q of A we have $G = EN_G(Q)$ by the Frattini argument. Hence $P = D_pN_p$ for some Sylow p-subgroups D_p of D and N_p of $N_G(Q)$. But P is cyclic and so $P \leq N_G(Q)$. Now let W be the Hall p'-subgroup of T such that $AW = AW$. Then

$$Q^G = Q^{AWP} = Q^{AW} \leq QAW = AW,$$

where $AW = WA$ is a p'-subgroup of G. Hence $Q^G \leq O_{p'}(G)$, which contradicts Claim (1). Now suppose that $G^{\mathcal{F}} \neq G$ and let $G^{\mathcal{F}} \leq M \leq G$, where M is a normal subgroup of G with simple quotient G/M. In view of Claim (7), p divides $|G/M|$. But then, since $\mathcal{F}$ consists of p-soluble groups, G/M is a p-group and hence $MP = G$. This contradiction shows that $G^{\mathcal{F}} = G$, so A permutes with the maximal subgroup Z of P. Since T is supersoluble by Claim (3), Z is normal in T. Hence

$$D = Z^G = Z^{AT} = Z^A \leq ZA.$$

By Lemma 1.4, $D = (A \cap D)(T \cap D)$. Assume that either $D \neq AT$ of $T \neq P$. Then D is p-soluble. Indeed, in the former case we have $D \cap A \neq A$ and so, by Claim (6), D is soluble. On the other hand, if $A \leq D$ and $T \neq P$, then the hypothesis still holds on DP. Since $|DP| < |G|$, DP is p-supersoluble by the choice of G. Now, let H be a minimal normal subgroup of G contained in D. Then since D is p-soluble, $|H| = p$ and $H \leq Z(G)$ by Claim (8). Let $N = N_G(P)$. If $P \leq Z(N)$, then G is p-nilpotent by the Burnside theorem [14], which contradicts the choice of G. Hence $N \neq C_N(P)$. Let $x \in N \setminus C_G(P)$ with $(|x|, |P|) = 1$ and $E = P \rtimes \langle x \rangle$. By [8, III, 13.4], $P = [E, P] \times (P \cap Z(E))$. Since $H \leq P \cap Z(E)$ and P is cyclic, it follows that $P = P \cap Z(E)$ and so $x \in C_G(P)$. This contradiction shows that $T = P$ and $D = M$. Let $q \neq p$ be a prime dividing $|A|$ and Q be any Sylow q-subgroup of A. Let $N = N_G(Q)$. Clearly, Q is a Sylow subgroup of D and so by the Frattini argument we have $G = DN$ and so $P = D_pN_p$ for some Sylow subgroup D_p of D and Sylow subgroup N_p of N. But P is cyclic and so $P = N_p$. Hence

$$Q^G = Q^{AP} = Q^A \leq A,$$

which contradicts Claim (1). Hence we have (9).

(10) $|P| \neq p^2$.

Suppose on the contrary that $|P| = p^2$. By Claim (9), P is not cyclic.

If Z is a maximal subgroup of P, then $Z^G \leq AZ$, so $p > 2$ by Claim (3). Therefore T has at least three different subgroups Z_1, Z_2, Z_3 of order p such that $G^{\mathcal{F}} \cap P \nleq Z_i$. Let $N_i = Z_i^G$ be the normal closure of Z_i in G. Then $N_i \leq AZ_i$ and so $N_i \cap N_j$ is contained in $O_{p'}(G) = 1$ for any different $i, j \in \{1,2,3\}$. Hence $P \leq C_i = C_G(N_i)$ for all i. Assume that for some i, $C_i \neq G$. Then C_i is p-soluble by Claim (6), and so G is p-soluble since G/C_i is a p'-group. This contradiction shows that $C_i = G$ for all i. It follows N_1, N_2 and N_3 are abelian groups and so $N_i = Z_i$ for all i and so P is normal in G. It follows that G is p-soluble, which contradicts Claim (4). Thus we have (10).

(11) $O_p(G) = 1$.

Let $D = O_p(G) \neq 1$ and H a minimal normal subgroup of G contained in D. Then $|H| = p$ by Claim (8) and so $|P| = p^2$ by Claim (5), which contradicts Claim (10).

Final contradiction.

Let V be a maximal subgroup of P and $N = V^G$ be the normal closure of V in G. Suppose that $G^{\mathcal{F}} \cap P \nleq V$. Then $N \leq AM$. If $N \cap A \neq A$, then N is p-soluble by Claim (5) and hence $O_p(G) \neq 1$, which contradicts Claim (11). Therefore $N \cap A = A$. Hence $N = AV$ and $|G/N| = p$, so $G^{\mathcal{F}} \leq N$. Therefore

$$G^{\mathcal{F}} \cap P \leq N \cap P = AV \cap P = V.$$

Thus $G^{\mathcal{F}} \cap P \leq \Phi(P)$. But then $G^{\mathcal{F}}$ is p-nilpotent by Tate's theorem [8]. It follows that G is p-soluble, contrary to Claim (4). □

REFERENCES

1. *Hall*, *P*. A characteristic property of soluble groups / P. Hall // J. London Math. Soc. – 1937. – Vol. 12, № 2. – P. 188–200.

2. *Huppert*, *B*. Zur Sylowstruktur auflosbarer Gruppen / B. Huppert // Arch. Math. – 1961. – Vol. 12. – P. 161–169.

3. *Sergienko*, *V.I.* A criterion for the p-solubility of finite groups / V.I. Sergienko // Mat. Zam. – 1971. – Vol. 9. – P. 375–383.

4. *Borovikov*, *M.T.* Groups with permutable subgroups of mutually simple orders / M.T. Borovikov // Questions of Alg. – 1990. – Vol. 5. – P. 80–82.

5. *Guo*, *W*. Criteria of Existence of Hall Subgroups in Non-soluble Finite Groups / W. Guo, A.N. Skiba // Acta Math. Sinica, English Ser. – 2010. – Vol. 26, № 2. – P. 295–304.

6. *Guo*, *W*. Finite groups with some given systems of X_m-semipermutable subgroups / W. Guo, K.P. Shum, A.N. Skiba // Math. Nachr. – 2010. – Vol. 283. – P. 1603–1612.

7. *Yi*, *X*. On some generalizations of permutability and *S*-permutability / X. Yi, A.N. Skiba // Problems of Physics, Mathematics and Technics. – 2013. – № 4 (17). – P. 47–54.

8. *Huppert*, *B*. Endliche Gruppen I / B. Huppert. – Berlin, Heidelberg, New York: Springer, 1967.

9. *Doerk*, *K*. Finite Soluble Groups / K. Doerk, T. Hawkes. – Berlin, New York: Walter de Gruyter, 1992.

10. *Guo*, *W*. The Theory of Classes of Groups / W. Guo. – Beijing, New York, Dordrecht, Boston, London: Science Press, Kluwer Academic Publishers, 2000.

11. *Ballester-Bolinches*, A. Classes of finite groups / A. Ballester-Bolinches, L.M. Ezquerro. – Dordrecht: Springer, 2006.

12. *Kegel*, *O.H*. Produkte nilpotenter Gruppen / O.H. Kegel // Arch. Math. – 1961. – Vol. 12. – P. 90–93.

13. *Knyagina*, *V.N.* On π'-properties of finite group having a Hall π-subgroup / V.N. Knyagina, V.S. Monakhov // Siberian Math. J. – 2011. – P. 234–243.

14. *Gorenstein*, *D*. Finite Groups / D. Gorenstein. – New York, Evanston, London: Harper & Row Publishers, 1968.

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