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METRIC ENERGY-MOMENTUM TENSOR FOR POLARIZABLE PARTICLES IN AN ELECTORMAGNETIC FIELD

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Within the covariant Lagrange formalism and the relativistic theory of continuous media, the metric energy-momentum tensor is obtained for spin polarizable particles interacting with an electromagnetic field. An equation of motion of the polarizable particles with a spin of 1/2 in an external electromagnetic field is derived.

INTRODUCTION

The theory of electromagnetic field interaction with structural particles in hadron electrodynamics starts from the basis principles of relativistic quantum field theory. In model representations, where the diagram technique is mainly used, a number of special features of photon interaction with hadrons have been established [1, 2]. However, the diagram technique is applicable mainly for a description of electromagnetic processes in the simplest quark systems. In the case of electromagnetic field interaction with complex low-energy quark-gluon systems, where the methods of quantum-chemical dynamics are inapplicable, low-energy theorems and rules of sum are being increasingly employed [3–5]. At present the low-energy electromagnetic hadron characteristics connected with their structure, such as the form factor and polarizability, can be obtained in the context of nonrelativistic theory [4, 5]. When going from relativistic electrodynamics of continuous media to relativistic and nonrelativistic quantum field theory, the correspondence principle can be used, but in this case, a correct relationship between the covariant Lagrange and Hamilton formalism must be established [6, 7].

With the help of the covariant Lagrangian of electromagnetic field interaction with the structural polarizable particle, the equation of motion is derived from which the current and charge densities can be found for a moving medium. The metric energy-momentum tensor is calculated, its theoretical field properties are established, and the Hamiltonian in the statistical limit is determined.

1. LAGRANGIAN

Let us consider the electromagnetic field interaction with a moving medium in the four-dimensional covariant formulation. In this case, we can take advantage of the Lagrangian of relativistic electrodynamics of continuous media [8]:

$$L' = -\frac{1}{2}(e^2 - b^2) - \frac{1}{2}e(\hat{\varepsilon} - 1)e + \frac{1}{2}b(\hat{\eta} - 1)b, \tag{1}$$

where $\eta_{\sigma\nu} = (\hat{\mu}^{-1})_{\sigma\nu}$.

If the vector

$$b^{\sigma} = \mu^{\sigma\nu} h_{\nu}$$

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is introduced, then Lagrangian (1) assumes the form

$$L'' = -\frac{1}{2}(e^2 - h^2) - \frac{1}{2}e(\hat{\varepsilon} - 1)e + \frac{1}{2}h(\hat{\mu} - 1)h. \tag{2}$$

In Eqs. (1) and (2), $\varepsilon_{\mu\nu}$ and $\mu_{\mu\nu}$ are the permittivity and magnetic permeability tensors of the medium at rest. The four-dimensional vectors e^{μ} and h^{μ} have the components [8, 9]

$$e^{\mu}\left\{\gamma(\boldsymbol{E}\boldsymbol{v}), \gamma(\boldsymbol{E} - [\boldsymbol{H}\boldsymbol{v}])\right\}, \tag{3}$$

$$h^{\mu}\left\{\gamma(\boldsymbol{H}\boldsymbol{v}), \gamma(\boldsymbol{H} + [\boldsymbol{E}\boldsymbol{v}])\right\}, \tag{4}$$

where $\gamma = \frac{1}{\sqrt{1 - \boldsymbol{v}^2}}$ and $\boldsymbol{v}$ is the velocity of the medium, and are related to the electromagnetic field tensors $e^{\mu} = F^{\mu\nu}U_{\nu}$ and $h^{\mu} = \widetilde{F}^{\mu\nu}U_{\nu}$. The four-dimensional velocity U has the components

$$U^{\mu}\left\{\gamma, \boldsymbol{v}\gamma\right\}.$$

Since we are interested in the electromagnetic field interaction with the polarizable particle, it is more convenient to use Lagrangian (2).

Let us express the electromagnetic field tensor through the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ in the following form [9]:

$$\widehat{F} = \begin{pmatrix} 0 & -\boldsymbol{E} \\ \boldsymbol{E} & \boldsymbol{H}^{\times} \end{pmatrix}, \tag{5}$$

where $(\boldsymbol{H}^{\times})_{ij} = \varepsilon_{ikj}H_k$ and $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field strength vectors.

The electric and magnetic polarizability tensors are introduced by the relationships

$$\hat{\varepsilon} = I + 4\pi\hat{\alpha}, \;\; \hat{\mu} = I + 4\pi\hat{\beta}. \tag{6}$$

In this case, Lagrangian (2) can be represented in form [10, 11]

$$L = L_0 + L_I, \tag{7}$$

where $L_0 = -\frac{1}{2}(e^2 - h^2) = -\frac{1}{4}F^2$,

$$L_I = -2\pi(e\hat{\alpha}e - h\hat{\beta}h). \tag{8}$$

In these expressions we have introduced the following designations: $F^2 = F_{\mu\nu}F^{\mu\nu}$, $e\hat{\alpha}e = e_{\mu}\alpha^{\mu\nu}e_{\nu}$, and $h\hat{\beta}h = h_{\mu}\beta^{\mu\nu}h_{\nu}$.

Let us consider the case in which $\alpha^{\mu\nu}$ and $\beta^{\mu\nu}$ can be expressed in a diagonal form with the help of the metric tensor $g^{\mu\nu}$:

$$\alpha^{\mu\nu} = g^{\mu\nu}\alpha, \;\; \beta^{\mu\nu} = g^{\mu\nu}\beta.$$

Then the Lagrangian of electromagnetic field interaction with the polarizable moving medium (8) assumes the form

$$L_I = -2\pi(\alpha e^2 - \beta h^2). \tag{9}$$

If we take into account the relationship

$$2(e^2 - h^2) = F^2, \tag{10}$$

then we obtain

$$L_I = -2\pi\left[(\alpha - \beta)\, e^2 + \frac{\beta}{2}F^2\right]. \tag{11}$$

Using Lagrangian (7), Eqs. (9) and (10), and the Lagrange–Euler equation

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu A_\nu)} - \frac{\partial L}{\partial A_\nu} = 0, \tag{12}$$

we obtain

$$\partial_\mu F^{\mu\nu} = j^{(M)\nu} = -\partial_\mu G_I^{\mu\nu}, \tag{13}$$

where

$$G_I^{\mu\nu} = 4\pi\left[(\alpha - \beta)(e^\mu U^\nu - e^\nu U^\mu) + \beta F^{\mu\nu}\right]. \tag{14}$$

Definitions of the charge and current densities for coupled charges and the medium at rest follow from Eqs. (13):

$$\rho^{(M)} = -4\pi\alpha(\nabla \boldsymbol{E}) = -4\pi\alpha \operatorname{div} \boldsymbol{E}, \tag{15}$$

$$\boldsymbol{j}^{(M)} = 4\pi(\alpha\partial_t \boldsymbol{E} - \beta \operatorname{rot} \boldsymbol{H}), \tag{16}$$

and the Maxwell equations for the medium have the form

$$\operatorname{rot}\boldsymbol{E} = -\partial_t \boldsymbol{H},\ \operatorname{div} \boldsymbol{H} = 0,$$

$$\operatorname{rot}\boldsymbol{B} = \frac{\partial \boldsymbol{D}}{\partial t},\ \operatorname{div} \boldsymbol{D} = 0, \tag{17}$$

where $\boldsymbol{D}$ and $\boldsymbol{B}$ are respectively the electric and magnetic induction vectors, $\boldsymbol{D} = \hat{\varepsilon}\boldsymbol{E}$, and $\boldsymbol{B} = \hat{\mu}\boldsymbol{H}$.

We can write Eqs. (15) and (16) for the rest medium in the covariant form if we introduce the tensor [8]

$$\widehat{M} = \begin{pmatrix} 0 & -\boldsymbol{P} \\ \boldsymbol{P} & \boldsymbol{M}^{\times} \end{pmatrix}, \tag{18}$$

whose components are the vectors of electric and magnetic polarization $\boldsymbol{M}$ and $\boldsymbol{P}$. In this case, the current and charge densities assume the form

$$\boldsymbol{j}^{(M)} = \frac{\partial \boldsymbol{P}}{\partial t} - \mathrm{rot}\,\boldsymbol{M},\ \ \rho^{(M)} = -\mathrm{div}\,\boldsymbol{P}. \tag{19}$$

Thus, the equation of motion

$$\partial_\mu (F^{\mu\nu} + G_I^{\mu\nu}) = j^\nu \tag{20}$$

is valid for the moving medium, and for the rest medium we have

$$\partial_\mu (F^{\mu\nu} + M^{\mu\nu}) = j^\nu, \tag{21}$$

where j^ν is the free-charge current density.

Let us introduce tensor [8, 12, 13]

$$G^{\mu\nu} = d^\mu U^\nu - d^\nu U^\mu + \varepsilon^{\mu\nu\rho\sigma} U_\rho b_\sigma, \tag{22}$$

where $d^\mu = \varepsilon^{\mu\sigma} e_\sigma$ and $b_\rho = \mu_{\rho\sigma} h^\sigma$.

Then Lagrangian (2) for the moving medium can be expressed in the following form:

$$L = -\frac{1}{4} F^{\mu\nu} G_{\mu\nu} = -\frac{1}{2}(e\hat{\varepsilon}e - h\hat{\mu}h). \tag{23}$$

2. ENERGY-MOMENTUM TENSOR AND EQUATION OF MOTION

The canonical energy-momentum tensor can be determined with the help of Lagrangian (2). As a matter of fact, from the Noether theorems it follows that [7]

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\rho)} (\partial^\nu A^\rho) - g^{\mu\nu} L. \tag{24}$$

From Eqs. (23) and (24) it follows that

$$T^{\mu\nu} = -G^{\mu\rho} (\partial^\nu A_\rho) - g^{\mu\nu} \frac{1}{4} (F_{\rho\sigma} G^{\rho\sigma}). \tag{25}$$

Now let us determine the metric energy-momentum tensor

$$\tilde{T}^{\mu\nu} = -G^{\mu\rho} (\partial^\nu A_\rho) - g^{\mu\nu} L + \partial_\rho (A^\nu G^{\mu\rho}). \tag{26}$$

Upon differentiation of Eq. (26) with the use of equation of motion (20), we obtain

$$\tilde{T}^{\mu\nu} = F_\rho^\nu G^{\mu\rho} + \frac{1}{4} g^{\mu\nu} (F_{\rho\sigma} G^{\rho\sigma}). \tag{27}$$

From Eq. (27) it follows that the energy density for the rest medium has the form

$$\tilde{T}^{00} = \omega = \frac{1}{2}(\varepsilon \boldsymbol{E}^2 + \mu \boldsymbol{H}^2). \tag{28}$$

Let us proceed now to a quantum-mechanical description of the electromagnetic field interaction with the polarizable structural particles. To this end, we take advantage of the correspondence principle [6, 7]. In Lagrangian (23), we consider separately the contribution of electric and magnetic polarizabilities and express it in the form

$$L_I = -2\pi(\alpha F_{\mu\rho} F^{\mu}_{\sigma} - \beta \tilde{F}_{\mu\rho} \tilde{F}^{\mu}_{\sigma}) U^{\rho} U^{\sigma}, \tag{29}$$

where $\tilde{F}_{\mu\rho} = \frac{1}{2}\varepsilon_{\mu\rho\sigma\kappa} F^{\sigma\kappa}$ and $\varepsilon_{0123} = -1$.

According to the correspondence principle, let us use the substitution

$$U^{\rho} U^{\sigma} \rightarrow \tilde{\Theta}^{\rho\sigma} = \frac{1}{2}(\Theta^{\rho\sigma} + \Theta^{\sigma\rho}) \tag{30}$$

in Eq. (29), where $\Theta^{\rho\sigma}$ has the form

$$\Theta^{\rho\sigma} = \frac{i}{2} \bar{\psi} \gamma^{\rho} \overleftrightarrow{\partial}^{\sigma} \psi. \tag{31}$$

In this relationship, $\overleftrightarrow{\partial}_{\mu} = \overrightarrow{\partial}_{\mu} - \overleftarrow{\partial}_{\mu}$ and γ^{ρ} are the Dirac matrices.

In the case of electromagnetic field interaction with the polarizable structural particle having a spin of 1/2, we obtain new Lagrangian

$$L_I = \frac{2\pi}{m}\left[\alpha F_{\mu\rho} F^{\mu}_{\sigma} - \beta \tilde{F}_{\mu\rho} \tilde{F}^{\mu}_{\sigma}\right] \tilde{\Theta}^{\rho\sigma}. \tag{32}$$

The explicit form of Lagrangian (32) is consistent with the normalization of the wave function $\psi(x)$.

Now let us write down the total Lagrangian to describe motion of the charged polarizable spinor particle in the electromagnetic field in the form

$$L = \frac{i}{2} \bar{\psi} \hat{\overleftrightarrow{\partial}} \psi - m \bar{\psi} \psi - e \bar{\psi} \hat{A} \psi - \frac{1}{4} F^2 - \frac{1}{4} F_{\mu\nu} G_I^{(S)\mu\nu}. \tag{33}$$

In the derivation of Eq. (33), we used the relationship

$$\tilde{F}_{\mu\rho} \tilde{F}^{\mu\sigma} = F_{\mu\rho} F^{\mu\sigma} - \frac{1}{2} \delta^{\sigma}_{\rho} F_{\mu\nu} F^{\mu\nu}.$$

Then tensor (14) is determined as

$$G_I^{(S)\mu\nu} = -\frac{4\pi}{m}\left\{(\alpha - \beta)\left[F^{\mu\sigma} \tilde{\Theta}^{\nu}_{\sigma} - F^{\nu\sigma} \tilde{\Theta}^{\mu}_{\sigma}\right] + \beta \tilde{\Theta}^{\rho}_{\rho} F^{\mu\nu}\right\}. \tag{34}$$

Using Lagrangian (33) and anti-symmetric tensor (34), we determine the metric energy-momentum tensor as follows:

$$\tilde{T}^{\mu\nu} = \tilde{\Theta}^{\mu\nu} + F_{\rho}^{\nu} F^{\mu\rho} + \frac{1}{4} g^{\mu\nu} F^2 - \frac{e}{2}\bar{\psi}(\gamma^{\mu} A^{\nu} + \gamma^{\nu} A^{\mu})\psi + \tilde{T}_I^{\mu\nu}, \tag{35}$$

where

$$\tilde{T}_I^{\mu\nu} = F_{\rho}^{\nu} G_I^{(S)\mu\rho} + \frac{1}{4} g^{\mu\nu} (F_{\rho\sigma} G_I^{(S)\rho\sigma}). \tag{36}$$

From Eq. (36) it follows that if the particle momentum is equal to zero, the interaction energy for the rest particle interacting with the electromagnetic field with allowance for the particle polarizability is equal to [5]

$$\tilde{T}_I^{\mu\nu} = -2\pi(\alpha \boldsymbol{E}^2 + \beta \boldsymbol{H}^2). \tag{37}$$

The equation of motion, which follows from Lagrangian (33), has the form

$$\partial_{\mu} F^{\mu\nu} = e\bar{\psi}\gamma^{\nu}\psi + j^{(M)\nu}, \tag{38}$$

where $j^{(M)\nu} = -\partial_{\mu} G_I^{(S)\mu\nu}$.

CONCLUSIONS

The equation of spinor particle motion has been obtained within the Lagrange covariant formalism based on the covariant Lagrangian of electromagnetic field interaction with polarizable particles. The relationships between the covariant Lagrangian and the metric energy-momentum tensor were established, which allowed us to derive the correct low-energy representation of the Hamilton function of electromagnetic field interaction with polarizable particles on the basis of the correspondence principle.

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