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ORDINARY DIFFERENTIAL EQUATIONS

Time Symmetry Preserving Perturbations of Differential Systems

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We derive an equation for perturbations of differential systems preserving the shift operator along solutions of these systems on a symmetric time interval $[-\omega, \omega]$. In particular, such perturbations preserve mappings for the period $[-\omega, \omega]$ of a periodic differential system. This simplifies the qualitative analysis of solution sets of differential systems.

Along with the original differential system

$$\frac{dx}{dt} = X(t, x), \qquad t \in R, \qquad x \in D \subset R^n, \tag{1}$$

we consider the set of perturbed systems

$$\frac{dx}{dt} = X(t, x) + \alpha(t)\Delta(t, x), \qquad t \in R, \qquad x \in D \subset R^n, \tag{2}$$

where $\alpha(t)$ is a continuous scalar odd function and $\Delta(t, x)$ is an arbitrary continuously differentiable vector function. Let us study the equivalence of the differential systems (1) and (2) in the sense of the coincidence of reflection functions [1, p. 11]. If the reflection functions of two systems coincide, then their shift operators [2, pp. 11–12] also coincide on a symmetric interval of the form $[-\tau, \tau]$ [1, p. 12]; and, therefore, the mappings for the period $[-\omega, \omega]$ coincide for periodic systems.

By [1, p. 11], a reflection function of system (1) satisfies the relation

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} X(t, x) + X(-t, F) \equiv 0. \tag{3}$$

If

$$V(x) = (V_1(x), V_2(x), \ldots, V_m(x))^{\mathrm{T}}$$

is a vector function and $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is a column vector, then, as usual, we set

$$\frac{\partial V}{\partial x} = \left(\frac{\partial V_i}{\partial x_j}\right), \qquad i = 1, \ldots, m, \qquad j = 1, \ldots, n.$$

Lemma 1. *The identity*

$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x} X\right) Y - \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x} Y\right) X \equiv \frac{\partial S}{\partial x}\left(\frac{\partial X}{\partial x} Y - \frac{\partial Y}{\partial x} X\right) \tag{4}$$

is valid for three arbitrary functions

$$S(t, x) = (S_1(t, x), S_2(t, x), \ldots, S_n(t, x))^{\mathrm{T}}, \qquad X(t, x) = (X_1(t, x), X_2(t, x), \ldots, X_n(t, x))^{\mathrm{T}},$$

$$Y(t, x) = (Y_1(t, x), Y_2(t, x), \ldots, Y_n(t, x))^{\mathrm{T}},$$

where S is twice continuously differentiable and X and Y are differentiable.

Proof. We transform the left-hand side of (4):

$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}X\right)Y-\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}Y\right)X\equiv\frac{\partial}{\partial x}\left(\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}X_i\right)Y-\frac{\partial}{\partial x}\left(\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}Y_i\right)X$$
$$\equiv\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}X_i\right)Y_j-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}Y_i\right)X_j$$
$$\equiv\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial^2 S}{\partial x_i\partial x_j}(X_iY_j-Y_iX_j)+\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}\frac{\partial X_i}{\partial x_j}Y_j(t,x)-\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial S}{\partial x_i}\frac{\partial Y_i}{\partial x_j}X_j(t,x)$$
$$\equiv\frac{\partial S}{\partial x}\frac{\partial X}{\partial x}Y-\frac{\partial S}{\partial x}\frac{\partial Y}{\partial x}X\equiv\frac{\partial S}{\partial x}\left(\frac{\partial X}{\partial x}Y-\frac{\partial Y}{\partial x}X\right).$$

The proof of the lemma is complete.

Lemma 2. *Let $F(t,x)$ be a reflection function of system* (1) *with a continuously differentiable right-hand side. Then, for each continuously differentiable vector function $\Delta(t,x)$, the function*

$$U(t,x):=\frac{\partial F}{\partial x}(t,x)\Delta(t,x)-\Delta(-t,F(t,x)) \tag{5}$$

satisfies the identity

$$\begin{aligned}\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}X+\frac{\partial X}{\partial x}(-t,F)U&\\ \equiv\frac{\partial F}{\partial t}\left(\frac{\partial \Delta}{\partial t}+\frac{\partial \Delta}{\partial x}X-\frac{\partial X}{\partial x}\Delta\right)&+\frac{\partial \Delta}{\partial t}(-t,F)\\ +\frac{\partial \Delta}{\partial x}(-t,F)X(-t,F)&-\frac{\partial X}{\partial x}(-t,F)\Delta(-t,F).\end{aligned} \tag{6}$$

Proof. By taking account of relation (3) and by performing simple computations, we obtain the identities

$$\begin{aligned}\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}X(t,x)&\equiv\frac{\partial^2 F}{\partial t\,\partial x}\Delta+\frac{\partial F}{\partial x}\frac{\partial \Delta}{\partial t}+\frac{\partial \Delta}{\partial t}(-t,F)-\frac{\partial \Delta}{\partial x}(-t,F)\frac{\partial F}{\partial t}\\ &+\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\Delta(t,x)\right)X-\frac{\partial \Delta}{\partial x}(-t,F(t,x))\frac{\partial F}{\partial x}(t,x)X(t,x)\\ &\equiv\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\Delta\right)X-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}X\right)\Delta-\frac{\partial X}{\partial x}(-t,F)\frac{\partial F}{\partial x}\Delta+\frac{\partial F}{\partial x}\frac{\partial \Delta}{\partial t}\\ &+\frac{\partial \Delta}{\partial t}(-t,F)-\frac{\partial \Delta}{\partial x}(-t,F)\frac{\partial F}{\partial t}-\frac{\partial}{\partial x}(\Delta(-t,F))X.\end{aligned}$$

We apply identity (4) to the first term in the last part of this identity. Then, after simple formal transformations, we obtain

$$\begin{aligned}\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}X\equiv\frac{\partial F}{\partial x}\left(\frac{\partial \Delta}{\partial t}+\frac{\partial \Delta}{\partial x}X-\frac{\partial X}{\partial x}\Delta\right)+\frac{\partial \Delta}{\partial t}(-t,F)+\frac{\partial \Delta}{\partial x}(-t,F)X(-t,F)\\ -\frac{\partial X}{\partial x}(-t,F)\Delta(-t,F)-\frac{\partial X}{\partial x}(-t,F)U.\end{aligned}$$

By adding the expression $\frac{\partial X}{\partial x}(-t,F)U$ to the left- and right-hand sides of this relation, we obtain identity (6) and complete the proof of the lemma.

Theorem 1. *Let a vector function* $\Delta(t,x)$ *be a solution of the partial differential equation*

$$\frac{\partial \Delta}{\partial t}(t,x) + \frac{\partial \Delta}{\partial x}(t,x)X(t,x) - \frac{\partial X}{\partial x}(t,x)\Delta(t,x) = 0. \tag{7}$$

Then the perturbed differential system (2), *where* $\alpha(t)$ *is an arbitrary continuous scalar odd function, is equivalent to the differential system* (1).

Proof. Let $F(t,x)$ be a reflection function of system (1). Consequently, this function satisfies the differential equation (3). We claim that it satisfies the identity

$$\frac{\partial F}{\partial x}(t,x)\Delta(t,x) \equiv \Delta(-t, F(t,x)) \tag{8}$$

as well. To prove this, we introduce the function $U(t,x)$ by formula (5). By Lemma 2, this function satisfies identity (6). By virtue of (7), under the assumptions of the theorem, this identity can be represented in the form

$$\frac{\partial U}{\partial t}(t,x) + \frac{\partial U}{\partial t}(t,x)X + \frac{\partial X}{\partial x}(-t,F)U \equiv 0.$$

Moreover, since the identity $F(0,x) = x$ [1, p. 11] is valid for any reflection function F, we have

$$U(0,x) \equiv \frac{\partial F}{\partial x}(0,x)\Delta(0,x) - \Delta(0,F(0,x)) \equiv 0.$$

Therefore, U is a solution of the Cauchy problem

$$\frac{\partial U}{\partial t}(t,x) + \frac{\partial U}{\partial t}(t,x)X(t,x) + \frac{\partial X}{\partial x}(-t,F)U = 0, \qquad U(0,x) = 0.$$

There exists a unique solution of this problem [3, p. 66]. Consequently, we have the identity $U(t,x) \overset{t,x}{\equiv} 0$, which implies identity (8).

Now we show that a reflection function $F(t,x)$ of system (1) is also a reflection function of system (2). To this end, we should verify the basic relation (3), which, in our case, should be rewritten in the form

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}(X + \alpha(t)\Delta) + X(-t,F) + \alpha(-t)\Delta(-t,F) \equiv 0. \tag{9}$$

Indeed, by successively transforming the left-hand side of the last relation and by taking account of the fact that $\alpha(t)$ is odd, we obtain the chain of identities

$$\begin{aligned}\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}&(X + \alpha(t)\Delta) + X(-t,F) + \alpha(-t)\Delta(-t,F) \\ &\equiv \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}X + X(-t,F) + \alpha(t)\frac{\partial F}{\partial x}\Delta - \alpha(t)\Delta(-t,F) \\ &\equiv \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}X + X(-t,F)\right] + \alpha(t)\left[\frac{\partial F}{\partial x}\Delta - \Delta(-t,F)\right].\end{aligned}$$

Both the terms in brackets identically vanish: the first is zero since identity (3) is valid for a reflection function of system (1), and the other vanishes since identity (8) is valid under the assumptions of the theorem. Consequently, identity (9) holds, and $F(t,x)$ is a reflection function of system (2). The proof of the theorem is complete.

Corollary. *Let functions* $\Delta_k(t,x)$ *be solutions of the partial differential equation* (6). *Then all differential systems of the form*

$$\frac{dx}{dt} = X(t,x) + \sum_{k=1}^{\infty} \alpha_k(t)\Delta_k(t,x), \tag{10}$$

where the $\alpha_k(t)$ are odd scalar continuous functions and the series occurring on the right-hand side converges to a continuously differentiable function, are equivalent to each other in the sense of coincidence of reflection functions, and all of them are equivalent to system (1).

Proof. The assertion is obvious.

Remark. It was shown in [1, p. 24] that if there exists a stationary system equivalent to system (1), then the right-hand side of the former can be found by the formula $Y(x) = X(0,x)$. This, together with the corollary, implies the importance of conditions under which the vector function $\Delta(t,x) := X(t,x) - X(0,x)$ can be represented in the form

$$\Delta(t,x) := \Delta^{(0)}(t,x) = \sum_{k=1}^{n} \alpha_k(t)\Delta_k(t,x), \tag{11}$$

where the Δ_k are solutions of Eq. (7). Forthcoming considerations are aimed at solving this problem. By solving it, one can reduce the analysis of properties of solutions of nonautonomous systems to the investigation of properties of solutions of autonomous systems of the form $dx/dt = X(0,x)$.

Lemma 3. *Let $\Delta(t,x)$ be the function given by* (11), *where the $\alpha_k(t)$ are s times differentiable functions and the Δ_k are differentiable functions satisfying Eq.* (7). *Let $\Delta^{(i)}(t,x)$ be the functions given by the formulas*

$$\Delta^{(i+1)}(t,x) := \frac{\partial \Delta^{(i)}(t,x)}{\partial t} + \frac{\partial \Delta^{(i)}(t,x)}{\partial x} X(t,x) - \frac{\partial X(t,x)}{\partial x}\Delta^{(i)}(t,x) \tag{12}$$

for each $i = 1, \dots, s-1$. Then

$$\Delta^{(i)}(t,x) = \sum_{k=1}^{m} \frac{d^i \alpha_k(t)}{dt^i}\Delta_k(t,x), \qquad i = 0, \dots, s. \tag{13}$$

Proof. We successively find the values of the expressions (12) for each $i = 1, \dots, s$:

$$\begin{aligned}
\Delta^{(0)} &= \sum_{k=1}^{m} \alpha_k \Delta_k, \\
\Delta^{(1)} &= \sum_{k=1}^{m} \left(\alpha_k \left[\frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x}\Delta_k \right] + \frac{d\alpha_k}{dt}\Delta_k \right) = \sum_{k=1}^{m} \frac{d\alpha_k}{dt}\Delta_k, \\
\Delta^{(2)} &= \sum_{k=1}^{m} \left(\frac{d\alpha_k}{dt} \left[\frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x}\Delta_k \right] + \frac{d^2\alpha_k}{dt^2}\Delta_k \right) = \sum_{k=1}^{m} \frac{d^2\alpha_k}{dt^2}\Delta_k, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Delta^{(s)} &= \sum_{k=1}^{m} \left(\frac{d^{s-1}\alpha_k}{dt^{s-1}} \left[\frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x}\Delta_k \right] + \frac{d^s\alpha_k}{dt^s}\Delta_k \right) = \sum_{k=1}^{m} \frac{d^s\alpha_k}{dt^s}\Delta_k,
\end{aligned}$$

which implies relation (13) and completes the proof of the lemma.

Theorem 2. *Let the following conditions be satisfied*:
(1) *the functions $\alpha_k(t)$ are odd, scalar, linearly independent on R, and m times differentiable*;
(2) *each of the m times differentiable vector functions $\Delta_k(t,x)$ is a solution of Eq.* (7).
Then there exist scalar continuous functions $a_s(t)$, $s = 0, \dots, m$, such that

$$a_0(t)\Delta + a_1(t)\Delta^{(1)} + \cdots + a_m(t)\Delta^{(m)} \overset{t,x}{\equiv} 0, \tag{14}$$

where the $\Delta^{(k)}(t,x)$ *are the functions given by* (12). *Moreover, the functions* $a_k(t)$ *can be found by the formulas*

$$a_0 = (-1)^{(m)} \begin{vmatrix} d\alpha_1/dt & d^2\alpha_1/dt^2 & \dots & d^m\alpha_1/dt^m \\ d\alpha_2/dt & d^2\alpha_2/dt^2 & \dots & d^m\alpha_2/dt^m \\ \dots & \dots & \dots & \dots \\ d\alpha_m/dt & d^2\alpha_m/dt^2 & \dots & d^m\alpha_m/dt^m \end{vmatrix},$$

$$a_1 = (-1)^{(m-1)} \begin{vmatrix} \alpha_1 & d^2\alpha_1/dt^2 & \dots & d^m\alpha_1/dt^m \\ \alpha_2 & d^2\alpha_2/dt^2 & \dots & d^m\alpha_2/dt^m \\ \dots & \dots & \dots & \dots \\ \alpha_m & {}^2\alpha_m/dt^2 & \dots & d^m\alpha_m/dt^m \end{vmatrix}, \qquad (15)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_m = \begin{vmatrix} \alpha_1 & d\alpha_1/dt & \dots & d^{m-1}\alpha_1/dt^{m-1} \\ \alpha_2 & d\alpha_2/dt & \dots & d^{m-1}\alpha_2/dt^{m-1} \\ \dots & \dots & \dots & \dots \\ \alpha_m & d\alpha_m/dt & \dots & d^{m-1}\alpha_m/dt^{m-1} \end{vmatrix}.$$

Proof. Formulas (15) define solutions of the system of algebraic equations

$$\sum_{s=0}^{m} a_s \frac{d^s\alpha_k}{dt^s} = 0, \qquad k = 1, \dots, m, \qquad (16)$$

for the variables $a_1, a_2, \dots, a_m$ with given (t,x).

We successively transform the left-hand side of (14). By Lemma 3, we have

$$\sum_{s=0}^{m} a_s \Delta^{(s)} \equiv \sum_{s=0}^{m} a_s \sum_{k=1}^{m} \frac{d^s\alpha_k}{dt^s}\Delta_k \equiv \sum_{k=1}^{m}\sum_{s=0}^{m}\left(a_s\frac{d^s\alpha_k}{dt^s}\right)\Delta_k.$$

By using (16), we find that the resulting expression identically vanishes; therefore, identity (14) is valid. The proof of the theorem is complete.

Theorem 3. *Suppose that for some* m *times differentiable vector function* $\Delta(t,x)$, *there exist scalar functions* $a_s(t)$, $s = 0, \dots, m$, *for which identity* (14) *is valid, where the* Δ_k *are given by* (12). *If the linear differential equation*

$$a_0(t)\alpha(t) + a_1(t)\frac{d\alpha(t)}{dt} + \dots + a_m(t)\frac{d^m\alpha(t)}{dt^m} = 0 \qquad (17)$$

has m *odd solutions* $\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t)$ *for which the Wronskian vanishes only at isolated points, then the system*

$$\frac{dx}{dt} = X(t,x) + \Delta(t,x), \qquad t \in R, \qquad x \in D \subset R^n,$$

is equivalent to system (1) [*in which, unlike* (2), *the function* Δ *is not multiplied by* $\alpha(t)$].

Proof. We write out a system of linear algebraic equations for the unknowns $\Delta_1, \Delta_2, \dots, \Delta_m$ with given (t,x):

$$\frac{d^i\alpha_1}{dt^i}\Delta_1 + \frac{d^i\alpha_2}{dt^i}\Delta_2 + \dots + \frac{d^i\alpha_m}{dt^i}\Delta_m = \Delta^{(i)}, \qquad i = 0, \dots, m-1. \qquad (18)$$

From this system, we find $\Delta_1, \Delta_2, \dots, \Delta_m$, which, by the assumptions of the theorem, can always be performed everywhere possibly except for isolated points $t = t_k$. Therefore, from the first equation in system (18), one can express the function $\Delta^{(0)}$ in the form

$$\Delta^{(0)} = \alpha_1\Delta_1 + \alpha_2\Delta_2 + \cdots + \alpha_m\Delta_m.$$

Let us show that

$$\frac{d^m\alpha_1}{dt^m}\Delta_1 + \frac{d^m\alpha_2}{dt^m}\Delta_2 + \cdots + \frac{d^m\alpha_m}{dt^m}\Delta_m = \Delta^{(m)}.$$

To this end, from identity (14), we express the function

$$\Delta^{(m)} \equiv -\frac{1}{a_m}\left(a_0\Delta^{(0)} + a_1\Delta^{(1)} + \cdots + a_{m-1}\Delta^{(m-1)}\right).$$

We find the functions $\Delta^{(i)}$, $i = 0, \dots, m-1$, from Eqs. (18) and substitute them into the left-hand side of the resulting identity:

$$\begin{aligned}\Delta^{(m)} = -\frac{1}{a_m}\Bigg[&a_1\left(\alpha_1\Delta_1 + \alpha_2\Delta_2 + \cdots + \alpha_m\Delta_m\right)\\ &+ a_0\left(\frac{d\alpha_1}{dt}\Delta_1 + \frac{d\alpha_2}{dt}\Delta_2 + \cdots + \frac{d\alpha_m}{dt}\Delta_m\right) + \cdots\\ &+ a_m\left(\frac{d^{m-1}\alpha_1}{dt^{m-1}}\Delta_1 + \frac{d^{m-1}\alpha_2}{dt^{m-1}}\Delta_2 + \cdots + \frac{d^{m-1}\alpha_m}{dt^{m-1}}\Delta_m\right)\Bigg],\\ \Delta^{(m)} = -\frac{1}{a_m}&\left(a_0\alpha_1 + a_1\frac{d\alpha_1}{dt} + \cdots + a_{m-1}\frac{d^{m-1}\alpha_1}{dt^{m-1}}\right)\Delta_1\\ -\frac{1}{a_m}&\left(a_0\alpha_2 + a_1\frac{d\alpha_2}{dt} + \cdots + a_{m-1}\frac{d^{m-1}\alpha_2}{dt^{m-1}}\right)\Delta_2 - \cdots\\ -\frac{1}{a_m}&\left(a_0\alpha_m + a_1\frac{d\alpha_m}{dt} + \cdots + a_{m-1}\frac{d^{m-1}\alpha_m}{dt^{m-1}}\right)\Delta_m.\end{aligned} \tag{19}$$

The functions $\alpha_1, \alpha_2, \dots, \alpha_m$ are solutions of Eq. (17). Consequently,

$$a_0\alpha_j + a_1\frac{d\alpha_j}{dt} + \cdots + a_{m-1}\frac{d^{m-1}\alpha_j}{dt^{m-1}} = -a_m\frac{d^m\alpha_j}{dt^m}, \qquad j = 1, \dots, m.$$

Therefore, relation (19) can be rewritten as follows:

$$\begin{aligned}\Delta^{(m)} &= -\frac{1}{a_m}\left(-a_m\frac{d^m\alpha_1}{dt^m}\right)\Delta_1 - \frac{1}{a_m}\left(-a_m\frac{d^m\alpha_2}{dt^m}\right)\Delta_2 - \cdots - \frac{1}{a_m}\left(-a_m\frac{d^m\alpha_m}{dt^m}\right)\Delta_m\\ &= \frac{d^m\alpha_1}{dt^m}\Delta_1 + \frac{d^m\alpha_2}{dt^m}\Delta_2 + \cdots + \frac{d^m\alpha_m}{dt^m}\Delta_m.\end{aligned}$$

We have thereby obtained the relations

$$\frac{d^s\alpha_1}{dt^s}\Delta_1 + \frac{d^s\alpha_2}{dt^s}\Delta_2 + \cdots + \frac{d^s\alpha_m}{dt^s}\Delta_m = \Delta^{(m)}, \qquad s = 0, \dots, m.$$

Let us now express $\Delta^{(1)}$ via α_k with regard to these relations:

$$\Delta^{(1)} = \frac{\partial\Delta^{(0)}}{\partial t} + \frac{\partial\Delta^{(0)}}{\partial x}X - \frac{\partial X}{\partial x}\Delta^{(0)} = \Delta^{(1)} + \sum_{k=1}^{m}\alpha_k\left(\frac{\partial\Delta_k}{\partial t} + \frac{\partial\Delta_k}{\partial x}X - \frac{\partial X}{\partial x}\Delta_k\right).$$

Hence we obtain

$$\sum_{k=1}^{m} \alpha_k \left(\frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x} \Delta_k \right) = 0.$$

In a similar way, by computing $\Delta^{(2)}$, we obtain the relation

$$\sum_{k=1}^{m} \frac{d\alpha_k}{dt} \left(\frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x} \Delta_k \right) = 0.$$

By continuing this procedure, we obtain the equations

$$\sum_{k=1}^{m} \frac{d^p\alpha_k}{dt^p} \left(\frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x} \Delta_k \right) = 0, \qquad p = 0, \dots, m-1,$$

which can be rewritten in the form

$$\frac{d^p\alpha_1}{dt^p} L\Delta_1 + \frac{d^p\alpha_2}{dt^p} L\Delta_2 + \dots + \frac{d^p\alpha_m}{dt^p} L\Delta_m = 0, \qquad p = 0, \dots, m-1,$$

where $L\Delta_k = \partial \Delta_k/\partial t + (\partial \Delta_k/\partial x)\, X - (\partial X/\partial x)\Delta_k$.

These relations imply that the functions $L\Delta_1, L\Delta_2, \dots, L\Delta_m$ with given (t, x) are solutions of the system of linear algebraic equations

$$\frac{d^p\alpha_1}{dt^p} y_1 + \frac{d^p\alpha_2}{dt^p} y_2 + \dots + \frac{d^p\alpha_m}{dt^p} y_m = 0, \qquad p = 0, \dots, m-1.$$

The zero solution is the unique solution of this system for $t \neq t_k$. Therefore, if $t \neq t_k$, then we have

$$L\Delta_k = \frac{\partial \Delta_k}{\partial t} + \frac{\partial \Delta_k}{\partial x} X - \frac{\partial X}{\partial x} \Delta_k = 0 \qquad \forall k = 1, \dots, m.$$

We have represented the perturbation $\Delta(t, x)$ in the form $\Delta = \sum \alpha_k \Delta_k(t, x)$, where the Δ_k are solutions of Eq. (7), i.e., we have reduced the perturbed system to the form (10). An application of Theorem 1 (more precisely, of its corollary) completes the proof of the theorem.

Lemma 4. *Let a* 2ω*-periodic differential system* (1) *with a solution* $x(t)$ *and a reflection function* $F(t, x)$ *be equivalent* (*in the sense of the coincidence of reflection functions*) *to some differential system with a solution* $y(t)$ *and a reflection function* $\Phi(t, x)$; *moreover, suppose that*

$$x(-\omega) = y(-\omega) \tag{20}$$

and $x(t)$ *and* $y(t)$ *can be continued to* $[-\omega, \infty)$. *Then*

$$x(2k\omega - \omega) = y(2k\omega - \omega) \tag{21}$$

for each positive integer k.

Proof. We prove the desired assertion by induction. First, we show that the assertion of the lemma is valid for $k = 1$. Indeed, by the basic property of a reflection function, we have

$$x(\omega) = F(-\omega, x(-\omega)), \qquad y(\omega) = \Phi(-\omega, y(-\omega)). \tag{22}$$

Since the differential systems mentioned in the statement of the lemma are equivalent, we also have $F(-\omega, x(-\omega)) = \Phi(-\omega, x(-\omega))$. It follows from (20) that the right-hand side of this relation can be represented in the form $\Phi(-\omega, x(-\omega)) = \Phi(-\omega, y(-\omega))$; therefore, the right-hand side of equations (22) coincide. Therefore, the assertion of the lemma is valid for $k = 1$.

Now we suppose that the assertion of the lemma holds for some k. This implies that relation (21) is valid for that k. Let us show that the assertion of the lemma is valid for $k+1$ as well, i.e.,

$$x(2k\omega + \omega) = y(2k\omega + \omega). \tag{23}$$

We introduce the function $z(t) = x(2k\omega + t)$, which is a solution of the differential system (1), since this system is 2ω-periodic. Likewise, the function $u(t) = y(2k\omega + t)$ is a solution of the system with the solution $y(t)$. Both functions can be continued to $[-\omega, \omega]$, which follows from the fact that the solutions $x(t)$ and $y(t)$ can be continued to $[-\omega, \infty)$. By our assumption, we have the chain of relations $z(-\omega) = x(2k\omega - \omega) = y(2k\omega - \omega) = u(-\omega)$.

Consequently, $z(\omega) = u(\omega)$, which, together with the definition of the functions z and u, implies (23) and completes the proof of the lemma.

Theorem 4. *Let a* 2ω*-periodic differential system* (1) *with a solution* $x(t)$ *be equivalent* (*in the sense of the coincidence of reflection functions*) *to the stationary system*

$$dy/dt = X(0, y) \tag{24}$$

with a solution $y(t)$, *and let the following conditions be satisfied*:

(A) *relation* (20) *is valid*;

(B) *the solution* $y(t)$ *is bounded on* $[-\omega, \infty)$;

(C) *there exists a number* a *such that* $\|y(2k\omega - 3\omega)\| \le a$ *for each positive integer* k;

(D) *all solutions* $z(t)$ *of system* (1) *satisfying the inequality* $\|z(-\omega)\| \le a$ *can be continued to* $[-\omega, \omega]$.

Then the solution $x(t)$ *can be continued to* $[-\omega, \infty)$ *and is bounded there.*

Proof. First, let us show that the solution $x(t)$ can be continued to $[-\omega, \infty)$. This solution can be continued to $[-\omega, \omega]$, which follows from condition (D), relation (20), and condition (C) (for $k = 1$): $\|x(-\omega)\| = \|y(-\omega)\| \le a$. Let us show that the solution $x(t)$ can be continued to $[\omega, 3\omega]$ as well. Note that the function $z(t) = x(t + 2\omega)$ is a solution of system (1) and satisfies the relations $\|z(-\omega)\| = \|x(\omega)\| = \|y(\omega)\| \le a$, whose validity follows from the basic property of a reflection function. Then, by the assumption of the theorem, $z(t)$ can be continued to $[-\omega, \omega]$, i.e., $x(t)$ can actually be continued to $[\omega, 3\omega]$. By induction over k, one can show that $x(t)$ can be continued to $[-\omega, 2k\omega + \omega]$. Since k is arbitrary, it follows that $x(t)$ can be continued to $[-\omega, \infty)$.

Now we show that $x(t)$ is bounded on $[-\omega, \infty)$. Since all solutions $z(t)$ of system (1) such that $\|z(-\omega)\| \le a$ can be continued to $[-\omega, \omega]$, it follows that there exists a number M such that $\|z(t)\| \le M$ for each $t \in [-\omega, \omega]$. Lemma 4 implies that $x(2k\omega - 3\omega) = y(2k\omega - 3\omega)$ for each positive integer k. Therefore, $\|z(-\omega)\| = \|x(2k\omega - \omega)\| = \|y(2k\omega - \omega)\| \le a$ for $z(t) := x(t + 2\omega k)$; consequently, $\|x(t + 2\omega k)\| = z(t) \le M$ for $t \in [-\omega, \omega]$. Therefore, for each positive integer k, we have an inequality that implies that the solution $x(t)$ is bounded on $[-\omega, \infty)$. The proof of the theorem is complete.

Theorem 5. *Let conditions* (A), (C), *and* (D) *in Theorem* 4 *be satisfied, and let a solution* $y(t)$ *of system* (24) *be* 2ω*-periodic and asymptotically stable* (*respectively, asymptotically unstable*). *Then the solution* $x(t)$ *of system* (1) *is also* 2ω*-periodic and asymptotically stable* (*respectively, asymptotically unstable*).

Proof. Let the solution $y(t)$ be 2ω-periodic. Then

$$\begin{aligned} x(\omega) &= F(-\omega, x(-\omega)) = \Phi(-\omega, x(-\omega)) = \Phi(-\omega, y(-\omega)) \\ &= y(\omega) = y(-\omega) = x(-\omega), \end{aligned}$$

i.e., $x(\omega) = x(-\omega)$. This implies that $x(-\omega)$ is a fixed point of the mapping for the period $[-\omega, \omega]$, whence we obtain the 2ω-periodicity of the solution $x(t)$.

The subsequent proof follows from the coincidence of the mappings $F(-\omega, x)$ and $\Phi(-\omega, x)$ for the period $[-\omega, \omega]$ for the two systems. The proof of the theorem is complete.

Example. Consider the system

$$dx/dt = a(t)x + b(t)y - a(t)x\left(x^2 + y^2\right),$$
$$dy/dt = -b(t)x + a(t)y - a(t)y\left(x^2 + y^2\right),$$

where $a(t)$ and $b(t)$ are continuous 2π-periodic functions such that $\alpha_1(t) := a(t) - 1$ and $\alpha_2(t) := b(t) - 1$ are odd functions.

This system is equivalent to the stationary system

$$dx/dt = x + y - x\left(x^2 + y^2\right), \qquad dy/dt = -x + y - y\left(x^2 + y^2\right).$$

Here

$$\Delta = \alpha_1\Delta_1 + \alpha_2\Delta_2, \qquad \Delta_1 = \left(x - x(x^2 + y^2), y - y(x^2 + y^2)\right)^{\mathrm{T}},$$
$$\Delta_2 = (y, -x)^{\mathrm{T}}, \qquad X = \left(x + y - x(x^2 + y^2), -x + y - y(x^2 + y^2)\right)^{\mathrm{T}}.$$

Since the stationary system has an asymptotically stable limit cycle $x^2 + y^2 = 1$ corresponding to 2π-periodic solutions, it follows that both solutions $x(t)$ and $y(t)$ of the system issuing from the circle $x^2(-\pi) + y^2(-\pi) = 1$ for $t = -\pi$ are 2π-periodic and each of the remaining solutions except for the zero solution tends to one of the above-mentioned periodic solutions as $t \to \infty$.

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