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КРИТЕРИЙ НИЛЬПОТЕНТНОСТИ КОММУТАНТА КОНЕЧНОЙ ГРУППЫ

В.С. Монахов

Гомельский государственный университет им. Ф. Скорины

THE NILPOTENCY CRITERION FOR THE DERIVED SUBGROUP OF A FINITE GROUP

V.S. Monakhov

F. Scorina Gomel State University

Доказывается, что коммутант конечной группы нильпотентен в точности тогда, когда $|ab| \ge |a||b|$ для всех примарных коммутаторов a и b взаимно простых порядков.

***Ключевые слова**: конечная группа, коммутатор, коммутант, нильпотентная подгруппа.*

It is proved that the derived subgroup of a finite group is nilpotent if and only if $|ab| \ge |a||b|$ for all primary commutators a and b of coprime orders.

***Keywords**: finite group, commutator, derived subgroup, nilpotent subgroup.*

1 A sufficient condition for nilpotency of the derived subgroup of a finite groups

Bastos and Shumyatsky [1] received the following sufficient condition for nilpotency of the derived subgroup.

***Theorem* 1.1.** *Let G be a finite group in which $|ab| = |a||b|$ whenever the elements a, b are commutators of coprime orders. Then G' is nilpotent.*

Here the symbol $|x|$ denotes the order of the element x in a group G, G' is the derived subgroup of G.

2 The nilpotency attribute of the derived subgroup

In Theorem 1.1 we can limit to only primary commutators, and we also obtain the nilpotency criterion for the derived subgroup. An element is said to be primary if it is of prime power order.

By $N_G(H)$ and $C_G(H)$ we denote the normalizer and centralizer of a subgroup H in a group G, respectively. If a group G has a normal Sylow p-subgroup, then G is p-closed; a group G is p-nilpotent if there is a normal complement for a Sylow p-subgroup in G. By $H \rtimes K$ we denote the semidirect product of a normal subgroup H and a subgroup K.

***Lemma* 2.1.** *Suppose that in a finite group G the product of any two primary commutators of coprime orders s and t is of order $\ge st$. Let H be a primary subgroup, x be a primary commutator, $(|x|, |H|) = 1$. If $x \in N_G(H)$, then $x \in C_G(H)$.*

Proof. Let y be an element of H. Since $[x, y] \in H$, the order of $[x, y]$ is relatively prime to the order of x. Hence $|x[x, y]| \ge |x||[x, y]|$. As $x[x, y] = y^{-1}xy$, we have $|x[x, y]| = |x|$. Now $[x, y] = 1$ and $x \in C_G(H)$. □

By $X_p(G)$ we denote the set of all commutators $x = [g, h]$ of G such that $|x| = p^n$ for a prime p and a non-negative integer n.

The aim of the present paper is to prove the following theorem.

***Theorem* 2.2.** *The derived subgroup of a finite group is nilpotent if and only if $|ab| \ge |a||b|$ when a and b are the primary commutators of coprime orders.*

Proof. Evidently, if the derived subgroup of a finite group is nilpotent, then $|ab| = |a||b|$ for all primary commutators a and b of coprime orders.

Conversely, suppose that G is a finite group such that $|ab| \ge |a||b|$ if elements a and b are the primary commutators of coprime orders. Firstly, we prove that G is soluble. Assume the converse and let G be a counterexample of least order. Then all proper subgroups of G are soluble, $G = G'$ and $G/\Phi(G)$ is a simple group. Here $\Phi(G)$ is the Frattini subgroup of G. Let P be a Sylow p-subgroup of $\Phi(G)$, Q be a Sylow q-subgroup of G, $p \ne q$. By the Focal Subgroup Theorem [2, 7.3.4], Q is generated by elements $x^{-1}y$, where $x, y \in Q$ and x conjugates to y in G. Therefore

$$x = y^g,\ g \in G, \qquad x^{-1}y = g^{-1}y^{-1}gy = [g, y] \in X_q(G), \tag{2.1}$$

i. e. Q is generated by elements of $Q \cap X_q(G)$. If $x \in Q \cap X_q(G)$, then $x \in C_G(P)$ in view of Lemma

2.1, and $Q \le C_G(P)$. This is true for any $q \ne p$, therefore $|G : C_G(P)| = p^a$. Since $C_G(P)$ is normal in G, we have $G = C_G(P)$. This is true for any Sylow subgroup of $\Phi(G)$, so $\Phi(G) = Z(G)$.

Since G is not 2-nilpotent, it follows that in G there is a 2-closed Schmidt subgroup (a nonnilpotent group all of whose proper subgroups are nilpotent) $S = P \leftthreetimes Q$ of even order [3, IV.5.4], where P is a normal in S Sylow 2-subgroup, $Q = \langle y \rangle$ is a nonnormal in S cyclic Sylow q-subgroup. The structure of P is well known, in particular, $P' = \Phi(P)$ is an elementary abelian 2-group. Hence the exponent of P is at most 4. Furthermore, $Z(S) = \Phi(P) \times \langle y^q \rangle$, $S' = P$ [3, III.5.2]. The last equality implies that $X_2(S)$ is not contained in $Z(G)$. If $\forall a, b, g \in P$, then $[a, g][b, g] = [ab, g]$, as $P' \le Z(P)$. Hence $P' \subseteq X_2(S)$. If $P' \nsubseteq Z(G)$, then there is the commutator $a \in P' \setminus Z(G)$ of order 2. Let $P' \le Z(G)$ and $a \in X_2(S) \setminus Z(G)$, $a^2 \in Z(G)$. Since $G / Z(G)$ is a nonabelian simple group, there is [2, 3.8.2] an element $g \in G$ such that $\langle aZ(G), a^g Z(G) \rangle$ is not a 2-group. But a subgroup generated by two involutions is a dihedral group. So there is an element $tZ(G) \in \langle aZ(G), a^g Z(G) \rangle$ of odd prime order r such that $\langle aZ(G), tZ(G) \rangle = D / Z(G)$ is a dihedral group of order $2r$. Let R be a Sylow r-subgroup of D. Since $RZ(G) = R \times Z(G)_{r'}$ is normal in D, it follows that R is normal in D. In view of Lemma 2.1, $a \in C_G(R)$ and $D / Z(G)$ is abelian. This is a contradiction, and G is soluble.

By induction, all proper subgroups of G have the nilpotent derived subgroups, in particular, G' is not nilpotent, but G'' is nilpotent. Suppose that S is a Schmidt subgroup in G', $S = P \leftthreetimes Q$, where P is a normal in S Sylow p-subgroup, Q is a nonnormal in S cyclic Sylow q-subgroup, $S' = P$. It is clear that $P \le G''$ and P is subnormal in G. This implies that $P \le O_p(G)$. Here $O_p(G)$ is the largest normal p-subgroup of G. Let Q_1 be a Sylow q-subgroup of G containing Q. In view of the Focal Subgroup Theorem [2, 7.3.4], $Q_1 \cap G'$ is generated by elements $x^{-1}y$, where $x, y \in Q_1$ and x conjugates to y in G. From (2.1) it follows that $Q_1 \cap G'$ is generated by elements of $Q_1 \cap X_q(G)$. If $x \in Q_1 \cap X_q(G)$, then $x \in C_G(O_p(G)) \le C_G(P)$ by Lemma 2.1, and $Q \le Q_1 \cap G' \le C_G(P)$. This is a contradiction since S is nilpotent. □

***Corollary* 2.2.1.** *The derived subgroup of a finite group is nilpotent if and only if all primary commutators of coprime orders are permutable.*

The following nilpotency criterion for a finite group was obtained by Baumslag and Wiegold [4].

***Theorem* 2.3.** *Let G be a finite group in which* $|ab| = |a||b|$ *for elements* a, b *of coprime orders. Then G is nilpotent.*

Here we can also limit to only primary elements. An element a of a group G is called a p-element if $|a| = p^n$ for a non-negative integer n. If the order of a is not divided by p, then a is a p'-element. If a group G has a normal Sylow p-subgroup and G p-nilpotent, then G is said to be p-decomposable.

***Theorem* 2.4.** *Let G be a finite group. Fix a prime* $p \in \pi(G)$. *A group* G *is p-decomposable if and only if* $|ab| \ge |a||b|$ *for a p-element* a *and a primary* p'*-element* b.

Proof. If a is a p-element and b is a primary p'-element of a finite p-decomposable group, then $|ab| = |a||b|$.

Conversely, assume that the result is not true and let G be a counterexample of least order. It is clear that the hypothesis holds for every subgroup of G. Hence all proper subgroups of G are p-decomposable. Suppose that G is not p-nilpotent. In view of Ito Theorem [3, IV.5.4], $G = P \leftthreetimes Q$ is a Schmidt group, where P is a normal in G Sylow p-subgroup, $Q = \langle y \rangle$ is a Sylow subgroup of order q^t. If x is an element of P, then $x^{-1}yx = ay^s$ for some $a \in P$ and a non-negative integer s. Since

$$a = x^{-1}yx(y^s)^{-1} = x^{-1}yxy^{-1}y(y^s)^{-1} = [x, y^{-1}]y(y^s)^{-1}, \quad [x, y^{-1}] \in G' \le P, \tag{2.2}$$

this implies that $y(y^s)^{-1} = 1$, $y = y^s$ and $x^{-1}yx = ay$. By the hypothesis,

$$|y| = |x^{-1}yx| = |ay| \ge |a||y|, \quad a = 1, \ x^{-1}yx = y. \tag{2.3}$$

Consequently, Q is normal in G, a contradiction. Thus, $G = H \leftthreetimes P$ is p-nilpotent.

If P is not cyclic, then in P there are two different maximal subgroups P_1 and P_2. Subgroups HP_1 and HP_2 are p-decomposable, it follows that G is p-decomposable. Let $P = \langle y \rangle$ be cyclic. If x is an element of H, then $x^{-1}yx = ay^s$ for some $a \in P$ and a non-negative integer s. Repeating (2.2) and (2.3), we have that G is p-decomposable. □

***Corollary* 2.4.1.** *A finite group is nilpotent if and only if* $|ab| \ge |a||b|$ *for any two primary elements a and b of coprime orders.*

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