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On some non-periodic groups whose cyclic subgroups are monopronormal

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The description of non-periodic locally generalized radical groups whose cyclic subgroups are monopronormal is given.
Keywords: normal subgroup, abnormal subgroup, pronormal subgroup, contranormal subgroup, monopronormal subgroup, locally nilpotent radical, locally finite radical, (generalized) radical group.

Получено описание непериодических локально обобщенно радикальных групп, циклические подгруппы которых монопронормальны.
Ключевые слова: нормальная подгруппа, абнормальная подгруппа, пронормальная подгруппа, контранормальная подгруппа, монопронормальная подгруппа, локально нильпотентный радикал, локально конечный радикал, (обобщенно) радикальная группа.

Introduction. Let *G* be a group. Recall that a subgroup *H* of *G* is called *abnormal* in *G* if $g \in \langle H, H^g \rangle$ for every element $g \in G$. Recall also that a subgroup *H* of *G* is *contranormal* in G if $H^G = G$, where H^G is a normal closure of *H* in *G*. Note that every abnormal subgroup is contranormal (see, for example, [1]). Clearly abnormal and contranormal subgroups are antipodes (in some sense) of normal subgroups. On the one hand, a subgroup *H* of *G* is both normal and abnormal in *G* if and only if $H = G$. On the other hand, if *H* is a normal subgroup of *G*, then $H^G = H$. These remarks show that the properties of normal subgroups and abnormal (respectively, contranormal) subgroups are diametrically opposite.

At the same time, there are subgroups that combine the concepts of normality and abnormality. One of the typical examples of such subgroups are pronormal subgroups. Recall that a subgroup *H* of a group *G* is called *pronormal* in *G* if for every element $g \in G$ the subgroups *H* and H^g are conjugate in $\langle H, H^g \rangle$. Thus, every normal and abnormal subgroup of *G* is pronormal in *G*. Note that the normalizer $N_G(H)$ of pronormal subgroup *H* is abnormal in *G* (see, for example, [2]), and hence contranormal in *G*.

In the paper [3] the authors introduced the following generalization of normal and abnormal subgroups.

Definition. A subgroup *H* of a group *G* is called *monopronormal* in *G* if for every element $g \in G$ either $H^g = H$ or $N_K(H)^K = K$, where $K = \langle H, g \rangle$.

Clearly every pronormal subgroup is monopronormal. Note that the converse statement in general does not hold.

In the paper [3], the authors obtained the description of locally finite groups whose all subgroups are monopronormal. Later, in the paper [4], the description of locally finite groups whose cyclic subgroups are monopronormal has been obtained.

In this article, we continue studying the influence of monopronormal subgroups on the group structure. More precisely, we investigate the structure of some non-periodic groups whose cyclic subgroups are monopronormal.

We recall some definitions. A *locally nilpotent radical* of a group *G* is a subgroup $\mathbf{Lnr}(G)$ generated by all normal locally nilpotent subgroups of *G*. We recall also that a *locally finite radical* of a group *G* is a subgroup $\mathbf{Lfr}(G)$ generated by all normal locally finite subgroups of *G*.

A group *G* is called *radical* if *G* has an ascending series whose factors are locally nilpotent. A group *G* is called *generalized radical* if *G* has an ascending series whose factors are locally nilpotent or locally finite.

It was also observed that a periodic generalized radical group is locally finite, and hence periodic locally generalized radical group is also locally finite.

The main result of this paper is the following

Theorem A. *Let G be a non-periodic locally generalized radical group. Suppose that R is a locally nilpotent radical of G. If every cyclic subgroup of G is monopronormal, then either G is abelian or* $G = R\langle b\rangle$, *where R is abelian,* $b^2 \in R$ *and* $a^b = a^{-1}$ *for each element* $a \in R$. *Moreover, in the second case, the following conditions hold:*

(i) if $b^2 = 1$, *then the Sylow* 2*-subgroup D of R is elementary abelian;*

(ii) if $b^2 \neq 1$, *then either D is elementary abelian or* $D = E \times \langle v\rangle$, *where E is elementary abelian and* $\langle b, v\rangle$ *is a quaternion group.*

Conversely, if a group G satisfies the above conditions, then every cyclic subgroup of G is monopronormal.

Preliminary results.

Lemma 0. *Let G be a group whose cyclic subgroups are monopronormal.*

(i) If H is a subgroup of G, then every cyclic subgroup of H is monopronormal.

(ii) If H is a normal subgroup of G, then every cyclic subgroup of G/H *is monopronormal.*

Proof. It follows from the definition of monopronormal subgroups.

Let *G* be a group and R^{LN} be a family of all normal subgroups *H* of *G* such that G/H is locally nilpotent. Then the intersection $\cap \mathsf{R}^{\mathsf{LN}} = R^{\mathsf{LN}}$ is called the *locally nilpotent residual* of *G*. It is not difficult to prove that if *G* is locally finite, then G/R^{LN} is locally nilpotent.

Lemma 1. *Let G be a locally finite group. If every cyclic subgroup of G is monopronormal, then the derived subgroup of G is abelian.*

Proof. Let *L* be the locally nilpotent residual of *G*. Since $L \leq [G,G]$,

$$G/[G,G] \cong (G/L)/([G,G]/L).$$

Since G/L is locally nilpotent,

$$G/L = \mathbf{Dr}_{p\in\Pi(G/L)} S_p/L,$$

where S_p/L is a Sylow *p*-subgroup of G/L. Then

$$[G,G]/L = [G/L, G/L] = \mathbf{Dr}_{p\in\Pi(G/L)}[S_p/L, S_p/L].$$

Put $D_p/L = [S_p/L, S_p/L]$. By Theorem 1 from [4], G/L is a Dedekind group. It follows that D_p/L is abelian for each $p \in \Pi(G/L)$. By [4, Corollary 12], $[G,G] \leq C_G(L)$, in particular, $[G,G]$ is nilpotent. Let $p \in \Pi([G,G]) \setminus \Pi(L)$. Choose in $[G,G]$ a Sylow *p*-subgroup *P*. By [4, Lemma 8], $P \cap L = \langle 1\rangle$, thus

$$P \cong P/(P\cap L) \cong PL/L \cong D_p/L.$$

Therefore, *P* is abelian. Since by Corollary 11 from [4], *L* is abelian, $[G,G]$ is abelian too.

In the paper [5], B.H. Neumann proved the following classical result: *if the factor-group* $G/\zeta(G)$ *is finite, then the derived subgroup* $[G,G]$ *is also finite.* As a corollary, we can come to the following generalization: *if the factor-group* $G/\zeta(G)$ *is locally finite, then the derived subgroup* $[G,G]$ *is also locally finite.*

Lemma 2. *Let G be a generalized radical group. If every cyclic subgroup of G is monopronormal, then G is soluble of class at most* 3.

Proof. Suppose that the locally finite radical $\mathbf{Lfr}(G) = F$ of *G* is non-identity. Then by Lemma 1 $[F,F]$ is abelian. It follows that in any case the locally nilpotent radical $\mathbf{Lnr}(G) = R$ of *G* is non-identity. We will prove that *G* is a radical group. Suppose the contrary. Then *G* includes the normal subgroups *T* and *S* such that $R \leq T \leq S$, *T* is radical, S/T is locally finite and $\mathbf{Lnr}(S/T) = \langle 1\rangle$. By [4, Corollary 4], *R* is a Dedekind group. Corollary 1 from [4] shows that every subgroup of *R* is *G*-invariant. Then $S/C_S(R)$ is abelian (see, for example [6, Theorem 1.5.1]). We observe that $C_S(R) \cap T \leq R$ (see [7, Lemma 4]). Suppose first that *R* is periodic. Then

$$C_S(R)/(C_S(R)\cap R) = C_S(R)/(C_S(R)\cap T) \cong C_S(R)T/T \leq S/T.$$

In particular, $C_S(R)/(C_S(R)\cap R)$ is locally finite. Since R is periodic and locally nilpotent, $C_S(R)$ is locally finite. Being locally finite, $C_S(R)$ is metabelian by Lemma 1. Since S/T does not include non-identity normal abelian subgroups, $C_S(R)\le T$. We have now

$$S/T\cong (S/C_S(R))/(T/C_S(R)).$$

We have remarked above that the factor-group $S/C_S(R)$ is abelian, and therefore S/T is abelian. Contradiction.

Suppose now that R is not periodic. Corollary 4 from [4] implies that R is abelian. Let V be the periodic part of R and put $C=C_S(R)$. By proved above, $C/R\cong C/(C\cap R)$ is locally finite. Also, the inclusion $R\le \zeta(C)$ implies that $[C,C]$ is a locally finite subgroup. Using Lemma 1, we obtain that C is soluble. It follows that $C_S(R)\le T$, and using the arguments from above, we again obtain a contradiction. This contradiction shows that G is a radical group.

Then $C_G(R)\le R$ [7, Lemma 4]. By [4, Corollary 4], R is a Dedekind group, in particular, R is metabelian. Corollary 1 from [4] shows that every subgroup of R is G-invariant. Then $G/C_G(R)$ is abelian (see, e. g., Theorem 1.5.1 in [6]). The inclusion $C_G(R)\le R$ implies that G/R is abelian, so that G is soluble and $\mathbf{scl}(G)\le 3$.

Corollary 1. *Let G be a locally generalized radical group. If every cyclic subgroup of G is monopronormal, then G is soluble of class at most* 3.

Lemma 3. *Let G be a group and A be a normal abelian subgroup of G. Suppose that* $G=A\langle b\rangle$ *where* $b^2\in A$ *and* $a^b=a^{-1}$ *for each element* $a\in A$. *If the subgroup* $\langle b\rangle$ *is monopronormal, then*

(i) if $b^2=1$, *then the Sylow* 2*-subgroup D of A is elementary abelian;*

(ii) if $b^2\ne 1$, *then either D is elementary abelian or* $D=E\times\langle v\rangle$, *where E is elementary abelian and* $\langle b,v\rangle$ *is a quaternion group.*

Proof. Suppose that $a\in C_A(b)$, then $a^b=a$. On the other hand, by our conditions, $a^b=a^{-1}$, that is $a^{-1}=a$ and $a^2=1$. Thus $C_A(b)$ is an elementary abelian 2-subgroup. If $c=b^2\ne 1$, then $c\in C_A(b)$, and by proved above, $1=c^2=b^4$. Conversely, if $|a|=2$, then $a\in C_A(b)$.

Note that if $a\in\langle b\rangle$, then $\langle b\rangle^a=\langle b\rangle$. Let a be an arbitrary element of A. Then $b^{-1}a^{-1}ba=aa=a^2$, and $b^a=a^{-1}ba=ba^2$. Furthermore, $b^{-1}ab=a^{-1}$ and $ab=ba^{-1}$. Then we have

$$(ba)(ba)=b(ab)a=b(ba^{-1})a=b^2.$$

Since this is valid for arbitrary element a, we obtain $(ba^2)^2=b^2$.

Since $\langle b\rangle$ is a monopronormal subgroup, we have two possibilities: either $\langle b\rangle^a=\langle b\rangle$ or $N_K(\langle b\rangle)^K=K$, where $K=\langle\langle b\rangle,a\rangle=\langle b,a\rangle$, $a\in A$. In the first case, we obtain that a subgroup

$$\langle b\rangle=\langle b\rangle^a=\langle ba^2\rangle=\langle b,a^2\rangle,$$

is a 2-subgroup, in particular, a^2 (and hence a) is a 2-element. In the second case, we have

$$\langle b,a^2\rangle=N_K(\langle b\rangle)^K=K=\langle b,a\rangle,$$

which is impossible.

Suppose first $|b|=2$. Then $\langle b\rangle\cap A=\langle 1\rangle$. Assume that A has an element u of order 4. By proved above $u^{-1}bu=bu^2$. Since $|u^2|=2$, $u^2\in C_A(b)$. It follows that $\langle b,u^2\rangle$ is abelian. On the one hand, $\langle b\rangle\ne\langle b\rangle^u$. On the other hand

$$N_K(\langle b\rangle)^K=\langle b,u^2\rangle\ne\langle b,u\rangle=K,$$

and we obtain a contradiction. This contradiction shows that a Sylow 2-subgroup of A is elementary abelian.

Suppose now that $c = b^2 \neq 1$. Let D be a Sylow 2-subgroup of A. Since the subgroup $\langle c \rangle$ is normal in G, its image in the factor-group $G/\langle c \rangle$ is a monopronormal subgroup. As proved above, $D/\langle c \rangle$ is an elementary abelian 2-subgroup. Then either D is elementary abelian or D has an element v of order 4 such that $v^2 = c = b^2$. Consider the last situation. Since v has a maximal order among all the elements of D, $D = E \times \langle v \rangle$. Since $\langle v \rangle$ is $\langle b \rangle$-invariant, we have

$$\left|\langle b \rangle \langle v \rangle\right| = \frac{\left|\langle b \rangle\right|\left|\langle v \rangle\right|}{\left|\langle b \rangle \cap \langle v \rangle\right|} = 8.$$

Furthermore, as proved above, $v^{-1}bv = bv^2 = bb^2 = b^3$. Hence $\langle b, v \rangle$ is a product of two normal cyclic subgroups of order 4. It follows that $\langle b, v \rangle$ is a quaternion group.

Corollary 2. *Let G be a group and A be a normal abelian non-periodic subgroup of G. Suppose that $G = A\langle b \rangle$ where $b^2 \in A$ and $a^b = a^{-1}$ for each element $a \in A$. Then G has a subgroup, which is not monopronormal.*

Proof. Indeed, let h be an element of A of infinite order. Put $H = \langle h^4 \rangle$. Then H is normal in G, the element hH has order 4, and $\langle hH \rangle \cap \langle bH \rangle = H$. Lemma 3 shows that the subgroup $\langle b, h^4 \rangle$ can not be monopronormal.

Lemma 4. *Let G be a non-periodic finitely generated soluble group. Suppose that R is a locally nilpotent radical of G. If every cyclic subgroup of G is monopronormal, then either G is abelian or $G = R\langle b \rangle$, where R is abelian, $b^2 \in R$, and $a^b = a^{-1}$ for each element $a \in R$.*

Proof. By [4, Corollary 4], R is a Dedekind group. Corollary 1 from [4] shows that every subgroup of R is G-invariant. Then $G/C_G(R)$ is abelian (see, for example [6, Theorem 1.5.1]). The inclusion $C_G(R) \leq R$ [7, Lemma 4] implies that G/R is abelian. Being abelian and finitely generated G/R is finitely presented. It follows that R has the elements $x_1, ..., x_k$ such that $R = \langle x_1 \rangle^G ... \langle x_k \rangle^G$ (see, for example, [8, p. 421]). Since every subgroup of R is G-invariant, $\langle x_j \rangle^G = \langle x_j \rangle$, $1 \leq j \leq k$. It follows that R is finitely generated. If we suppose that R is periodic, then R is finite. The inclusion $C_G(R) \leq R$ [7, Lemma 4] implies that G/R is also finite, and hence G is finite. This contradiction proves that R is non-periodic.

Then Corollaries 2 and 3 from [4] shows that R is abelian. Suppose that the center $\zeta(G)$ contains every element of R of infinite order. Clearly, R is generated by elements of infinite order, so that $R \leq \zeta(G)$. Then the fact that G/R is abelian implies that G is nilpotent. Using again Corollaries 2 and 3 from [4] we obtain that G is abelian. Therefore, we consider the case when a subgroup R contains an element of infinite order, which is not central. Since R is abelian and finitely generated,

$$R = \langle u_1 \rangle \times ... \times \langle u_n \rangle \times \langle v_1 \rangle \times ... \times \langle v_t \rangle,$$

where the elements $u_1, ..., u_n$ have infinite orders and the elements $v_1, ..., v_t$ have finite orders. Suppose that $u_j \in \zeta(G)$ for all j, $1 \leq j \leq n$. Since $\zeta(G)$ does not include R, there exists an index m such that $v_m \notin \zeta(G)$. Then there exists an element g such that $v_m^g = v_m^r \neq v_m$ where r is a certain positive integer. Consider the element $u_1 v_m$. We have

$$(u_1 v_m)^g = u_1^g v_m^g = u_1 v_m^r \neq u_1 v_m.$$

We remark that $u_1 v_m$ has infinite order. By [4, Corollary 1], a subgroup $\langle u_1 v_m \rangle$ is G-invariant. Then the fact that $g \notin C_G(u_1 v_m)$ implies $\langle u_1 v_m \rangle^g = \langle u_1 v_m \rangle^{-1} = u_1^{-1} v_m^{-1}$. On the other hand, we have $(u_1 v_m)^g = u_1 v_m^r$, which implies that $u_1 = u_1^{-1}$. Contradiction. So, there exists an index j such that $u_j \notin \zeta(G)$. Without loss of generality we can suppose that $j = 1$. Let b be an element of G such that $G = \langle b \rangle C_G(\langle u_1 \rangle)$. Then $u_1^b = u_1^{-1}$, and $b^2 \in C_G(\langle u_1 \rangle)$. Suppose now that there exists an index s, $1 < s \leq n$, such that $[b, u_s] = 1$. Then

$$(u_1 u_s)^b = u_1^b u_s^b = u_1^{-1} u_s \neq u_1 u_s.$$

On the other hand, an infinite cyclic subgroup $\langle u_1 u_s \rangle$ is *G*-invariant by [4, Corollary 1]. Then it follows that

$$(u_1 u_s)^b = (u_1 u_s)^{-1} = u_1^{-1} u_s^{-1}.$$

Hence $u_s = u_s^{-1}$, and we obtain a contradiction. This contradiction shows that $u_j^b = u_j^{-1}$ for all *j*, $1 \le j \le n$. Using the same arguments we can prove that $v_j^b = v_j^{-1}$ for all *j*, $1 \le j \le t$. It follows that $a^b = a^{-1}$ for all elements $a \in R$.

With the help of similar arguments, we can prove that

$$C_G(\langle u_1 \rangle) = C_G(R) = R.$$

Hence $G = R\langle b \rangle$ and $b^2 \in R$.

Corollary 3. *Let G be a non-periodic locally generalized radical group. Suppose that R is a locally nilpotent radical of G. If every cyclic subgroup of G is monopronormal, then either G is abelian or* $G = R\langle b \rangle$, *where R is abelian,* $b^2 \in R$, *and* $a^b = a^{-1}$ *for each element* $a \in R$.

Proof. By Corollary 1, *G* is soluble. Suppose that *G* is not abelian. Then *G* includes a non-periodic finitely generated non-abelian subgroup *K*. By Lemma 4, $K = \mathbf{Lnr}(K)\langle b \rangle$, where $\mathbf{Lnr}(K)$ is abelian, $b^2 \in \mathbf{Lnr}(K)$, $b^4 = 1$, and $a^b = a^{-1}$ for each element $a \in \mathbf{Lnr}(K)$.

Choose in *G* a local family $\mathbb{L}$ of finitely generated subgroups containing *K*, and let $L \in \mathbb{L}$. Using again Lemma 4 we obtain that $L = \mathbf{Lnr}(L)\langle b_1 \rangle$, where $\mathbf{Lnr}(L)$ is abelian, $b_1^2 \in \mathbf{Lnr}(L)$, $b_1^4 = 1$, and $a^{b_1} = a^{-1}$ for each element $a \in \mathbf{Lnr}(L)$. Since *K* is not locally nilpotent, $\mathbf{Lnr}(L) \cap K \ne K$. On the other hand,

$$|K : \mathbf{Lnr}(L) \cap K| \le |L : \mathbf{Lnr}(L)| = 2,$$

so that $\mathbf{Lnr}(K) = \mathbf{Lnr}(L) \cap K$. In particular, $b \notin \mathbf{Lnr}(L)$. It follows that $b = b_1 u$ for some element $u \in \mathbf{Lnr}(L)$. As in the proof of Lemma 3, we can show that $b^2 = (b_1 u)^2 = b_1^2$. So, instead of b_1 we can put *b*. In other words, if *L* is an arbitrary subgroup of the family $\mathbb{L}$, then $L = \mathbf{Lnr}(L)\langle b \rangle$, where $\mathbf{Lnr}(L)$ is abelian, $b^2 \in \mathbf{Lnr}(L)$, $b^4 = 1$, and $a^b = a^{-1}$ for each element $a \in \mathbf{Lnr}(L)$. Since $\mathbb{L}$ is a local family, $G = \mathbf{Lnr}(G)\langle b \rangle$, where $\mathbf{Lnr}(G)$ is abelian, $b^2 \in \mathbf{Lnr}(G)$, $b^4 = 1$, and $a^b = a^{-1}$ for each element $a \in \mathbf{Lnr}(G)$.

Proof of the main result.

Proof of Theorem A. The necessity of the theorem conditions follows from Lemma 3 and Corollary 3.

Conversely, let a group *G* satisfies the theorem conditions and let *x* be an arbitrary element of *G*. If $x \in R$, then $\langle x \rangle$ is normal in *G*, in particular, $\langle x \rangle$ is monopronormal. Suppose that $x \notin R$. Then $x = bu$ for some element $u \in R$. In this case, $G = R\langle x \rangle$. As in the proof of Lemma 3, we can show that $x^2 = (bu)^2 = b^2$. Since *R* is abelian, $a^x = a^b = a^{-1}$ for each element $x \in R$.

Let *g* be an arbitrary element of *G*, then $g = x^k a$ for some element $a \in R$. It follows that $g^{-1}xg = a^{-1}xa$. We have $x^{-1}a^{-1}xa = aa = a^2$, and $a^{-1}xa = xa^2$. Furthermore, $x^{-1}ax = a^{-1}$, and $ax = xa^{-1}$. Then we have $(xa)(xa) = x(ax)a = x(xa^{-1})a = x^2$.

Consider $\langle x \rangle^a$. We have $\langle x \rangle^a = \langle xa^2 \rangle = \langle x, a^2 \rangle$. In particular, it shows that $\langle x \rangle^a$ is a 2-subgroup. In turn, it follows that a^2 is a 2-element, so that *a* is also a 2-element. Then $a = vc$ where $c^2 = 1$. A subgroup $\langle b, v \rangle$ is a quaternion group, so that $\langle b \rangle$ is $\langle v \rangle$-invariant. It follows that $\langle x \rangle$ is $\langle v \rangle$-invariant. Since $c^2 = 1$, $[c, x] = 1$. It follows that $\langle x \rangle^a = \langle x \rangle$, which shows that $\langle x \rangle$ is a monopronormal subgroup.

The following result follows directly from Theorem A and Corollary 2.

Corollary 4. *Let G be a non-periodic locally generalized radical group. Then every subgroup of G is monopronormal if and only if G is abelian.*

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