**E.M. Ovsiyuk[1], Y.A. Voynova[2], A.D. Koral'kov[1]**
[1]Mozyr State Pedagogical University named after I. P. Shamyakin, Mozyr, Belarus
[2]Minsk Suvorov Military School, Minsk, Belarus

# P-NONINVARIANT EQUATION FOR SPIN 1/2 PARTICLE, TAKING INTO ACCOUNT OF THE EXTERNAL COULOMB FIELD

Within the theory of relativistic wave equations with extended sets of Lorentz group representations, a new P-noninvariant 20-component wave equation for spin 1/2 particle was proposed. The quantum mechanical Dirac-like $P$-noninvariant equation is solved in presence of external Coulomb field. It is shown that the energy spectrum for $P$-noninvariant spin 1/2 particle in external Coulomb field coincides with that for ordinary particle, though explicit form of wave function is different.

## 1. *P*-noninvariant equation and spherical solutions

Within the theory of relativistic wave equations with extended sets of Lorentz group representations, a new *P*-noninvariant 20-component wave equation for spin 1/2 particle was proposed [1-3]. Let us study an elementary example of this model, first a free $P$-noninvariant equation for particle in spherical tetrad of Minkowski space in absence of external fields:

$$\left( i\gamma^0 \partial_t + i\gamma^3 \partial_r + \frac{1}{r}\Sigma_{\theta\phi} + i\gamma^5 M \right)\psi(x) = 0, \quad (1)$$

where the angular operator is determined by the formula

$$\Sigma_{\theta,\phi} = i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + i\sigma^{12}}{\sin\theta}.$$

We take substitution for wave function with quantum number $\varepsilon, j, m$: (using
$D$-functions $D^j_{-m,\sigma}(\phi,\theta,0) \equiv D_\sigma$):

$$\Psi_{\varepsilon jm}(x) = \frac{e^{-i\varepsilon t}}{r} \begin{vmatrix} f_1(r) D_{-1/2} \\ f_2(r) D_{+1/2} \\ f_3(r) D_{-1/2} \\ f_4(r) D_{+1/2} \end{vmatrix}. \quad (2)$$

We find four radial equations

$$\varepsilon f_3 - i\frac{d}{dr} f_3 - i\frac{\nu}{r} f_4 + iMf_1 = 0, \quad \varepsilon f_4 + i\frac{d}{dr} f_4 + i\frac{\nu}{r} f_3 + iMf_2 = 0,$$

$$\varepsilon f_1 + i\frac{d}{dr} f_1 + i\frac{\nu}{r} f_2 + i\sigma M f_3 = 0, \quad \varepsilon f_2 - i\frac{d}{dr} f_2 - i\frac{\nu}{r} f_1 + i\sigma M f_4 = 0, \quad (3)$$

in comparison with ordinary Dirac equation here the signs at $M$ in equation 3 and 4 are different (note that $\sigma = -1$).

For ordinary Dirac equation, we can diagonalize additionally space reflection operator. Conventional $P$-reflection operator in Cartesian tetrad $\hat{\Pi}_C = i\gamma^0 \otimes \hat{P}$ after transforming to spherical basis takes the form

$$\hat{\Pi}_{sph} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \hat{P}\,. \qquad (4)$$

From eigenvalues equation $\hat{\Pi}_{sph}\Psi_{jm} = \Pi\Psi_{jm}$ we find two eigenvalues $\Pi = \delta(-1)^j$ and corresponding restrictions on radial functions:

$$\Pi = \delta\,(-1)^{j+1},\ \ \delta = \pm 1, \quad f_4 = \delta f_1, \quad f_3 = \delta f_2. \qquad (5)$$

However, these restrictions are not consistent with the system (3). Let us try to diagonalize another discrete operator $\hat{\Delta}_{sph}$ (adding multiplier $\gamma^5$):

$$\hat{\Delta}_{sph} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \hat{P}. \qquad (6)$$

The eigenvalues equation $\hat{\Delta}_{sph}\psi = \Delta\psi$ gives two eigenvalues and respective restrictions on radial functions:

$$\delta = \pm i, \quad \Delta = \delta(-1)^j, \quad f_4 = -\delta f_1, \quad f_3 = -\delta f_2. \qquad (7)$$

Let us impose these constraints in the main radial system (3), they turn out to be consistent. They lead to 2-nd order equations:

$\delta = +i$,

$$\frac{d^2 f_1}{dr^2} + \left(\frac{-iM}{\nu + iMr} + \frac{1}{r}\right)\frac{df_1}{dr} + \left(-\frac{i\varepsilon}{r} - \frac{\nu^2}{r^2} - M^2 - \frac{\varepsilon M}{\nu + iMr} + \varepsilon^2\right) f_1 = 0,$$

$$\frac{d^2 f_2}{dr^2} + \left(+\frac{iM}{\nu - iMr} + \frac{1}{r}\right)\frac{df_2}{dr} + \left(\frac{i\varepsilon}{r} - \frac{\nu^2}{r^2} - M^2 - \frac{\varepsilon M}{\nu - iMr} + \varepsilon^2\right) f_2 = 0; \qquad (8)$$

$\delta = -i$,

$$\frac{d^2 f_1}{dr^2} + \left(\frac{iM}{\nu - iMr} + \frac{1}{r}\right)\frac{df_1}{dr} + \left(-\frac{i\varepsilon}{r} - \frac{\nu^2}{r^2} - M^2 + \frac{\varepsilon M}{\nu - iMr} + \varepsilon^2\right) f_1 = 0,$$

$$\frac{d^2 f_2}{dr^2} + \left(\frac{-iM}{\nu + iMr} + \frac{1}{r}\right)\frac{df_2}{dr} + \left(\frac{i\varepsilon}{r} - \frac{\nu^2}{r^2} - M^2 + \frac{\varepsilon M}{\nu + iMr} + \varepsilon^2\right) f_2 = 0. \qquad (9)$$

Here we have differential equations for confluent Heun functions.

Now, let us take into account the presence of external Coulomb field, assuming that anomalous magnetic moment equals to zero. Then we get the following 2-nd order equations

$\delta = +i$,

$$\frac{d^2 f_1}{dr^2}+\left[\frac{1}{r}+\frac{-iM}{iMr+\nu}\right]\frac{df_1}{dr}+$$

$$+\left[\frac{2\varepsilon\alpha\nu-\alpha M-i\varepsilon\nu}{\nu r}+\frac{\alpha^2-\nu^2}{r^2}+\varepsilon^2-M^2-\frac{M\left(-iM\alpha+\varepsilon\nu\right)}{\nu\left(iMr+\nu\right)}\right]f_1=0,$$

$$\frac{d^2 f_2}{dr^2}+\left[\frac{1}{r}+\frac{iM}{-iMr+\nu}\right]\frac{df_2}{dr}+$$

$$+\left[\frac{2\varepsilon\alpha\nu-\alpha M+i\varepsilon\nu}{\nu r}+\frac{\alpha^2-\nu^2}{r^2}+\varepsilon^2-M^2-\frac{M\left(iM\alpha+\varepsilon\nu\right)}{\nu\left(-iMr+\nu\right)}\right]f_2=0; \quad (10)$$

$\delta=-i,$

$$\frac{d^2 f_1}{dr^2}+\left(\frac{1}{r}+\frac{iM}{-iMr+\nu}\right)\frac{df_1}{dr}+$$

$$+\left[\frac{2\varepsilon\alpha\nu+\alpha M-i\varepsilon\nu}{\nu r}+\frac{\alpha^2-\nu^2}{r^2}+\varepsilon^2-M^2+\frac{M\left(iM\alpha+\varepsilon\nu\right)}{\nu\left(-iMr+\nu\right)}\right]f_1=0,$$

$$\frac{d^2 f_2}{dr^2}+\left(\frac{1}{r}+\frac{-iM}{iMr+\nu}\right)\frac{df_2}{dr}+$$

$$+\left[\frac{2\varepsilon\alpha\nu+\alpha M+i\varepsilon\nu}{\nu r}+\frac{\alpha^2-\nu^2}{r^2}+\varepsilon^2-M^2+\frac{M\left(-iM\alpha+\varepsilon\nu\right)}{\nu\left(iMr+\nu\right)}\right]f_2=0. \quad (11)$$

Here again we have differential equations for confluent Heun functions.

**2. Studying differential equation**

Note the symmetry $M \to -M$ between two pairs of equations. Also we see that equations for functions $f_1$ and $f_2$ are complex conjugate to each other. Therefore, in fact we van study only one equation. Let it be equation for $f_1$

$$\frac{d^2 f_1}{dr^2}+\left[\frac{1}{r}+\frac{-iM}{iMr+\nu}\right]\frac{df_1}{dr}+$$

$$+\left[\frac{2\varepsilon\alpha\nu-\alpha M-i\varepsilon\nu}{\nu r}+\frac{\alpha^2-\nu^2}{r^2}+\varepsilon^2-M^2-\frac{M\left(-iM\alpha+\varepsilon\nu\right)}{\nu\left(iMr+\nu\right)}\right]f_1=0.$$

It is convenient to use the variable $x=Mr$, then we get

$$\frac{d^2 f_1}{dx^2}+\left(\frac{1}{x}-\frac{1}{x-i\nu}\right)\frac{df_1}{dx}+\left(\frac{2E\alpha-\gamma-iE}{x}-\frac{\Gamma^2}{x^2}+E^2-1+\frac{\gamma+iE}{x-i\nu}\right)f_1=0, \quad (12)$$

where $E=\dfrac{\varepsilon}{M}$, $\Gamma^2=\nu^2-\alpha^2$, $\gamma=\alpha/\nu$. We find behavior of solutions near singular points:

$$x\to i\nu,\quad f_1=(x-i\nu)^{\rho},\quad \rho=0,2;$$
$$x\to 0,\quad f_1=x^{A},\quad A=\pm\Gamma^2.$$

In the variable $y=x^{-1}$, equation in vicinity of $y=0$ reads

$$\left(\frac{d^2}{dy^2}+\frac{2}{y}\frac{d}{dy}+\frac{E^2-1}{y^4}\right)f_1=0;$$

therefore, we conclude that the singular point $y=0$ is irregular of the rank 2.

In eq. (12) we change the variable $y=-ix/\nu$:

$$\frac{d^2 f_1}{dy^2}+\left(\frac{1}{y}-\frac{1}{y-1}\right)\frac{df_1}{dy}+$$
$$+\left[\frac{\nu\left(E+2iE\alpha-i\gamma\right)}{y}-\frac{\Gamma^2}{y^2}-\nu^2\left(-1+E^2\right)-\frac{\nu\left(-i\gamma+E\right)}{y-1}\right]f_1=0. \quad (13)$$

We search solutions in the form $f_1=y^{A}(y-1)^{B}e^{Cy}f(y)$, getting for $f(y)$ the following equation

$$\frac{d^2 f}{dy^2}+\left(2C+\frac{2A+1}{y}+\frac{2B-1}{y-1}\right)\frac{df}{dy}+$$
$$+\left[\frac{-2AB-B+A+\nu E-i\nu\gamma+2i\nu E\alpha+2AC+C}{y}+\frac{A^2-\Gamma^2}{y^2}+\right.$$
$$\left.+C^2-\nu^2E^2+\nu^2+\frac{B-\nu E+2BC-A+2AB+i\nu\gamma-C}{y-1}+\frac{B\left(B-2\right)}{\left(y-1\right)^2}\right]f=0.$$

We should impose restrictions

$$A=\pm\Gamma,\quad B=0,2,\quad C=\pm\nu\sqrt{-1+E^2}$$

to bound states may correspond the values

$$B=0,\quad A=+\Gamma,\quad C=-i\nu\sqrt{1-E^2},\quad f_1\sim x^{+\Gamma}e^{-\sqrt{1-E^2}x}f(y). \quad (14)$$

Equation for $f(y)$ becomes simpler

$$\frac{d^2 f}{dy^2}+\left(2C+\frac{2A+1}{y}+\frac{2B-1}{y-1}\right)\frac{df}{dy}+$$

$$+\left[\frac{-2\,AB-B+A+\nu\,E-i\nu\,\gamma+2i\nu\,E\alpha+2\,AC+C}{y}+\right.$$
$$\left.+\frac{B-\nu\,E+2\,BC-A+2\,AB+i\nu\,\gamma-C}{y-1}\right]f=0;$$

it may be identified with canonical form of confluent Heun equation

$$\frac{d^2H}{dz^2}+\left(-t+\frac{c}{z}+\frac{d}{z-1}\right)\frac{dH}{dz}+\left(-\frac{\lambda}{z}+\frac{\lambda-ta}{z-1}\right)H=0. \quad (15)$$

Its parameters are given by relations

$$t=-2C=2i\nu\sqrt{1-E^2}\,,\quad c=2A+1=2\Gamma+1,\quad d=2B-1=-1, \quad (16)$$

and

$$\lambda=2\,AB+B-A-\nu\,E+i\nu\,\gamma-2i\nu\,E\alpha-2\,AC-C=$$
$$=-\Gamma-\nu\,E+i\nu\,\gamma-2i\nu\,E\alpha+2\Gamma i\nu\sqrt{1-E^2}+i\nu\sqrt{1-E^2}\,. \quad (17)$$

From identity

$$\lambda-ta=B-\nu\,E+2\,BC-A+2\,AB+i\nu\,\gamma-C \quad \Rightarrow$$
$$2i\nu\sqrt{1-E^2}\,a=-2i\nu\,E\alpha+2\Gamma i\nu\sqrt{1-E^2} \quad (18)$$

we find expression for parameter $a$:

$$a=-\frac{E\alpha}{\sqrt{1-E^2}}+\Gamma,\qquad \Gamma=\sqrt{(j+1/2)^2-\alpha^2}\,. \quad (19)$$

We may determine so called transcendental confluent Heun function, by imposing the constraint $a=-n$, it give the quantization rule

$$\frac{E\alpha}{\sqrt{1-E^2}}=n+\sqrt{(j+1/2)^2-\alpha^2}\equiv N \quad \Rightarrow \quad E=\frac{1}{\sqrt{1+\frac{\alpha^2}{N^2}}}. \quad (20)$$

We conclude that the energy spectrum for $P$-noninvariant spin 1/2 particle in external Coulomb field coincides with that for ordinary particle, though explicit form of wave function is different.

**References**

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