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SPINORS OF THE LORENTZ GROUP AND JONES FORMALISM FOR A PARTLY POLARIZED LIGHT

Introduction

The main line of evolution in theoretical methods of polarization optics seems to be quite independent of that in relativistic symmetry methods, developed, for example, in particle physics. In the paper a technique of working with the Lorentz is used, the systematic construction of that was given by Fedorov [1], also see a quaternionic approach [2]. This technique is specified for looking at the problems of light polarization optics in the frames of vector Stokes-Mueller and spinor Jones formalism.

Remembering on great differences between properties of isotropic and time-like vectors in Special Relativity we should expect the same principal differences in describing polarized and partly polarized light. So below we will be considering these two cases separately: a polarized light and a partly polarized light. In particular, substantial differences will be revealed when turning to spinor techniques – also see [4]. Let us start with some basic definitions concerning the polarization of the light (at this we have used [3], though it might be another from many). For the Stokes vector of the partly polarized light we have

$$S^a = (I, I\ p\,\mathbf{n}), \qquad S_a S^a = I^2(1-p^2) \geq 0, \tag{1}$$

where I is a general intensity, p is a degree of polarization which runs within $[0,1]$ interval: $0 \leq p \leq 1$, $\mathbf{n}$ stands for any 3-vector. Behavior of Stokes 4-vectors for polarized and partly polarized light under acting optics devices may be considered as isomorphic to behavior of respectively isotropic and time-like vectors with respect to Lorentz group in Special Relativity. This simple observation leads to many consequences, some of them will be discussed below.

1. Spinor representation of Stokes 4-vector for a completely polarized light

Let start with the well-known relations between 2-rank bi-spinors and simplest tensors. Bi-spinor of second rank $U = \Psi \otimes \Psi$ can be resolved into scalar Φ, vector Φ_b; pseudoscalar $\tilde{\Phi}$, pseudovector $\tilde{\Phi}_b$, and antisymmetric tensor Φ_{ab}

$$U = \Psi \otimes \Psi = \left[-i\Phi + \gamma^b\ \Phi_b + i\sigma^{ab}\Phi_{ab} + \gamma^5\tilde{\Phi} + i\gamma^b\gamma^5\tilde{\Phi}_b\right]E^{-1}; \tag{2}$$

let us refer all consideration to the spinor basis

$$U = \begin{vmatrix} \xi^{\alpha\beta} & \Delta^{\alpha}{}_{\dot{\beta}} \\ H_{\dot{\alpha}}{}^{\beta} & \eta_{\dot{\alpha}\dot{\beta}} \end{vmatrix}, \quad \gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix}, \quad \gamma^5 = \begin{vmatrix} -I & 0 \\ 0 & +I \end{vmatrix},$$

$$\sigma^{ab} = \frac{1}{4}\begin{vmatrix} \bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a & 0 \\ 0 & \sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a \end{vmatrix}, E = \begin{vmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{vmatrix}. \tag{3}$$

Inverse to (2) relations look

$$\Phi_a = \frac{1}{4}\,\mathrm{Sp}\,[E\gamma_a U], \qquad \tilde{\Phi}_a = \frac{1}{4i}\,\mathrm{Sp}\,[E\gamma^5\gamma_a U],$$

$$\Phi = \frac{i}{4}\,\mathrm{Sp}\,[EU], \quad \tilde{\Phi} = \frac{1}{4}\,\mathrm{Sp}\,[E\gamma^5 U], \quad \Phi_{mn} = -\frac{1}{2i}\,\mathrm{Sp}\,[E\sigma_{mn} U]. \qquad (4)$$

First, we are interested in two vectors obtained from spinors:

$$\Phi_0 = \xi^1\eta_{\dot 2} - \xi^2\eta_{\dot 1}, \qquad \Phi_1 = \xi^1\eta_{\dot 1} - \xi^2\eta_{\dot 2},$$

$$\Phi_2 = i\,(\xi^1\eta_{\dot 1} + \xi^2\eta_{\dot 2}), \qquad \Phi_3 = -\,(\xi^1\eta_{\dot 2} + \xi^2\eta_{\dot 1});$$

for pseudovector, scalar and pseudoscalar $\tilde{\Phi}_0 = 0$, $\tilde{\Phi}_1 = 0$, $\tilde{\Phi}_2 = 0$, $\tilde{\Phi}_3 = 0$, $\Phi = 0$, $\tilde{\Phi} = 0$; and for antisymmetric tensor

$$\Phi^{01} = \frac{i}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) + (\eta_{\dot 1}\eta_{\dot 1} - \eta_{\dot 2}\eta_{\dot 2})\,],$$

$$\Phi^{23} = \frac{1}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) - (\eta_{\dot 1}\eta_{\dot 1} - \eta_{\dot 2}\eta_{\dot 2})\,],$$

$$\Phi^{02} = -\frac{1}{4}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) + (\eta_{\dot 1}\eta_{\dot 1} + \eta_{\dot 2}\eta_{\dot 2})\,],$$

$$\Phi^{31} = -\frac{1}{4i}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) - (\eta_{\dot 1}\eta_{\dot 1} + \eta_{\dot 2}\eta_{\dot 2})\,],$$

$$\Phi^{03} = -\frac{i}{2}\,[\,\xi^1\xi^2 + \eta_{\dot 1}\eta_{\dot 2}\,], \quad \Phi^{12} = -\frac{1}{2}\,[\,\xi^1\xi^2 - \eta_{\dot 1}\eta_{\dot 2}\,],$$

Collecting results together:

$$\Psi = \left|\begin{matrix}\xi^{\alpha}\\ \eta_{\dot\alpha}\end{matrix}\right|, \qquad \Psi \otimes \breve{\Psi} \quad\Rightarrow\quad \Phi = 0,\ \tilde{\Phi} = 0,\ \tilde{\Phi}_a = 0,\ \Phi_a \neq 0,\ \Phi_{mn} \neq 0,$$

we see that to have real vector and tensor one should impose additional restriction: let it be

$$\eta = +i\,\sigma^2\,\xi^* \qquad \Rightarrow \qquad \eta_{\dot 1} = +\,\xi^{2*},\ \eta_{\dot 2} = -\,\xi^{1*}; \qquad (5)$$

which results in

$$\Phi_0 = -(\xi^1\,\xi^{1*} + \xi^2\,\xi^{2*}) < 0, \qquad \Phi_3 = (\xi^1\,\xi^{1*} - \xi^2\,\xi^{2*}),$$

$$\Phi_1 = (\xi^1\,\xi^{2*} + \xi^2\,\xi^{1*}), \qquad \Phi_2 = i\,(\xi^1\,\xi^{2*} - \xi^2\,\xi^{1*});$$

$$\Phi^{01} = \frac{i}{4}\,[\,(\xi^1\,\xi^1 - \xi^2\,\xi^2) + (\xi^{2*}\,\xi^{2*} - \xi^{1*}\,\xi^{1*})\,],$$

$$\Phi^{23} = \frac{1}{4}\,[\,(\xi^1\,\xi^1 - \xi^2\,\xi^2) - (\xi^{2*}\,\xi^{2*} - \xi^{1*}\,\xi^{1*})\,],$$

$$\Phi^{02} = -\frac{1}{4}\,[\,(\xi^1\,\xi^1 + \xi^2\,\xi^2) + (\xi^{2*}\,\xi^{2*} + \xi^{1*}\,\xi^{1*})\,],$$

$$\Phi^{31} = -\frac{1}{4i} [(\xi^1 \xi^1 + \xi^2 \xi^2) - (\xi^{2*} \xi^{2*} + \xi^{1*} \xi^{1*})],$$

$$\Phi^{03} = -\frac{i}{2} (\xi^1 \xi^2 - \xi^{2*} \xi^{1*}), \qquad \Phi^{12} = -\frac{1}{2} [\xi^1 \xi^2 + \xi^{2*} \xi^{1*}]. \qquad (6)$$

There exists alternative additional restriction:

$$\eta = -i \sigma^2 \xi^* \qquad \Rightarrow \qquad \eta_1 = -\xi^{2*}, \ \eta_2 = +\xi^{1*}, \qquad (7)$$

which results in

$$\Phi_0 = (\xi^1 \xi^{1*} + \xi^2 \xi^{2*}) > 0, \qquad \Phi_3 = -(\xi^1 \xi^{1*} - \xi^2 \xi^{2*}),$$

$$\Phi_1 = -(\xi^1 \xi^{2*} + \xi^2 \xi^{1*}), \qquad \Phi_2 = -i (\xi^1 \xi^{2*} - \xi^2 \xi^{1*});$$

$$\Phi^{01} = \frac{i}{4} [(\xi^1 \xi^1 - \xi^2 \xi^2) + (\xi^{2*} \xi^{2*} - \xi^{1*} \xi^{1*})],$$

$$\Phi^{23} = \frac{1}{4} [(\xi^1 \xi^1 - \xi^2 \xi^2) - (\xi^{2*} \xi^{2*} - \xi^{1*} \xi^{1*})],$$

$$\Phi^{02} = -\frac{1}{4} [(\xi^1 \xi^1 + \xi^2 \xi^2) + (\xi^{2*} \xi^{2*} + \xi^{1*} \xi^{1*})],$$

$$\Phi^{31} = -\frac{1}{4i} [(\xi^1 \xi^1 + \xi^2 \xi^2) - (\xi^{2*} \xi^{2*} + \xi^{1*} \xi^{1*})],$$

$$\Phi^{03} = -\frac{i}{2} (\xi^1 \xi^2 - \xi^{2*} \xi^{1*}), \qquad \Phi^{12} = -\frac{1}{2} [\xi^1 \xi^2 + \xi^{2*} \xi^{1*}]. \qquad (8)$$

The last case (7)–(8) seems to be appropriate to describe Stokes 4-vector and determine Stokes 2-rank tensor:

$$\Psi = \begin{vmatrix} \xi \\ \eta = -i \sigma^2 \xi^* \end{vmatrix}, \qquad \Psi \otimes \Psi \qquad \Rightarrow \qquad S_a \neq 0, \ S_{mn} \neq 0,$$

$$S_0 = (\xi^1 \xi^{1*} + \xi^2 \xi^{2*}) > 0, \qquad S_3 = -(\xi^1 \xi^{1*} - \xi^2 \xi^{2*}),$$

$$S_1 = -(\xi^1 \xi^{2*} + \xi^2 \xi^{1*}), \qquad S_2 = -i (\xi^1 \xi^{2*} - \xi^2 \xi^{1*}),$$

$$a^1 = S^{01} = \frac{i}{4} [(\xi^1 \xi^1 - \xi^2 \xi^2) + (\xi^{2*} \xi^{2*} - \xi^{1*} \xi^{1*})],$$

$$b^1 = S^{23} = \frac{1}{4} [(\xi^1 \xi^1 - \xi^2 \xi^2) - (\xi^{2*} \xi^{2*} - \xi^{1*} \xi^{1*})],$$

$$a^2 = S^{02} = -\frac{1}{4} [(\xi^1 \xi^1 + \xi^2 \xi^2) + (\xi^{2*} \xi^{2*} + \xi^{1*} \xi^{1*})],$$

$$b^2 = S^{31} = -\frac{1}{4i} [(\xi^1 \xi^1 + \xi^2 \xi^2) - (\xi^{2*} \xi^{2*} + \xi^{1*} \xi^{1*})],$$

$$a^3 = S^{03} = -\frac{i}{2} (\xi^1 \xi^2 - \xi^{2*} \xi^{1*}), \qquad b^3 = S^{12} = -\frac{1}{2} (\xi^1 \xi^2 + \xi^{2*} \xi^{1*}). \qquad (9)$$

Let us calculate the main invariant – it turns to equal to zero:

$$S_0 S_0 - S_j S_j = 0\,, \tag{10}$$

so S_a may be considered as a Stokes 4-vector for a completely polarized light.

In turn, 4-tensor S_{mn}, being constructed from Jones bi-spinor Ψ, is a Stokes 2-rank tensor. Let us calculate two invariants for S_{mn}:

$$I_1 = -\frac{1}{2} S^{mn} S_{mn} = \mathbf{a}^2 - \mathbf{b}^2 = 0\,, \qquad I_2 = \frac{1}{4} \varepsilon_{abmn} S^{ab} S^{mn} = 0\,. \tag{11}$$

Finally, let us specify Stokes 4-vector and 4-tensor in parameters $(M, N, \Delta = \alpha - \beta)$:

$$\Psi = \begin{vmatrix} N\, e^{i\alpha} \\ +M\, e^{i\beta} \\ -M\, e^{-i\beta} \\ N\, e^{-i\alpha} \end{vmatrix}, \qquad \Psi \otimes \Psi \qquad \Rightarrow \qquad S_a \neq 0,\; S_{mn} \neq 0\,,$$

$$S_0 = M^2 + N^2\,, \qquad S_3 = M^2 - N^2\,,$$

$$S_1 = -2MN \cos(\alpha - \beta)\,, \qquad S_2 = 2MN \sin(\alpha - \beta)\,,$$

and

$$a^1 = S^{01} = -\frac{1}{2}(N^2 \sin 2\alpha - M^2 \sin 2\beta)\,, b^1 = S^{23} = +\frac{1}{2}(N^2 \cos 2\alpha - M^2 \cos 2\beta)\,,$$

$$a^2 = S^{02} = -\frac{1}{2}(N^2 \cos 2\alpha + M^2 \cos 2\beta)\,, b^2 = S^{31} = -\frac{1}{2}(N^2 \sin 2\alpha + M^2 \sin 2\beta)\,,$$

$$a^3 = S^{03} = +NM \sin(\alpha + \beta)\,, \qquad b^3 = S^{12} = -NM \cos(\alpha + \beta)\,. \tag{12}$$

Two vectors $\mathbf{a}, \mathbf{b}$ are determined by 4 parameters N, M, α, β, additional identities hold

$$\mathbf{a}^2 = \mathbf{b}^2 = \frac{(N^2 + M^2)^2}{4}\,, \qquad \mathbf{ab} = 0\,;$$

therefore the quantities $\mathbf{a}, \mathbf{b}$ depend in fact upon 4 independent parameters $N, M, \beta - \alpha, \beta + \alpha$; whereas Stokes 4-vector depends upon only three ones $N, M, \beta - \alpha$.

Instead of Stokes 4-tensor S_{ab} one may introduce a complex Stokes 3-vector $\mathbf{s} = \mathbf{a} + i\, \mathbf{b}$ with the components

$$s_1 + i s_2 = -i\, \xi^2 \xi^2\,, \qquad s_1 - i s_2 = +i\, \xi^1 \xi^1\,, \qquad s^3 = -i\; \xi^1\, \xi^2\,. \tag{13}$$

The quantity $\mathbf{s}$ transforms as a vector under complex rotation group $SO(3.C)$, isomorphic to Lorentz group $L_+^{\uparrow}$. The later permits to introduce additionally to Jones spinor and Mueller vector formalisms one other technique based on the use of complex 3-vector

$$\mathbf{s} = \mathbf{a} + i\,\mathbf{b} = \frac{1}{2}\begin{vmatrix} i\,(N^2 e^{2i\alpha} - M^2 e^{2i\beta}) \\ -(N^2 e^{2i\alpha} + M^2 e^{2i\beta}) \\ -2i\,NM\,e^{i(\alpha+\beta)} \end{vmatrix}; \qquad (14)$$

evidently this complex vector is isotropic $\mathbf{s}^2 = 0$, the later condition provide us with two additional condition, so $\mathbf{s}$ depends on 4 parameters.

2. On possible Jones 4-spinor for a partly polarized light

Now let us examine else one possibility

$$\Psi \otimes (-i\Psi^c) = \begin{vmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot 1} \\ \eta_{\dot 2} \end{vmatrix} \otimes \begin{vmatrix} +\eta^*_{\dot 2} \\ -\eta^*_{\dot 1} \\ -\xi^{2*} \\ +\xi^{1*} \end{vmatrix} = \begin{vmatrix} +\xi^1\eta^*_{\dot 2} & -\xi^1\eta^*_{\dot 1} & -\xi^1\xi^{2*} & +\xi^1\xi^{1*} \\ +\xi^2\eta^*_{\dot 2} & -\xi^2\eta^*_{\dot 1} & -\xi^2\xi^{2*} & +\xi^2\xi^{1*} \\ +\eta_{\dot 1}\eta^*_{\dot 2} & -\eta_{\dot 1}\eta^*_{\dot 1} & -\eta_{\dot 1}\xi^{2*} & +\eta_{\dot 1}\xi^{1*} \\ +\eta_{\dot 2}\eta^*_{\dot 2} & -\eta_{\dot 2}\eta^*_{\dot 1} & -\eta_{\dot 2}\xi^{2*} & +\eta_{\dot 2}\xi^{1*} \end{vmatrix}. \qquad (15)$$

Corresponding 4-vector is determined by

$$\Phi_0 = \frac{1}{2}[\,(\eta_{\dot 2}\eta^*_{\dot 2} + \eta_{\dot 1}\eta^*_{\dot 1}) + (\xi^2\xi^{2*} + \xi^1\xi^{1*})\,] > 0\,,$$

$$\Phi_3 = -\frac{1}{2}[\,(\eta_{\dot 2}\eta^*_{\dot 2} - \eta_{\dot 1}\eta^*_{\dot 1}) + (-\xi^2\xi^{2*} + \xi^1\xi^{1*})\,],$$

$$\Phi_1 = \frac{1}{2}[\,(\eta_{\dot 1}\eta^*_{\dot 2} + \eta_{\dot 2}\eta^*_{\dot 1}) - (\xi^1\xi^{2*} + \xi^2\xi^{1*})\,],$$

$$\Phi_2 = \frac{i}{2}[(\eta_{\dot 1}\eta^*_{\dot 2} - \eta_{\dot 2}\eta^*_{\dot 1}) + (-\xi^1\xi^{2*} + \xi^2\xi^{1*})\,].$$

We readily derive

$$\Phi^a\Phi_a = \eta_{\dot 1}\eta^*_{\dot 1}\;\xi^1\xi^{1*} + \eta_{\dot 2}\eta^*_{\dot 2}\;\xi^2\xi^{2*} + \eta_{\dot 1}\eta^*_{\dot 2}\;\xi^2\xi^{1*} + \eta_{\dot 2}\eta^*_{\dot 1}\xi^1\xi^{2*}\,. \qquad (16)$$

Let us demonstrate that this vector is time-like. With the notation

$$\xi = \begin{vmatrix} N_1 e^{in_1} \\ N_2 e^{in_2} \end{vmatrix}, \qquad \eta = \begin{vmatrix} M_1 e^{im_1} \\ M_2 e^{im_2} \end{vmatrix}, \qquad (17)$$

we get

$$\Phi^a\Phi_a = N_1^2 M_1^2 + N_2^2 M_2^2 + 2N_1M_1\;N_2M_2\,\cos[(n_1 - n_2) - (m_1 - m_2)];$$

therefore

$$(N_1M_1 - N_2M_2)^2 < \Phi_0^2 - \Phi_1^2 - \Phi_2^2 - \Phi_1^3 < (N_1M_1 + N_2M_2)^2\,. \qquad (18)$$

This means that we have ground to consider 4-vector Φ_a as Stokes 4-vector S_a:

$$(N_1M_1 - N_2M_2)^2 < S_0^2 - \mathbf{S}^2 < (N_1M_1 + N_2M_2)^2 \,, \qquad (19)$$

and two 2-spinors (15) as making up a Jones bi-spinor corresponding a partly polarized light.

It remains to find explicit form for corresponding (real) Stokes 4-tensor S_{ab}:

$$\Phi^{01} = \frac{i}{4}[\,(\xi^1\eta^*_{\dot 2} + \xi^2\eta^*_{\dot 1}) - (\eta_{\dot 1}\xi^{2*} + \eta_{\dot 2}\xi^{1*})\,],$$

$$\Phi^{23} = \frac{1}{4}[\,(\xi^1\eta^*_{\dot 2} + \xi^2\eta^*_{\dot 1}) + (\eta_{\dot 1}\xi^{2*} + \eta_{\dot 2}\xi^{1*})\,],$$

$$\Phi^{02} = -\frac{1}{4}[\,(\xi^1\eta^*_{\dot 2} - \xi^2\eta^*_{\dot 1}) + (-\eta_{\dot 1}\xi^{2*} + \eta_{\dot 2}\xi^{1*})\,],$$

$$\Phi^{31} = \frac{i}{4}[\,(\xi^1\eta^*_{\dot 2} - \xi^2\eta^*_{\dot 1}) - (-\eta_{\dot 1}\xi^{2*} + \eta_{\dot 2}\xi^{1*})\,],$$

$$\Phi^{03} = -\frac{i}{4}[\,(\xi^2\eta^*_{\dot 2} - \xi^1\eta^*_{\dot 1}) + (-\eta_{\dot 2}\xi^{2*} + \eta_{\dot 1}\xi^{1*})],$$

$$\Phi^{12} = -\frac{1}{4}[\,(\xi^2\eta^*_{\dot 2} - \xi^1\eta^*_{\dot 1}) - (-\eta_{\dot 2}\xi^{2*} + \eta_{\dot 1}\xi^{1*})],$$

$$s^1 = a^1 + ib^1 = \frac{i}{2}(\xi^1\eta^*_{\dot 2} + \xi^2\eta^*_{\dot 1}), \quad s^2 = a^2 + ib^2 = -\frac{1}{2}(\xi^1\eta^*_{\dot 2} - \xi^2\eta^*_{\dot 1}),$$

$$s^3 = a^3 + ib^3 = -\frac{i}{2}(\xi^2\eta^*_{\dot 2} - \xi^1\eta^*_{\dot 1}); \qquad (20)$$

besides this complex 3-vector is not isotropic: $\mathbf{s}^2 = -\frac{1}{4}(\xi^1\eta^*_{\dot 1} - \xi^2\eta^*_{\dot 1})^2 \neq 0$.

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