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The structure of invariant measures on locally compact semigroups with open translations

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In this note we extend the results of [1] to the case of semigroups that are not necessarily embeddable in groups. For measures we use the terminology of [2]. All measures are assumed to be non-zero and Borel. Throughout what follows S is a left ideal of a semigroup T. A measure μ on S is said to be left-invariant with respect to T if $\mu(aA) = \mu(A)$ under the condition that A and aA are measurable ($a \in T$). A semigroup S is called a left-semitopological subsemigroup of T if S is a Hausdorff space and the left translations λ_a: $x \longmapsto ax$ are continuous maps for every $a \in T$. If all the λ_a ($a \in T$) are open maps, then S is called a semigroup with open left translations.
Definition. A semigroup S is called a *left-homogeneous* subsemigroup of T if S cannot be split into two left ideals of T.

***Lemma* 1.** a) *Every semigroup can be split uniquely into left ideals each of which is a left-homogeneous subsemigroup of it.*

b) *For any splitting of a left-semitopological semigroup with open right translations into left ideals the ideals occurring in the splitting are opened-and-closed.*

The elements of the splitting in Lemma 1 a) are called the left homogeneous components of the semigroup in question. The existence of this splitting can be proved in the same way as in [1] (see Theorem 2 there).

***Lemma* 2.** *Let S be a left-homogeneous locally compact left-semitopological subsemigroup of T with open left translations. For any measure μ on S that is left-invariant with respect to T its support coincides with S, and left cancellation by elements of T holds in S.*

Sketch of proof. If we assume that S contains nonempty μ-negligible open subsets, then the union of all these subsets and also its complement in S are left ideals of T, which is impossible. The validity of left cancellation by elements of T can now be proved as in [4].

Lemma 2 shows, in particular, that in [4] the condition on measures of open sets to be positive is superfluous.

***Theorem* 1.** *In a left-homogeneous metrizable locally compact left-semitopological subsemigroup S of a semigroup T with open left translations any two measures that are left-invariant with respect to T are proportional.*

Proof. Let μ and μ_1 be two non-proportional measures on S, both left invariant with respect to T. Then the measures μ and $\nu = \mu + \mu_1$ are also non-proportional and $\mu \ll \nu$. Therefore, $d\mu/d\nu = f \neq$ const ν-almost everywhere. Then there is a c such that the sets $M_1 = \{x \mid f(x) < c\}$ and $M_2 = S \setminus M_1$ are not locally ν-negligible. Since μ and ν are left-invariant with respect to T, we see that $f(zx) = f(x)$ ν-almost everywhere for every $z \in T$. Consequently, $\nu(zA_i \cap M_i) = \nu(A_i)$ ($i = 1$ or 2) for any Borel set $A_i \subset M_i$ and any $z \in T$. We use the concepts and results of [3], §15. For a regular sequence $\mathfrak{M}$ of nets in a separable metric subspace $X \subset S$ and a Borel set $A \subset S$, let $D_{\mathfrak{M}}(A, x)$ denote $(\nu, \mathfrak{M})D(\chi \cdot \nu)(x)$ if $x \in X$ and zero otherwise. From [3], Theorem (15.7), it follows that $D_{\mathfrak{M}}(A, x) = \chi_{A \cap X}(x)$ ν-almost everywhere. Let L_1 be the set of $x \in S$ for which there is a precompact neighbourhood of X and a regular sequence $\mathfrak{M}$ of nets in it such that $D_{\mathfrak{M}}(M_1, x) = 1$ and $L_2 = S \setminus L_1$. By the corollary to Theorem (15.7) of [3] mentioned above, $L_i \neq \phi$ ($i = 1$ or 2). We claim that the L_i are left ideals of T. Let $x \in L_1$, $z \in T$, and let X be that neighbourhood of x with a regular sequence $\mathfrak{M}$ of nets for which $D_{\mathfrak{M}}(M_1, x) = 1$. In a neighbourhood zX of zx we choose a regular sequence $\mathfrak{N}$ of nets so that $\mathfrak{N}_n$ has the nucleus zU_n (U_n being the nucleus of the net $\mathfrak{M}_n$ containing x). The existence of such a net follows from the continuity of λ_z. Then $\nu(M_1 \cap U_n) = \nu(M_1 \cap z(M_1 \cap U_n)) \leqslant \nu(M_1 \cap zU_n)$. Therefore, $D_{\mathfrak{N}}(M_1, zx) = 1$, that is $zx \in L_1$.
We claim that L_2 is a left ideal of T. We assume that there exist an $x \in L_2$ and a $z \in T$ such that $zx \in L_1$. Then we can find a neighbourhood Y of zx and a regular sequence $\mathfrak{N}$ of nets in it, such that for every $\epsilon > 0$ there exists an N such that $\nu(M_1 \cap V_n) \geqslant (1 - \epsilon)\nu(V_n)$ for every $n > N$ (here V_n is the nucleus of $\mathfrak{N}_n$ containing zx). In the precompact neighbourhood $X \subset \lambda_z^{-1}(Y)$ of x we choose a

regular sequence $\mathfrak{M}$ of nets so that the nucleus U_n of $\mathfrak{M}_n$ containing x is such that $zU_n = V_n$. The existence of such a net follows from the fact that λ_z is open. Now $\nu(M_2 \cap U_n) \leqslant \nu(M_2 \cap zU_n) = \nu(M_2 \cap V_n) \leqslant \epsilon\nu(V_n) = \epsilon\nu(U_n)$ for $n > N$. Consequently, $D_{\mathfrak{M}}(M_2, x) = 0$, that is, $D_{\mathfrak{M}}(M_1, x) = 1$, which contradicts the choice of x. Thus, $\{L_1; L_2\}$ is a splitting of S into left ideals of T, which is impossible. The theorem is now proved.

Theorem 2. *Let S be a locally compact left-semitopological semigroup with open left and right translations. Every left-invariant measure μ on S has the form $\mu(B) = \sum_{\alpha\in\Lambda} \mu_\alpha(B)$, where $(L_\alpha)_{\alpha\in\Lambda}$ is a family of left-homogeneous components of S with left cancellation by elements of S, μ_α is a measure in $^r{}_\alpha$ that is left-invariant with respect to S (unique up to proportionality), and for every Borel set B in the sum on the right-hand side at most countably many terms are different from zero.*

Proof. Let $(L_\alpha)_{\alpha\in\Lambda}$ be the family of all not locally μ-negligible left-homogeneous components of S. By Lemmas 1 and 2, the L_α are open and have left cancellation by elements of S. We denote by μ_α the restriction of μ to L_α. Since the complement of $\cup\{L_\alpha \mid \alpha \in \Lambda\}$ is locally μ-negligible and any Borel set B has a non-empty intersection with at most countably many L_α, we see that $\mu(B) = \sum_{\alpha\in\Lambda} \mu_\alpha(B)$, which was to be proved.

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