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# On the $\mathfrak{F}$-hypercenter and the intersection of $\mathfrak{F}$-maximal subgroups of a finite group

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**Abstract.** Let $\mathfrak{X}$ be a class of groups. A subgroup $U$ of a group $G$ is called $\mathfrak{X}$-*maximal* in $G$ provided that (a) $U \in \mathfrak{X}$, and (b) if $U \le V \le G$ and $V \in \mathfrak{X}$, then $U = V$. A chief factor $H/K$ of $G$ is called $\mathfrak{X}$-eccentric in $G$ provided $(H/K) \rtimes G/C_G(H/K) \notin \mathfrak{X}$. A group $G$ is called a quasi-$\mathfrak{X}$-group if for every $\mathfrak{X}$-eccentric chief factor $H/K$ and every $x \in G$, $x$ induces an inner automorphism on $H/K$. We use $\mathfrak{X}^*$ to denote the class of all quasi-$\mathfrak{X}$-groups. In this paper we describe all hereditary saturated formations $\mathfrak{F}$ containing all nilpotent groups such that the $\mathfrak{F}^*$-hypercenter of $G$ coincides with the intersection of all $\mathfrak{F}^*$-maximal subgroups of $G$ for every group $G$.

## Introduction

Throughout this paper, all groups are finite, $G$, $p$ and $\mathfrak{X}$ always denote a finite group, a prime and a class of groups, respectively.

A *formation* is a class $\mathfrak{X}$ of groups with the following properties:

(a) Every homomorphic image of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.

(b) If $G/M$ and $G/N$ are $\mathfrak{X}$-groups, then also $G/(M \cap N) \in \mathfrak{X}$.

A formation $\mathfrak{X}$ is said to be

- *saturated* (respectively *solubly saturated*) if $G \in \mathfrak{X}$ whenever $G/\Phi(N) \in \mathfrak{X}$ for some normal (respectively for some soluble normal) subgroup $N$ of $G$.
- *hereditary* (respectively *normally hereditary*) if $H \in \mathfrak{X}$ whenever $H \le G \in \mathfrak{X}$ (respectively whenever $H \trianglelefteq G \in \mathfrak{X}$).

A subgroup $U$ of $G$ is called $\mathfrak{X}$-*maximal* in $G$ provided that the following hold (see [4, p. 288]):

(a) $U \in \mathfrak{X}$.

(b) If $U \le V \le G$ and $V \in \mathfrak{X}$, then $U = V$.

We use the symbol $\mathrm{Int}_{\mathfrak{X}}(G)$ to denote the intersection of all $\mathfrak{X}$-maximal subgroups of $G$. A chief factor $H/K$ of $G$ is called *$\mathfrak{X}$-central* in $G$ provided that $(H/K) \rtimes G/C_G(H/K) \in \mathfrak{X}$; otherwise it is *$\mathfrak{X}$-eccentric* (see [11, pp. 127–128]). A normal subgroup $N$ of $G$ is said to be *$\mathfrak{X}$-hypercentral* in $G$ if $N = 1$ or $N \neq 1$ and every chief factor of $G$ below $N$ is $\mathfrak{X}$-central. The symbol $\mathrm{Z}_{\mathfrak{X}}(G)$ denotes the *$\mathfrak{X}$-hypercenter* of $G$, that is, the largest normal $\mathfrak{X}$-hypercentral subgroup of $G$ (see [4, p. 389]). If $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups, then $\mathrm{Z}_{\mathfrak{N}}(G)$ is the hypercenter of $G$.

It is well known that the intersection of maximal abelian subgroups of $G$ is the center of $G$. Baer [1] showed that the intersection of maximal nilpotent subgroups of $G$ is the hypercenter of $G$. Nevertheless, the intersection of maximal supersoluble subgroups of $G$ does not necessarily coincide with the supersoluble hypercenter of $G$ (see [13, Example 5.17]). Shemetkov posed the following questions at the Gomel Algebraic Seminar in 1995:

**Question 1.** (1) For what non-empty hereditary saturated formations $\mathfrak{X}$ does the equality $\mathrm{Int}_{\mathfrak{X}}(G) = \mathrm{Z}_{\mathfrak{X}}(G)$ hold for every group $G$?

(2) For what non-empty normally hereditary solubly saturated formations $\mathfrak{X}$ does the equality $\mathrm{Int}_{\mathfrak{X}}(G) = \mathrm{Z}_{\mathfrak{X}}(G)$ hold for every group $G$?

The solution to the first question was obtained by Skiba in [12, 13] (for the soluble case, see also Beidleman and Heineken [3]). It is necessary to note that the methods of the papers [12, 13] are not applicable for non-saturated or non-hereditary formations. Thus, the answer to the second of Shemetkov's questions was not known even in such an important special case, when $\mathfrak{X} = \mathfrak{N}^*$ is the class of all quasinilpotent groups. The aim of this paper is to give the answer to that second question for some wide class of solubly saturated formations that contains $\mathfrak{N}^*$.

In [8, 9] Guo and Skiba introduced the concept of quasi-$\mathfrak{F}$-group for a saturated formation $\mathfrak{F}$. Recall that $G$ is called a *quasi-$\mathfrak{F}$-group* if for every $\mathfrak{F}$-eccentric chief factor $H/K$ and every $x \in G$, $x$ induces an inner automorphism on $H/K$. We use $\mathfrak{F}^*$ to denote the class of all quasi-$\mathfrak{F}$-groups. If $\mathfrak{N} \subseteq \mathfrak{F}$ is a normally hereditary saturated formation, then $\mathfrak{F}^*$ is a normally hereditary solubly saturated formation by [8, Theorem 2.6].

Our main result is the following:

**Theorem 1.** *Suppose that $\mathfrak{F}$ is a hereditary saturated formation containing all nilpotent groups. Then* $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *holds for every group $G$ if and only if* $\mathrm{Int}_{\mathfrak{F}^*}(G) = \mathrm{Z}_{\mathfrak{F}^*}(G)$ *holds for every group $G$.*

**Corollary 1.** *The intersection of all maximal quasinilpotent subgroups of $G$ is equal to* $\mathrm{Z}_{\mathfrak{N}^*}(G)$.

Recall [2, Theorem 3.4.5] that every solubly saturated formation $\mathfrak{F}$ contains the greatest saturated subformation $\mathfrak{F}_l$ with respect to set inclusion. The proof of Theorem 1 contains many steps. Two of them are given by the following theorems.

**Theorem 2.** *Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups. Then $(\mathfrak{F}^*)_l = \mathfrak{F}$.*

Recall that $C^p(G)$ is the intersection of the centralizers of all abelian $p$-chief factors of $G$ ($C^p(G) = G$ if $G$ has no such chief factors). Let $f(p)$ be the intersection of all formations containing $(G/C^p(G) \mid G \in \mathfrak{F}$ and has an abelian $p$-chief factor) and $F(p) = (G \mid G/\mathrm{O}_p(G) \in f(p))$.

**Theorem 3.** *Suppose that $\mathfrak{F}$ is a non-empty solubly saturated formation such that* $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *holds for every group $G$, $F(p) \subseteq \mathfrak{F}_l$ for all $p \in \mathbb{P}$ and $\mathfrak{F}_l$ is hereditary. Then* $\mathrm{Int}_{\mathfrak{F}_l}(G) = \mathrm{Z}_{\mathfrak{F}_l}(G)$ *holds for every group $G$.*

**Remark 1.** The function $F$ of the form $\mathbb{P} \cup \{0\} \to \{\text{formations}\}$, where $\mathbb{P}$ is the set of all primes, such that $F(0) = \mathfrak{F}$ and $F(p)$ is the same as in Theorem 3 for every $p \in \mathbb{P}$ is called the *canonical composition definition* of $\mathfrak{F}$.

## 1 Preliminaries

The notation and terminology agree with the books [4, 7]. We refer the reader to these books for the results on formations. Recall that $G_{\mathfrak{S}}$ is the soluble radical of $G$, $\mathfrak{N}_p\mathfrak{F} = (G \mid G/\mathrm{O}_p(G) \in \mathfrak{F})$ is a formation for a formation $\mathfrak{F}$, $\pi(G)$ is the set of all prime divisors of $G$, $\pi(\mathfrak{X}) = \bigcup_{G\in\mathfrak{X}} \pi(G)$, $G$ is called $s$-critical for $\mathfrak{X}$ if all proper subgroups of $G$ are $\mathfrak{X}$-groups and $G \notin \mathfrak{X}$.

A function of the form $f : \mathbb{P} \to$ {*formations*} is called a *formation function*. Recall [4, Chapter IV, Definition 3.1] that a formation $\mathfrak{F}$ is called *local* if one has $\mathfrak{F} = (G \mid G/C_G(H/K) \in f(p)$ for every $p \in \pi(H/K)$ and every chief factor $H/K$ of $G$) for some formation function $f$. In this case $f$ is called a *local definition* of $\mathfrak{F}$. By the Gaschütz–Lubeseder–Schmid theorem, a formation is local if and only if it is non-empty and saturated. Recall [4, Chapter IV, Proposition 3.8] that if $\mathfrak{F}$ is a local formation, there exists a unique formation function $F$, defining $\mathfrak{F}$, such that $F(p) = \mathfrak{N}_pF(p) \subseteq \mathfrak{F}$ for every $p \in \mathbb{P}$. In this case $F$ is called the *canonical local definition* of $\mathfrak{F}$. Recall [13] that $\mathfrak{F}$ is said to satisfy the *boundary condition* if $\mathfrak{F}$ contains every group $G$ whose maximal subgroups all belong to $F(p)$ for some $p$.

**Lemma 1** ([4, Chapter IV, Proposition 3.16]). *Let $F$ be the canonical local definition of a local formation $\mathfrak{F}$. Then $\mathfrak{F}$ is hereditary if and only if $F(p)$ is hereditary for all $p \in \mathbb{P}$.*

**Lemma 2** ([7, Chapter 1, Proposition 1.15]). *Let $\mathfrak{F}$ be a local formation and let $F$ be its canonical local definition. Then a chief factor $H/K$ of $G$ is $\mathfrak{F}$-central if and only if $G/C_G(H/K) \in F(p)$ for all $p \in \pi(H/K)$.*

**Theorem 4** ([13, Theorem A]). *Let $\mathfrak{F}$ be a hereditary saturated formation. Then* $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *holds for every group $G$ if and only if $\mathfrak{F}$ satisfies the boundary condition.*

Suppose that $\mathfrak{F}$ is a normally hereditary solubly saturated formation. Then $G \in \mathfrak{F}$ if and only if $G = \mathrm{Z}_{\mathfrak{F}}(G)$ (see [7, Chapter 1, Theorem 2.6]). Also note that $\mathrm{Z}_{\mathfrak{F}}(G/\mathrm{Z}_{\mathfrak{F}}(G)) \simeq 1$.

Let $f$ be a function of the form $f : \mathbb{P} \cup \{0\} \to \{\text{formations}\}$. Recall [7, p. 4] that $\mathrm{CLF}(f) = (G \mid G/G_{\mathfrak{S}} \in f(0)$ and $G/C^p(G) \in f(p)$ for all $p \in \pi(G)$ such that $G$ has an abelian $p$-chief factor). If a formation $\mathfrak{F} = \mathrm{CLF}(f)$ for some $f$, then $\mathfrak{F}$ is called *composition* or *Baer-local*. A formation is solubly saturated if and only if it is composition (Baer-local) [4, Chapter IV, Theorem 4.17]. If $\mathfrak{F} = \mathrm{CLF}(f)$, then, as follows from [7, Chapter 5, Lemma 1.8] and [7, Chapter 1, Theorem 1.6], the function $F$ such that $F(0) = \mathfrak{F}$ and $F(p) = \mathfrak{N}_p(\mathfrak{F} \cap f(p))$ coincides with the canonical composition definition of $\mathfrak{F}$.

**Theorem 5** ([7, Chapter 1, Proposition 3.6 (b)]). *Suppose that $\mathfrak{F}$ is a saturated formation containing all nilpotent groups with the canonical local definition $F$. Then $\mathfrak{F}^* = \mathrm{CLF}(f)$, where $f(0) = \mathfrak{F}^*$ and $f(p) = F(p)$ for all $p \in \mathbb{P}$.*

**Theorem 6** ([2, Theorem 3.4.5]). *Let $F$ be the canonical composition definition of a non-empty solubly saturated formation $\mathfrak{F}$. Then $f$ is a local definition of $\mathfrak{F}_l$, where $f(p) = F(p)$ for all $p \in \mathbb{P}$.*

The following lemma directly follows from [10, Chapter X, Lemma 13.16 (a)].

**Lemma 3.** *Let a normal subgroup $N$ of $G$ be a direct product of isomorphic simple non-abelian groups. Then $N$ is a direct product of minimal normal subgroups of $G$.*

**Lemma 4.** *Let $\mathfrak{F}$ be a hereditary saturated formation. Then* $\mathrm{Z}_{\mathfrak{F}^*}(G) \leq \mathrm{Int}_{\mathfrak{F}^*}(G)$.

*Proof.* Let $\mathfrak{F}$ be a hereditary saturated formation with the canonical local definition $F$, let $M$ be an $\mathfrak{F}^*$-maximal subgroup of $G$ and $N = M\mathrm{Z}_{\mathfrak{F}^*}(G)$. We will show that $N \in \mathfrak{F}^*$. It is sufficient to show that for every chief factor $H/K$ of $N$ below $\mathrm{Z}_{\mathfrak{F}^*}(G)$ either $H/K$ is $\mathfrak{F}$-central in $N$ or every $x \in N$ induces an inner automorphism on $H/K$. Let $1 = Z_0 \trianglelefteq Z_1 \trianglelefteq \dots \trianglelefteq Z_n = \mathrm{Z}_{\mathfrak{F}^*}(G)$ be a chief series of $G$ below $\mathrm{Z}_{\mathfrak{F}^*}(G)$. Then we may assume that $Z_{i-1} \leq K \leq H \leq Z_i$ for some $i$ by the Jordan–Hölder theorem.

If $Z_i/Z_{i-1}$ is an $\mathfrak{F}$-central chief factor of $G$, then $G/C_G(Z_i/Z_{i-1}) \in F(p)$ for all $p \in \pi(Z_i/Z_{i-1})$ by Lemma 2. Since $F(p)$ is hereditary by Lemma 1, we have for all $p \in \pi(Z_i/Z_{i-1})$,

$$NC_G(Z_i/Z_{i-1})/C_G(Z_i/Z_{i-1}) \simeq N/C_N(Z_i/Z_{i-1}) \in F(p).$$

Note that $N/C_N(H/K)$ is a quotient group of $N/C_N(Z_i/Z_{i-1})$. Thus $H/K$ is an $\mathfrak{F}$-central chief factor of $N$ by Lemma 2.

If $Z_i/Z_{i-1}$ is an $\mathfrak{F}$-eccentric chief factor of $G$, then $G/C_G(Z_i/Z_{i-1}) \notin F(p)$ for some $p \in \pi(F(p))$ by Lemma 2. Hence $Z_i/Z_{i-1}$ is an $\mathfrak{F}$-eccentric chief factor of

$$(Z_i/Z_{i-1}) \rtimes G/C_G(Z_i/Z_{i-1}) \in \mathfrak{F}^*.$$

So every element of $G/C_G(Z_i/Z_{i-1})$ induces an inner automorphism on it. Therefore every element of $G$ induces an inner automorphism on $Z_i/Z_{i-1}$. It means that $Z_i/Z_{i-1}$ is a simple group. Hence it is also a chief factor of $N$. From $N \leq G$ it follows that every element of $N$ induces an inner automorphism on $Z_i/Z_{i-1}$.

Thus $N \in \mathfrak{F}^*$. So $N = M\mathrm{Z}_{\mathfrak{F}^*}(G) = M$. Therefore $\mathrm{Z}_{\mathfrak{F}^*}(G) \leq M$ for every $\mathfrak{F}^*$-maximal subgroup $M$ of $G$. □

## 2 Proofs of the main results

*Proof of Theorem* 2. Let $\mathfrak{N} \subseteq \mathfrak{F}$ be a hereditary saturated formation, and $F$ the canonical local definition of $\mathfrak{F}$. From Theorem 5 it follows that $F^*$ is the composition definition of $\mathfrak{F}^*$, where $F^*(0) = \mathfrak{F}^*$ and $F^*(p) = F(p)$ for all $p \in \mathbb{P}$. Since $F$ is the canonical local definition of $\mathfrak{F}$ and $\mathfrak{F} \subseteq \mathfrak{F}^*$, we see that $F^*$ is the canonical composition definition of $\mathfrak{F}^*$. Therefore $(\mathfrak{F}^*)_l$ is defined by $F$ by Theorem 6. Thus $(\mathfrak{F}^*)_l = \mathfrak{F}$. □

*Proof of Theorem* 3. If we set $F(0) = \mathfrak{F}$, then we may assume that $F$ is the canonical composition definition of $\mathfrak{F}$ by Remark 1. Now $f$ is a local definition of $\mathfrak{F}_l$, where $F(p) = f(p)$ for all $p \in \mathbb{P}$ by Theorem 6. From $f(p) = F(p) \subseteq \mathfrak{F}_l$ and $\mathfrak{N}_p f(p) = \mathfrak{N}_p F(p) = F(p) = f(p)$ it follows that $f$ is the canonical local definition of $\mathfrak{F}_l$. Note that $f(p)$ is hereditary for all $p \in \mathbb{P}$ by Lemma 1.

We show that $\mathrm{Int}_{\mathfrak{F}_l}(G) = \mathrm{Z}_{\mathfrak{F}_l}(G)$ holds for every group $G$. Assume the contrary. Then there exists an $s$-critical for $f(p)$ group $G \notin \mathfrak{F}_l$ for some $p \in \mathbb{P}$ by Theorem 4. We may assume that $G$ is a minimal group with this property. Then one has $\mathrm{O}_p(G) = \Phi(G) = 1$ and $G$ has a unique minimal normal subgroup by [13, Lemma 2.10]. Note that $G$ is also $s$-critical for $\mathfrak{F}_l$.

Assume that $G \notin \mathfrak{F}$. Then there exists a simple $\mathbb{F}_p G$-module $V$ which is faithful for $G$ by [4, Theorem 10.3B]. Let $T = V \rtimes G$. Note that $T \notin \mathfrak{F}$. Let $M$ be a maximal subgroup of $T$. If $V \leq M$, then $M = M \cap VG = V(M \cap G)$, where $M \cap G$ is a maximal subgroup of $G$. From $M \cap G \in f(p)$ and $f(p) = \mathfrak{N}_p f(p)$ it follows that $V(M \cap G) = M \in f(p) \subseteq \mathfrak{F}_l \subseteq \mathfrak{F}$. Hence $M$ is an $\mathfrak{F}$-maximal subgroup of $G$. If $V \not\leq M$, then $M \simeq T/V \simeq G \notin \mathfrak{F}_l$. Now it is clear that the sets of all maximal $\mathfrak{F}_l$-subgroups and all $\mathfrak{F}$-maximal subgroups of $T$ coincide. Thus $V$ is the intersection of all $\mathfrak{F}$-maximal subgroups of $T$. From $T \simeq V \rtimes T/C_T(V) \notin \mathfrak{F}$ it follows that $V \not\leq \mathrm{Z}_{\mathfrak{F}}(T)$, a contradiction.

Assume that $G \in \mathfrak{F}$. Let $N$ be a minimal normal subgroup of $G$. If $N$ is abelian, then $N$ is a $p$-group. Hence $G/C_G(N) \in F(p) = f(p)$. So $N$ is an $\mathfrak{F}_l$-central chief factor of $G$ by Lemma 2. Therefore $N \leq \mathrm{Z}_{\mathfrak{F}_l}(G)$. Since $N$ is a unique minimal normal subgroup of the $s$-critical for $\mathfrak{F}_l$ group $G$ and $\Phi(G) = 1$, we see that $G/N \in \mathfrak{F}_l$. Hence $G \in \mathfrak{F}_l$, a contradiction. Thus $N$ is non-abelian.

Let $p \in \pi(N)$. According to [6], there is a Frattini $\mathbb{F}_p G$-module $A$ which is faithful for $G$. By the known Gaschütz theorem [5], there exists a Frattini extension $A \rightarrowtail R \twoheadrightarrow G$ such that

$$A \overset{G}{\simeq} \Phi(R) \quad \text{and} \quad R/\Phi(R) \simeq G.$$

Assume that $R \in \mathfrak{F}$. Note that $C^p(R) \leq C_R(\Phi(R))$. So

$$R/\Phi(R) = R/C_R(\Phi(R)) \simeq G \in F(p) = f(p) \subseteq \mathfrak{F}_l,$$

a contradiction. Hence $R \notin \mathfrak{F}$.

Let $M$ be a maximal subgroup of $R$. Then $M/\Phi(R)$ is isomorphic to a maximal subgroup of $G$. So $M/\Phi(R) \in f(p)$. From $\mathfrak{N}_p f(p) = f(p)$ it follows that $M \in f(p) \subseteq \mathfrak{F}_l \subseteq \mathfrak{F}$. Hence the sets of maximal and $\mathfrak{F}$-maximal subgroups of $R$ coincide. Thus $\Phi(R) = \mathrm{Z}_{\mathfrak{F}}(R)$. From $R/\mathrm{Z}_{\mathfrak{F}}(R) \simeq G \in \mathfrak{F}$ it follows that $R \in \mathfrak{F}$, the final contradiction. □

*Proof of Theorem* 1. Assume that $\mathfrak{F}$ is a hereditary saturated formation containing all nilpotent groups with the canonical local definition $F$. Then $F(p)$ is a hereditary formation for all $p \in \mathbb{P}$ by Lemma 1.

Suppose that $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group $G$. We will show that $\mathrm{Int}_{\mathfrak{F}^*}(G) = \mathrm{Z}_{\mathfrak{F}^*}(G)$ also holds for every group $G$. Let $H/K$ be a chief factor of $G$ below $\mathrm{Int}_{\mathfrak{F}^*}(G)$.

**Step 1.** *If $H/K$ is an abelian group, then $MC_G(H/K)/C_G(H/K) \in F(p)$ for every $\mathfrak{F}^*$-maximal subgroup $M$ of $G$.*

If $H/K$ is abelian, then it is an elementary abelian $p$-group for some $p$ and $H/K \in \mathfrak{F}$. Let $M$ be an $\mathfrak{F}^*$-maximal subgroup of $G$ and let

$$K = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = H$$

be a part of a chief series of $M$. If $H_i/H_{i-1}$ is $\mathfrak{F}$-eccentric for some $i$, then every element of $M$ induces an inner automorphism on $H_i/H_{i-1}$. So

$$M/C_M(H_i/H_{i-1}) \simeq 1 \in F(p).$$

Therefore $H_i/H_{i-1}$ is an $\mathfrak{F}$-central chief factor of $M$, a contradiction. Hence $H_i/H_{i-1}$ is an $\mathfrak{F}$-central chief factor of $M$ for all $i = 1, \dots, n$. It follows from Lemma 2 that $M/C_M(H_i/H_{i-1}) \in F(p)$ for all $i = 1, \dots, n$. Therefore we have $M/C_M(H/K) \in \mathfrak{N}_p F(p) = F(p)$ by [14, Lemma 1]. Now

$$MC_G(H/K)/C_G(H/K) \simeq M/C_M(H/K) \in F(p)$$

for every $\mathfrak{F}^*$-maximal subgroup $M$ of $G$.

**Step 2.** *If $H/K \in \mathfrak{F}$ is non-abelian, then $MC_G(H/K)/C_G(H/K) \in F(p)$ for every $p \in \pi(H/K)$ and every $\mathfrak{F}^*$-maximal subgroup $M$ of $G$.*

If $H/K \in \mathfrak{F}$ is non-abelian, then it is a direct product of isomorphic non-abelian simple $\mathfrak{F}$-groups. Let $M$ be an $\mathfrak{F}^*$-maximal subgroup of $G$. By Lemma 3, $H/K = H_1/K \times \cdots \times H_n/K$ is a direct product of minimal normal subgroups $H_i/K$ of $M/K$. From $H_i/K \in \mathfrak{F}$ it follows that

$$(H_i/K)/\mathrm{O}_{p',p}(H_i/K) \simeq H_i/K \in F(p)$$

for all $p \in \pi(H_i/K)$ and all $i = 1, \dots, n$. Assume that $H_i/K$ is an $\mathfrak{F}$-eccentric chief factor of $M/K$ for some $i$. It means that every element of $M$ induces an inner automorphism on $H_i/K$. So $M/C_M(H_i/K) \simeq H_i/K \in F(p)$, a contradiction.

Now $H_i/K$ is the $F$-central chief factor of $M/K$ for all $i = 1, \dots, n$. Therefore $M/C_M(H_i/K) \in F(p)$ for all $p \in \pi(H_i/K)$ by Lemma 2. Note that

$$C_M(H/K) = \bigcap_{i=1}^{n} C_M(H_i/K).$$

Since $F(p)$ is a formation, we have

$$M/\bigcap_{i=1}^{n} C_M(H_i/K) = M/C_M(H/K) \in F(p)$$

for all $p \in \pi(H/K)$. It means that

$$MC_G(H/K)/C_G(H/K) \simeq M/C_M(H/K) \in F(p)$$

for every $p \in \pi(H/K)$ and every $\mathfrak{F}^*$-maximal subgroup $M$ of $G$.

**Step 3.** *If $H/K \in \mathfrak{F}$, then all $\mathfrak{F}$-subgroups of $G/C_G(H/K)$ are $F(p)$-groups for all $p \in \pi(H/K)$.*

Let $Q/C_G(H/K)$ be an $\mathfrak{F}$-maximal subgroup of $G/C_G(H/K)$. Then there is an $\mathfrak{F}$-maximal subgroup $N$ of $G$ with $NC_G(H/K)/C_G(H/K) = Q/C_G(H/K)$ by [7, Chapter 1, Lemma 5.7]. From the inclusion $\mathfrak{F} \subseteq \mathfrak{F}^*$ it follows that there exists an $\mathfrak{F}^*$-maximal subgroup $L$ of $G$ with $N \leq L$. So

$$Q/C_G(H/K) \leq LC_G(H/K)/C_G(H/K) \in F(p)$$

by Steps 1 and 2. Since $F(p)$ is hereditary, we have $Q/C_G(H/K) \in F(p)$. It means that all $\mathfrak{F}$-maximal subgroups of $G/C_G(H/K)$ are $F(p)$-groups. Hence all $\mathfrak{F}$-subgroups of $G/C_G(H/K)$ are $F(p)$-groups.

**Step 4.** *If $H/K \in \mathfrak{F}$, then it is $\mathfrak{F}$-central in $G$.*

Assume that $H/K$ is not $\mathfrak{F}$-central in $G$. So $G/C_G(H/K) \notin F(p)$ for some $p \in \pi(H/K)$ by Lemma 2. It means that $G/C_G(H/K)$ contains an $s$-critical for $F(p)$ subgroup $S/C_G(H/K)$ for some $p \in \pi(H/K)$. Since $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group $G$, it follows that $S/C_G(H/K) \in \mathfrak{F}$ by Theorem 4. Hence $S/C_G(H/K) \in F(p)$ by Step 3, a contradiction. Thus $H/K$ is $\mathfrak{F}$-central in $G$.

**Step 5.** *If $H/K \notin \mathfrak{F}$ is non-abelian, then every element of $G$ induces an inner automorphism on it.*

Let $M$ be an $\mathfrak{F}^*$-maximal subgroup of $G$. Then $H/K = H_1/K \times \cdots \times H_n/K$ is a direct product of minimal normal subgroups $H_i/K$ of $M/K$ by Lemma 3. Since $H_i/K \notin \mathfrak{F}$ for all $i = 1, \ldots, n$, it is an $\mathfrak{F}$-eccentric chief factor of $M$ for all $i = 1, \ldots, n$. So every element of $M$ induces an inner automorphism on $H_i/K$ for all $i = 1, \ldots, n$. Hence every element of $M$ induces an inner automorphism on $H/K = H_1/K \times \cdots \times H_n/K$.

It means that for every $\mathfrak{F}^*$-maximal subgroup $M$ of $G$ every element of $M$ induces an inner automorphism on $H/K$. Since $\mathfrak{N} \subseteq \mathfrak{F}$, $\langle x \rangle \in \mathfrak{F}$ for every $x \in G$. From $\mathfrak{F} \subseteq \mathfrak{F}^*$ it follows that for every element $x$ of $G$ there is an $\mathfrak{F}^*$-maximal subgroup $M$ of $G$ with $x \in M$. Thus every element of $G$ induces an inner automorphism on $H/K$.

**Step 6.** *The final step.*

If $H/K \in \mathfrak{F}$, then from $\mathfrak{F} \subseteq \mathfrak{F}^*$ and Step 4 it follows that $H/K$ is $\mathfrak{F}^*$-central in $G$. Assume that $H/K \notin \mathfrak{F}$. By Step 5 every element of $G$ induces an inner automorphism on it. Hence $H/K$ is a simple non-abelian group. Since $G/C_G(H/K)$ is isomorphic to some subgroup of $\mathrm{Inn}(H/K)$, we see that $G/C_G(H/K) \simeq H/K$. It is straightforward to check that $H/K \rtimes G/C_G(H/K) \simeq H/K \times H/K$. From $H/K \in \mathfrak{F}^*$ it follows that $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}^*$. Hence $H/K$ is $\mathfrak{F}^*$-central in $G$.

Thus every chief factor of $G$ below $\mathrm{Int}_{\mathfrak{F}^*}(G)$ is $\mathfrak{F}^*$-central in $G$. Hence we have $\mathrm{Int}_{\mathfrak{F}^*}(G) \leq \mathrm{Z}_{\mathfrak{F}^*}(G)$. According to Lemma 4, $\mathrm{Z}_{\mathfrak{F}^*}(G) \leq \mathrm{Int}_{\mathfrak{F}^*}(G)$. Therefore $\mathrm{Z}_{\mathfrak{F}^*}(G) = \mathrm{Int}_{\mathfrak{F}^*}(G)$.

Suppose that $\mathrm{Int}_{\mathfrak{F}^*}(G) = \mathrm{Z}_{\mathfrak{F}^*}(G)$ holds for every group $G$. Let $F^*$ be the canonical composition definition of $\mathfrak{F}^*$. From Theorem 2 and its proof it follows that $(\mathfrak{F}^*)_l = \mathfrak{F}$ and $F^*(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$. Hence $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group $G$ by Theorem 3. □

Now Corollary 1 directly follows from Theorem 1 when $\mathfrak{F} = \mathfrak{N}$.

**Remark 2.** Note that every $\mathfrak{N}$-central chief factor is central. From Steps 4 and 5 of the proof of Theorem 1 it follows that $\mathrm{Z}_{\mathfrak{N}^*}(G)$ is the greatest normal subgroup of $G$ such that every element of $G$ induces an inner automorphism on every chief factor of $G$ below $\mathrm{Z}_{\mathfrak{N}^*}(G)$.

## Final remarks

It is natural to ask if (2) of Question 1 can be reduced to (1) of Question 1. That is why A. F. Vasil'ev suggested the following question at the Gomel Algebraic Seminar in 2015:

**Question 2.** (1) Let $\mathfrak{H}$ be a normally hereditary saturated formation. Assume that $\mathrm{Int}_{\mathfrak{H}}(G) = \mathrm{Z}_{\mathfrak{H}}(G)$ holds for every group $G$. Describe all normally hereditary solubly saturated formations $\mathfrak{F}$ with $\mathfrak{F}_l = \mathfrak{H}$ such that $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group $G$.

(2) Let $\mathfrak{F}$ be a normally hereditary solubly saturated formation. Assume that $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group $G$. Does $\mathrm{Int}_{\mathfrak{F}_l}(G) = \mathrm{Z}_{\mathfrak{F}_l}(G)$ hold for every group $G$?

The partial answer to part (2) of Question 2 is given in Theorem 3. Note that the converse of Theorem 3 is false.

**Example 1.** Recall that $\mathfrak{N}_{\mathrm{ca}}$ is the class of groups whose abelian chief factors are central and non-abelian chief factors are simple groups. It is straightforward to check that $\mathfrak{N}_{\mathrm{ca}}$ is a normally hereditary solubly saturated formation (see [15]). Note that if $F$ is the canonical composition definition of $\mathfrak{N}_{\mathrm{ca}}$, then $F(p) = \mathfrak{N}_p$ for all $p \in \mathbb{P}$. It means that $(\mathfrak{N}_{\mathrm{ca}})_l = \mathfrak{N}$. Recall that $\mathrm{Int}_{\mathfrak{N}}(G) = \mathrm{Z}_{\mathfrak{N}}(G)$ holds for every group $G$.

Let $G \simeq D_4(2)$ be a Chevalley orthogonal group and let $H$ be an $\mathfrak{N}_{\mathrm{ca}}$-maximal subgroup of $\mathrm{Aut}(G)$. We may assume that $G \simeq \mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. Since $G$ is simple and $HG/G \in \mathfrak{N}_{\mathrm{ca}}$, we have $HG \in \mathfrak{N}_{\mathrm{ca}}$ by the definition of $\mathfrak{N}_{\mathrm{ca}}$. Hence $HG = H$. So $G$ lies in the intersection of all $\mathfrak{N}_{\mathrm{ca}}$-maximal subgroups of $\mathrm{Aut}(G)$. It is clear that $\mathrm{Aut}(G)/C_{\mathrm{Aut}(G)}(G) \simeq \mathrm{Aut}(G)$. In case that $G \leq \mathrm{Z}_{\mathfrak{N}_{\mathrm{ca}}}(\mathrm{Aut}(G))$, we obtain

$$G \rtimes (\mathrm{Aut}(G)/C_{\mathrm{Aut}(G)}(G)) \simeq G \rtimes \mathrm{Aut}(G) \in \mathfrak{N}_{\mathrm{ca}}.$$

Note that $\mathrm{Out}(G) \simeq S_3 \notin \mathfrak{N}_{\mathrm{ca}}$ is the quotient group of $G \rtimes \mathrm{Aut}(G)$. Therefore $G \rtimes \mathrm{Aut}(G) \notin \mathfrak{N}_{\mathrm{ca}}$. Thus $G \not\leq \mathrm{Z}_{\mathfrak{N}_{\mathrm{ca}}}(\mathrm{Aut}(G))$ and $\mathrm{Int}_{\mathfrak{N}_{\mathrm{ca}}}(\mathrm{Aut}(G)) \neq \mathrm{Z}_{\mathfrak{N}_{\mathrm{ca}}}(\mathrm{Aut}(G))$.

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