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On σ-Solubility Criteria for Finite Groups

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Abstract
In this paper, the structure of finite groups in which maximal subgroups of some Sylow subgroups have a σ-soluble or σ-nilpotent supplement, where σ is a partition of the set of all prime numbers, is investigated. Some solubility, σ-solubility and σ-nilpotency criteria leading to some significant improvements of earlier results are given.

Keywords Finite group · Supplementation · σ-Solubility · σ-Nilpotency

Mathematics Subject Classification 20D10 · 20D20

1 Introduction

A subgroup B of a group G is supplemented in G if there exists a subgroup A of G such that $G = AB$; in this case, we say that A is a supplement of B in G. For every subgroup of G there is always the trivial supplement $A = G$. Non-trivial supplements of some distinguished families of subgroups of G may be useful in determining the structure

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of G. For instance, if B is normal in G, the existence of a non-trivial supplement of B in G could be important for the extension property, i.e., for the description of G by means of B and the factor group G/B. Generally, a supplement A is the more useful the smaller the intersection $A \cap B$. If we have even that $A \cap B = 1$, then A is called a complement of B in G. While groups which satisfy certain complementation properties have been extensively studied, less has been done to investigate groups which satisfy certain supplementation properties.

The topic of this paper is an investigation of supplementation in finite groups. Therefore, all groups considered in the paper will be finite. To make our results more precise, we introduce the following notions according to the notation of Skiba in [8].

Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set of all primes $\mathbb{P}$, i.e., $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

Definition 1.1 A group G is said to be σ-primary if G is a σ_i-group for some $i \in I$. We say that G is σ-soluble if every chief factor of G is σ-primary. G is said to be σ-nilpotent if it is a direct product of σ-primary groups.

Note that a group G is soluble (respectively, nilpotent) if and only if, it is σ-soluble (respectively, σ-nilpotent) for the partition $\sigma = \{\{2\}, \{3\}, \{5\}, \ldots\}$. If π is a set of primes and $\sigma = \{\pi, \pi'\}$, then a group G is σ-soluble if and only if, G is π-separable. In this case, G is σ-nilpotent if and only if, G is π-decomposable. If $\pi = \{p_1, \ldots, p_n\}$, and $\sigma = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$, then G is σ-soluble if and only if, G is π-soluble, and furthermore, G is σ-nilpotent if and only if, G has a normal Hall π'-subgroup and a normal Sylow p_i-subgroup, for all $i = 1, \ldots, n$.

In the sequel, σ will be a partition of the set of all prime numbers.

Guo et al. proved in [3] that a group G is nilpotent (respectively, supersoluble) if every maximal subgroup of every Sylow subgroup of G has a nilpotent (respectively, supersoluble) supplement in G. This result motivated the following question which appears as Problem 19.87 in Kourovka Notebook [7]:

> Suppose that every maximal subgroup of every Sylow subgroup of a group G have a σ-soluble supplement in G. Is G σ-soluble?

This question was answered affirmatively in [5, 6].

Theorem 1.2 *Suppose that every maximal subgroup of every Sylow subgroup of a group G have a σ-soluble supplement in G. Then G is σ-soluble.*

The goal of this paper is to take these studies further. In particular, a significant improvement of Theorem 1.2 (Theorem 1.6) is showed. Our first main result is a solubility criteria in terms of supplements.

Theorem 1.3 *Let p be a prime dividing the order of a group G, and $P \in Syl_p(G)$. Assume that $p \notin \{7, 13\}$. If every maximal subgroup of P has a soluble supplement in G, then G is soluble.*

The hypothesis $p \notin \{7, 13\}$ in Theorem 1.3 cannot be removed (see Lemma 2.2 below).

Our second main result provides sufficient conditions for the σ-solubility of a group G in which every maximal subgroup of a Sylow p-subgroup of G, for a fixed prime p, has a σ-soluble supplement.

Theorem 1.4 *Assume that p is a prime number dividing the order of a group G, and $P \in Syl_p(G)$. Suppose that G has no composition factors isomorphic to $L_n(q)$, where n is a prime and $(q^n - 1)/(q - 1)$ is a prime-power number. If every maximal subgroup of P has a σ-soluble supplement in G, then G is σ-soluble.*

Corollary 1.5 *Let π be a set of primes. Assume that p is a prime dividing the order of a group G, and $P \in Syl_p(G)$. Suppose that G has no composition factors isomorphic to $L_n(q)$, where n is a prime and $(q^n - 1)/(q - 1)$ is a prime-power number. If every maximal subgroup of P has a π-soluble supplement in G, then G is π-soluble.*

Our third main result shows that it is enough to assume the existence of σ-soluble supplements of the maximal subgroups of the Sylow subgroups corresponding to two primes in Theorem 1.2 to guarantee σ-solubility.

Theorem 1.6 *Assume that p and q are two different primes dividing the order of a group G, and $P \in Syl_p(G)$, $Q \in Syl_q(G)$. If every maximal subgroup of P and Q has a σ-soluble supplement in G, then G is σ-soluble.*

Corollary 1.7 *Let π be a set of primes. Assume that p and q are two different primes dividing the order of a group G, and $P \in Syl_p(G)$, $Q \in Syl_q(G)$. If every maximal subgroup of P and Q has a π-soluble supplement in G, then G is π-soluble.*

We end the paper with a σ-nilpotent version of Theorem 1.6.

Theorem 1.8 *Assume that p and q are two different primes dividing the order of a group G, and $P \in Syl_p(G)$, $Q \in Syl_q(G)$. If every maximal subgroup of P and Q has a σ-nilpotent supplement in G, then G is σ-nilpotent.*

2 Preliminaries

Our first lemma collects some basic properties which are useful in induction arguments.

Recall that a formation is a class of groups $\mathfrak{F}$ which is closed under taking homomorphic images and subdirect products. We say that $\mathfrak{F}$ is subgroup-closed if every subgroup of every group in $\mathfrak{F}$ is also in $\mathfrak{F}$; $\mathfrak{F}$ is saturated if it is closed under taking Frattini extensions.

A subgroup T of a group G is said to be an $\mathfrak{F}$-supplement of a subgroup V of G if T is a supplement of V in G and $T \in \mathfrak{F}$.

Lemma 2.1 *Let $\mathfrak{F}$ be a subgroup-closed formation. Let N be a normal subgroup of a group G and $p \in \pi(G)$. Suppose that every maximal subgroup of a Sylow p-subgroup P of G has an $\mathfrak{F}$-supplement in G. Then:*

1. *If $p \notin \pi(G/N)$, then G/N is an $\mathfrak{F}$-group.*

2. *If $p \in \pi(G/N)$, then every maximal subgroup of PN/N has an $\mathfrak{F}$-supplement in G/N.*
3. *If H is a subgroup of G containing P, then every maximal subgroup of P has an $\mathfrak{F}$-supplement in H.*

Proof (1) If $p \notin \pi(G/N)$, then P is contained in N. Let V be a maximal subgroup of P. By hypothesis, there exists an $\mathfrak{F}$-subgroup T of G such that $G = VT$. Then, $G = NT$ and so $G/N = TN/N \cong T/T \cap N \in \mathfrak{F}$.

(2) Let W/N be a maximal subgroup of PN/N. Then, $W = W \cap PN = (W \cap P)N$ and $W \cap P$ is a maximal subgroup of P. By hypothesis, there exists an $\mathfrak{F}$-subgroup T of G such that $G = (W \cap P)T$. Hence, $G/N = (W \cap P)T/N = ((W \cap P)N/N)(TN/N) = (W/N)(TN/N)$ and $TN/N \cong T/T \cap N \in \mathfrak{F}$. Therefore, TN/N is an $\mathfrak{F}$-supplement of W/N in G/N.

(3) Let V be a maximal subgroup of P and let T be an $\mathfrak{F}$-supplement of V in G. Then, $H = V(H \cap T)$ and $H \cap T \in \mathfrak{F}$ because $\mathfrak{F}$ is subgroup-closed. Therefore, every maximal subgroup of P has an $\mathfrak{F}$-supplement in H. □

The following consequence of Guralnick's classification of non-abelian simple groups with a subgroup of prime-power index [4, Theorem 1] turns out to be crucial.

Lemma 2.2 *Suppose that G is a non-abelian simple group and H is a soluble subgroup of G of prime-power index p^k, $k \geq 1$. Then, one of the following holds:*

1. *$G = A_5$, $H \cong A_4$ and $p^k = 5$.*
2. *$G = L_2(8)$, H is a Borel subgroup of G and $p^k = 9$.*
3. *$G = L_2(q)$ with some prime $q \geq 5$, H is a Borel subgroup of G and $p^k = q + 1$ is a 2-power.*
4. *$G = L_2(2^{2^m})$ with some $m \geq 2$, H is a Borel subgroup of G, $k = 1$ and p is a Fermat prime.*
5. *$G = L_3(2)$, and either $H \cong S_4$ and $p^k = 7$ or $H \cong 7 : 3$ and $p^k = 8$.*
6. *$G = L_3(3)$, H is a parabolic subgroup of G and $p^k = 13$.*

It is not difficult to see that in the above lemma, H is a Hall p'-subgroup of G and a maximal subgroup of G. Moreover, G has a single conjugacy class of Hall p'-subgroups in cases (1)–(4) and case (5) and $p = 2$, and two conjugacy classes of Hall p'-subgroups in case (5) and $p = 7$ and case (6) (see [1]).

Lemma 2.3 *Assume that a group N is a direct product of subgroups that are isomorphic to a non-abelian simple group. Suppose that N is not σ-soluble and has a σ-soluble subgroup X such that $|N : X|$ is a power of a prime p. Then, X is a Hall p'-subgroup of N.*

Proof Assume that $N = N_1 \times N_2 \times \cdots \times N_t$, where $t \geq 1$ and $N_1, N_2, \ldots, N_t$ are isomorphic copies of a non-abelian simple group.

Let $f_i : N \to N_i$ be the projection of N onto N_i, and write $X_i = f_i(X)$, $i = 1, 2, \ldots, t$. Since the subgroup X is σ-soluble, we have that X_i is a proper subgroup of N_i for every $1 \leq i \leq t$, by [8, Lemma 2.1]. Obviously, $X \subseteq X_1 X_2 \ldots X_t$. Since $|N : X| = p^l$, for some $l \geq 1$, it follows that $|N : X_1 X_2 \ldots X_t|$ is a power of p.

Now $|N : X_1X_2 \dots X_t| = (|N_1||N_2|\dots|N_t|)/(|X_1||X_2|\dots|X_t|) = |N_1 : X_1||N_2 : X_2|\dots|N_t : X_t|$. Hence, $|N_i : X_i|$ is a power of p, for every $i = 1, 2, \dots, t$. By [4, Theorem 1], for every $1 \le i \le t$, it follows that either X_i is a Hall p'-subgroup of N_i or N_i and X_i satisfy one of the following cases:

1. $N_i = A_n$ and $X_i \cong A_{n-1}$ with $n = p^a$, for certain p prime and $a > 1$.
2. $N_i = \mathrm{PSU}_4(2)$ and X_i is a parabolic subgroup of index 27.

Assume, arguing by contradiction, that one of both cases holds for some $1 \le i \le t$. If Case 1 holds, then X_i is σ-primary as it is simple. Since $\pi(N_i) = \pi(X_i)$, we also have that N_i is σ-primary and therefore N is σ-soluble, contrary to our assumption. On the other hand, if case 2 holds, then X_i is isomorphic to $2^4 : A_5$ by [1]. Then, we have that A_5 is σ-primary as X_i is σ-soluble. Since $\pi(N_i) = \pi(A_5)$, we also get that N_i is σ-primary which implies that N is σ-soluble, contrary to our assumption.

Hence, for every $1 \le i \le t$, X_i is a Hall p'-subgroup of N_i. Therefore $X = X_1X_2 \dots X_t$ is a Hall p'-subgroup of N. □

3 The Proof of Theorem 1.3 and Some Consequences

Proof of Theorem 1.3 Assume that the result is not true and let G be a counterexample of minimal order. Then $G \neq 1$. Let p be a prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Let V be a maximal subgroup of P. By hypothesis, there exists a soluble subgroup T of G such that $G = VT$. Then $|G : T| = p^k$ for some $k \ge 1$, and so every Hall p'-subgroup of T is a Hall p'-subgroup of G.

Assume that G is a simple group. Then, G is non-abelian. By Lemma 2.2, T is a Hall p'-subgroup of G. Then, $P = V(T \cap P) = V$, contrary to our supposition.

Consequently, G is not simple. Let N be a minimal normal subgroup of G. Then, N is a proper subgroup of G. Assume that p does not divide $|G/N|$. Then, G/N is soluble by Statement (1) of Lemma 2.1. Furthermore, N contains P and so N satisfies the hypotheses of the theorem by Statement (3) of Lemma 2.1. By minimality of G, N is soluble. Hence, G is soluble, which is a contradiction. Therefore, p divides $|G/N|$. By Statement (2) of Lemma 2.1, G/N satisfies the hypotheses of the theorem. The minimal choice of G implies that G/N is soluble. Since G is not soluble, it follows that N is non-abelian.

Since the class of all soluble groups is a formation, $N = \mathrm{Soc}(G)$ is a minimal normal subgroup of G. By Statement (3) of Lemma 2.1, PN satisfies the hypotheses of the theorem. Since PN cannot be soluble because N is non-abelian, it follows that $G = PN$. Note that $T \cap N$ is a soluble subgroup of N such that $|N : T \cap N| = p^l$ for some $l \ge 1$. By Lemma 2.3, $X = T \cap N$ is a Hall p'-subgroup of N. Assume that $N = N_1 \times N_2 \times \cdots \times N_t$, where $t \ge 1$ and $N_1, N_2, \dots, N_t$ are isomorphic copies of a non-abelian simple group. Then, $X_i = X \cap N_i$ is a Hall p'-subgroup of N_i for all $i = 1, 2, \dots, t$. Since $p \notin \{7, 13\}$, we can apply Lemma 2.2 to conclude that $N_i \cong A_5$ or $N_i \cong L_2(8)$ or $N_i \cong L_2(q)$, with $q > 3$ or $N_i \cong L_2(2^{2^m})$ for some $m \ge 2$ or $N_i \cong L_3(2)$ with $p = 2$. Hence, N_i has a single conjugacy class of Hall p'-subgroups. Consequently, N has a single conjugacy class of Hall p'-subgroups.

Since $G = PN$, every Hall p'-subgroup of N is a Hall p'-subgroup of G and G has also a single conjugacy class of Hall p'-subgroups.

Let W be any maximal subgroup of P. Then, by hypothesis, G has a proper soluble subgroup S such that $G = WS$. Then, as before, we have that $S \cap N$ is a soluble subgroup of N such that $|N : S \cap N|$ is a power of p. By Lemma 2.3, $S \cap N$ is a Hall p'-subgroup of N and a Hall p'-subgroup of G. Thus, $T \cap N = (S \cap N)^g$ for some element $g \in G$. Because $G = (T \cap N)P$, we can assume that $g \in P$. Since $S \subseteq N_G(S \cap N)$ and W is normal in P, we have that

$$\begin{aligned} G = G^g = (WS)^g = (WN_G(S \cap N))^g = W^g(N_G(S \cap N))^g \\ = W^g(N_G((S \cap N)^g) = WN_G(T \cap N). \end{aligned}$$

Since $G = VT$, $G = VN_G(T \cap N)$ and

$$|G| = (|N_G(T \cap N)| \cdot |P|)/|N_G(T \cap N) \cap P|),$$

we have that $N_G(T \cap N) \cap P$ is a Sylow subgroup of $N_G(T \cap N)$. Assume that $N_G(T \cap N) \cap P$ is a proper subgroup of P. Let U be a maximal subgroup of P containing $N_G(T \cap N) \cap P$. Then, $G = UN_G(T \cap N) = U(T \cap N)$. This shows $P = U$, which is a contradiction. Hence, $N_G(T \cap N) \cap P = P$ and so $T \cap N$ is a normal subgroup of G. Hence, $N \subseteq T$ and N is soluble. This contradiction completes the proof of the theorem. □

Corollary 3.1 *Let p and q be two different primes dividing the order of a group G. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. If every maximal subgroup of P and Q has a soluble supplement in G, then G is soluble.*

Proof We adhere closely to the pattern of the proof of Theorem 1.3.

Assume that the result is not true and let G be a counterexample of minimal order. By Theorem 1.3, $\{p, q\} = \{7, 13\}$. Let N be a minimal normal subgroup of G. Applying Lemma 2.1 and minimality of G, we have that G/N is soluble. Consequently, $N = \mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G.

Let V and W be maximal subgroups of P and Q, respectively. Then, there exists soluble subgroups S and T of G such that $G = VS = WT$. By Lemma 2.3, $X = S \cap N$ is a Hall p'-subgroup of N and $Y = T \cap N$ is a Hall q'-subgroup of N. Therefore, if A is a non-abelian simple group which is a direct factor of N, it follows that A has a subgroup of p-power index and a subgroup of q-power index. This contradicts Lemma 2.2. □

Theorem 1.3 and the methods used in the proof of Corollary 3.1 enable us to prove the following results.

Recall that a group X is S_4-*free* if the symmetric group of degree 4 does not appear as a quotient of any subgroup of X.

Corollary 3.2 *Let $\mathfrak{F}$ be a soluble subgroup-closed formation. Assume that $\mathfrak{F}$ is composed of either S_4-free groups or 3-closed groups. Let p be a prime dividing the order of a group G, and $P \in Syl_p(G)$. If any maximal subgroup of P has an $\mathfrak{F}$-supplement in G, then G is soluble.*

Corollary 3.3 *Let p be a prime dividing the order of a group G, and $P \in Syl_p(G)$. If every maximal subgroup of P has a supersoluble supplement in G, then G is soluble.*

The group G in Corollary 3.3 is not supersoluble in general as the following example shows.

Example 3.4 Let Q be the quaternion group of order 8 and let V be a faithful and irreducible Q-module over the field of 5 elements. Let $G = [V]Q$ be the corresponding semidirect product. Let W be a maximal subgroup of Q and let T be a maximal subgroup of G containing V such that $G = WT$. It is clear that T is supersoluble. Therefore, G satisfies the hypotheses of Corollary 3.3 for the prime $p = 2$. However, G is not supersoluble because V is a minimal normal subgroup of G of order 25.

However, for the subgroup-closed formation of all nilpotent groups, we have the following result.

Corollary 3.5 *Let p be a prime number dividing the order of a group G, and $P \in Syl_p(G)$. If every maximal subgroup of P has a nilpotent supplement in G, then G is nilpotent.*

Proof Assume that the result is false and let G be a counterexample of minimal order. By Corollary 3.3, G is soluble. Let N be a minimal normal subgroup of G. By Lemma 2.1 and the minimal choice of G, it follows that G/N is nilpotent. Therefore, $N = \mathrm{Soc}(G)$ is an abelian minimal normal subgroup of G and $\mathrm{C}_G(N) = N$. Let q be a prime dividing $|N|$ and let M be a maximal subgroup of G such that $G = NM$ and $N \cap M = 1$. Assume that $p = q$. Since N is a faithful and irreducible M-module and M is nilpotent, it follows that M is a p'-group by [2, Lemma A. 13.6]. Hence, $N = P$. Let A be a maximal subgroup of N and let B be a nilpotent supplement of A in G. Then, $G = NB$ and so B is a maximal subgroup of G. Moreover, $B \cap N = 1$. Hence, $N = A(B \cap N) = A$, which is a contradiction. Therefore, $p \neq q$. Let V be a maximal subgroup of P. Then, by hypothesis, $G = VT$ for some nilpotent subgroup of G. Note that N is contained in T and so the Hall q'-subgroup of T is contained in $\mathrm{C}_T(N) = N$. Hence, T is a q-group. Then, $P = V(P \cap T) = V$. This final contradiction completes the proof. □

4 The Proof of Theorem 1.4

Proof of Theorem 1.4 Suppose that the result is false. Let G be a counterexample of the smallest possible order.

Let V be a maximal subgroup of P and let T be a σ-soluble supplement of V in G. Then, $|G : T| = p^k$ for some $k \geq 1$.

Assume that G is a non-abelian simple group. Then, by [4, Theorem 1] and [1], it follows that either T is a Hall p'-subgroup of G or $T/S(T)$ is a non-abelian simple group, with $\pi(G) = \pi(T/S(T))$ (here, $S(T)$ is the soluble radical of T). If T were a Hall p'-subgroup of G, we would have $P = V(P \cap T) = T$ and if $T/S(T)$ were non-abelian simple and $\pi(G) = \pi(T/S(T))$, G would be σ-soluble. In both cases, we get a contradiction. Therefore, G is not simple.

Note that the class of all σ-soluble groups is an extensible subgroup-closed formation by [8, Lemma 2.1]. Let N be a minimal normal subgroup of G. Applying Lemma 2.1 and minimality of G, we have that G/N is σ-soluble. Consequently, $N = \mathrm{Soc}(G)$ is a non-abelian and non-σ-soluble minimal normal subgroup of G. Furthermore, $G = PN$ by Statement (3) of Lemma 2.1. Since $|N : N \cap T|$ is a power of p and N is not contained in T, it follows that $p \in \pi(N)$.

Suppose that $N = N_1 \times N_2 \times \cdots \times N_t$, where $t \geq 1$ and $N_1, N_2, \ldots, N_t$ are isomorphic copies of a non-abelian simple group. Following the proof of Lemma 2.3, we get that $T_1 = T \cap N_1$ is a σ-soluble subgroup of N_1 of p-power index. Applying [4, Theorem 1], we have that N_1 is one of the following groups.

1. $N_1 = A_n$ and $T_1 \cong A_{n-1}$, where $n = p^k$;
2. $N_1 = L_2(11)$ and $T_1 \cong A_5$;
3. $N_1 = M_{23}$ and $T_1 \cong M_{22}$ or $N_1 = M_{11}$ and $T_1 \cong M_{10}$;
4. $N_1 = PSU_4(2) \cong PSp_4(3)$ and T_1 is the parabolic subgroup of index 27.

Assume that $N_1 \cong A_n$ and $T_1 \cong A_{n-1}$, where $n = p^k > p$. Then, T_1 is a σ-soluble simple group. Therefore, T_1 is a σ_i-group, where σ_i is the member of σ such that $2 \in \sigma_i$. Since $\pi(N) = \pi(N_1) = \pi(T_1)$, it follows that N is σ-soluble. This is not possible.

Assume that N_1 is a group of type (4). By [1], $T_1 \cong 2^4 : A_5$. Since T_1 is σ-soluble, it follows that A_5 is a σ_i-group. But $\pi(N) = \pi(N_1) = \pi(A_5)$. Therefore, N is σ-soluble. This is a contradiction.

Assume that either $N_1 = A_n$, where n is a prime, or $N_1 = L_2(11)$, or $N_1 = M_{23}$, or $N_1 = M_{11}$. Then, T_1 is a Hall p'-subgroup of N_1 and, by [4, Theorem 1], N_1 has a single conjugacy class of Hall p'-subgroups. By Lemma 2.3, $T \cap N$ is a Hall p'-subgroup of N and all of them are conjugate in N. We can now argue as in Theorem 1.3 to show that $T \cap N$ is a normal subgroup of G. This final contradiction completes the proof. □

5 The Proof of Theorems 1.6 and 1.8

Proof of Theorem 1.6 We argue by induction on $|G|$. Let N be a minimal normal subgroup of G. By Lemma 2.1, G/N is σ-soluble. Assume that N is non-abelian and non-σ-soluble. The hypotheses on G imply that N has σ-soluble subgroups X and Y of p-power index and q-power index, respectively. Let N_1 be a non-abelian direct factor of N. Then, following the proof of Lemma 2.3, we have that N_1 has subgroups T_1 and T_2 of p-power index and q-power index, respectively. By [4, Theorem 1], $N_1 \cong L_3(2)$, $T_1 \cong S_4$ and $T_2 \cong 7 : 3$, and $p = 7$ and $q = 2$.

Let Q be a Sylow 2-subgroup of G, and consider the subgroup QN. Let V be a maximal subgroup of Q. By hypothesis and Lemma 2.1, there exists a σ-soluble subgroup S such that $VS = QN$. By Lemma 2.3, $S \cap N$ is a soluble Hall $2'$-subgroup of QN which is a direct product of subgroups isomorphic to $7 : 3$. Therefore, S is soluble. By Theorem 1.3, QN is soluble. This contradiction shows that N is σ-soluble. Applying [8, Lemma 2.1], we have that G is σ-soluble, as desired. □

Proof of Theorem 1.8 Assume the theorem is false and let a counterexample G of smallest order be chosen. According to [8, Corollary 2.4 and Lemma 2.5], the class of all σ-nilpotent groups is a subgroup-closed saturated formation. Therefore, by Lemma 2.1, G is a primitive group. Let $N = \mathrm{Soc}(G)$ be the minimal normal subgroup of G. By Theorem 1.6, G is σ-soluble. Consequently, N is a σ_i-group for some $\sigma_i \in \sigma$.

Since N has σ-nilpotent subgroups of p-power index and q-power index, we can argue as in Theorem 1.6 to conclude that N is soluble. Then, N is r-elementary abelian for some prime $r \in \sigma_i$, and $\mathrm{C}_G(N) = N$.

Assume that $p \notin \sigma_i$. Let V a maximal subgroup of P and let T be a σ-nilpotent supplement of V in G. Then, T contains N. Following the proof of Corollary 3.5, since $\mathrm{C}_G(N) = N$, the Hall σ_i'-subgroup of T is trivial and so T is a σ_i-group. Then, $T \cap P = 1$ and $P = V$, which is a contradiction. Hence, $p \in \sigma_i$. Analogously, $q \in \sigma_i$.

Note that either $r \neq p$ or $r \neq q$. Suppose the latter case holds and let W be a maximal subgroup of Q. There exists a σ-nilpotent subgroup S of G such that $G = WS$. Since N is contained in S and $\mathrm{C}_G(N) = N$, as before, it follows that S is a σ_i-subgroup of G. Therefore, G is a σ_i-group. But this leads to the contradiction that G is σ-nilpotent. □

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