Bull. Malays. Math. Sci. Soc.
https://doi.org/10.1007/s40840-022-01369-y

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Finite Groups with σ-Subnormal Schmidt Subgroups

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Received: 11 April 2022 / Revised: 16 July 2022 / Accepted: 25 July 2022

Abstract

If $\sigma = \{\sigma_i : i \in I\}$ is a partition of the set $\mathbb{P}$ of all prime numbers, a subgroup H of a finite group G is said to be σ-*subnormal* in G if H can be joined to G by means of a chain of subgroups $H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ such that either H_{i-1} normal in H_i or $H_i / \mathrm{Core}_{H_i}(H_{i-1})$ is a σ_j-group for some $j \in I$, for every $i = 1, \ldots, n$. If $\sigma = \{\{2\}, \{3\}, \{5\}, ...\}$ is the minimal partition, then the σ-subnormality reduces to the classical subgroup embedding property of subnormality. A finite group X is said to be a *Schmidt group* if X is not nilpotent and every proper subgroup of X is nilpotent. Every non-nilpotent finite group G has Schmidt subgroups and a detailed knowledge of their embedding in G can provide a deep insight into its structure. In this paper, a complete description of a finite group with σ-subnormal Schmidt subgroups is given. It answers a question posed by Guo, Safonova and Skiba.

Keywords Finite group · Schmidt subgroup · σ-subnormal subgroup

Mathematics Subject Classification 20D10 · 20D20

Communicated by Peyman Niroomand.

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Published online: 08 August 2022

1 Introduction and Statements of Results

All groups considered in this paper are finite.

Recall that a group G is said to be a *minimal non-nilpotent* group or *Schmidt* group if G is not nilpotent and every proper subgroup of G is nilpotent. It is clear that every non-nilpotent group contains Schmidt subgroups and so a detailed knowledge of their embedding in the group can provide deep insight into its structure.

The following classical result describes the structure of Schmidt groups.

Lemma 1 *([3, 8]) Let S be a Schmidt group. Then S satisfies the following properties:*

1. *The order of S is divisible by exactly two prime numbers p and q;*
2. *S is a semidirect product $S = [P]\langle a\rangle$, where P is a normal Sylow p-subgroup of S, and $\langle a\rangle$ is a non-normal Sylow q-subgroup of S and $\langle a^q\rangle \subseteq \mathrm{Z}(S)$;*
3. *P is the nilpotent residual of S, i.e., the smallest normal subgroup of S with nilpotent quotient;*
4. *$P/\Phi(P)$ is a non-central chief factor of S, and $\Phi(P) = P^{'} \subseteq \mathrm{Z}(S)$;*
5. *$\Phi(S) = \mathrm{Z}(S) = P' \times \langle a^q\rangle$;*
6. *$\mathrm{C}_P(a) = \Phi(P)$;*
7. *if $\mathrm{Z}(S) = 1$, then $|S| = p^m q$, where m is the order of p modulo q.*

Knyagina and Monakhov proved in [6] that a group G is metanilpotent if every Schmidt subgroup of G is subnormal and Vedernikov [10] proved that the commutator subgroup of such a group G is nilpotent, and obtained the complete description of groups with all Schmidt subgroups subnormal.

Skiba [9] extended the concept of subnormality introducing σ-subnormality associated with a partition σ of the set $\mathbb{P}$, the set of all primes. Hence, $\mathbb{P} = \bigcup_{i\in I}\sigma_i$, with $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We say that σ is a *binary* partition if $\sigma = \{\pi, \pi^{'}\}$ for some set π of primes.

From now on let σ denote a partition of $\mathbb{P}$.

A group G is called σ*-primary* if the prime divisors of $|G|$ all belong to the same member of σ.

Definition 1 A subgroup H of a group G is called σ-subnormal in G if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G,$$

where, for every $i = 1, \ldots, n$, H_{i-1} normal in H_i or $H_i/Core_{H_i}(H_{i-1})$ is σ-primary.

It is rather clear that a subgroup H is subnormal in a group G if and only if it is σ-subnormal in G for the *minimal* partition $\sigma = \{\{2\}, \{3\}, \{5\}, \ldots\}$.

Bearing in mind the above-mentioned results and the strong structural impact of the σ-subnormality, the following interesting problem posed by Guo, Safonova and Skiba in [4, Problem 2.10] naturally emerges.

Problem 1 *Describe the groups whose Schmidt subgroups are σ-subnormal.*

This problem is closely related to the Question 4.7 proposed by Skiba in [9] and its solution will give us clues on how to solve Skiba's question.

The aim of the present paper is to solve Problem 1.

Our first main result shows that Problem 1 can be reduced to binary partitions. As usual, we denote $\pi(G)$ the set of all primes dividing the order of G, and $\sigma(G) = \{\sigma_i : \sigma_i \cap \pi(G) \neq \emptyset, i \in I\}$.

Theorem 1 *Let G be a group and suppose that $\sigma(G) = \{\sigma_1, \sigma_2, ..., \sigma_k\}$, $k \geq 2$. Then all Schmidt subgroups of G are σ-subnormal in G if and only if the following statements hold:*

1. *every Schmidt $(\sigma_i \cup \sigma_j)$-subgroup of G is contained in $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$, for all $i \neq j \in \{1, 2, ..., k\}$;*
2. *every Schmidt subgroup of $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$ is $\{\sigma_i, \sigma_i^{'}\}$-subnormal in $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$, for all $i \neq j \in \{1, 2, ..., k\}$.*

Our next two main results solve Problem 1 for binary partitions. In order to state them, we need to introduce some notation.

Following [6], we denote by $S_{\langle p,q\rangle}$ the class of all Schmidt groups with a normal Sylow p-subgroup and a non-normal cyclic Sylow q-subgroup.

An $F_{<p,d>}$-group is a Frobenius group whose kernel is an elementary abelian group of order p^m with cyclic complement of order d, where m is the exponent of p modulo q, for any prime $q \in \pi(d)$, where $\pi(d)$ is the set of all primes dividing d.

A chief factor H/K of a group G is said to be *σ-central* in G if the semidirect product $[H/K](G/\mathrm{C}_G(H/K))$ is σ-primary. G has a largest normal subgroup $\mathrm{Z}_\sigma(G)$ such that every chief factor of G below $\mathrm{Z}_\sigma(G)$ is σ-central in G; $\mathrm{Z}_\sigma(G)$ is called the *σ-hypercentre* of G. If σ is the minimal partition, then $\mathrm{Z}_\sigma(G) = \mathrm{Z}_\infty(G)$ is the hypercentre of G.

The relevant properties of the σ-hypercenter are proved in [5, Proposition 2.5].

Theorem 2 *Suppose that $\sigma = \{\pi, \pi^{'}\}$ for a set of primes π. Assume that G is a group such that $\mathrm{Z}_\sigma(G) = 1$. Then all Schmidt subgroups of G are σ-subnormal in G if and only if the following statements hold:*

1. *Every minimal normal subgroup of G is abelian and $\mathrm{Soc}(G)$ is complemented in G by a self-normalizing π-decomposable subgroup H of G, that is, $H = \mathrm{O}_\pi(H) \times \mathrm{O}_{\pi'}(H)$;*
2. *every $S_{<r,s>}$-subgroup of G is contained in $\mathrm{O}_\pi(G)$, for all $r, s \in \pi$;*
3. *every $S_{<r,s>}$-subgroup of G is contained in $\mathrm{O}_{\pi'}(G)$, for all $r, s \in \pi^{'}$;*
4. *Let $\mathrm{Soc}_\pi(G) = \mathrm{O}_\pi(\mathrm{Soc}(G))$. Then $\mathrm{Soc}_\pi(G)H_{\pi'}/(\mathrm{O}_{\pi'}(G) \cap H_{\pi'}) \in \{1, A_1 \times A_2 \times ... \times A_n\}$, where A_i is a $F_{<p_i,c_i>}$-group with $p_i \in \pi$, $\pi(c_i) \subseteq \pi^{'}$ and $(c_i, c_j) = 1$ for all $i \neq j \in \{1, 2, ..., n\}$;*
5. *Let $\mathrm{Soc}_{\pi'}(G) = \mathrm{O}_{\pi'}(\mathrm{Soc}(G))$. Then $\mathrm{Soc}_{\pi'}(G)H_\pi/(\mathrm{O}_\pi(G) \cap H_\pi) \in \{1, B_1 \times B_2 \times ... \times B_m\}$, where B_i is a $F_{<q_i,d_i>}$-group with $q_i \in \pi^{'}$, $\pi(d_i) \subseteq \pi$ and $(d_i, d_j) = 1$ for all $i \neq j \in \{1, 2, ..., m\}$.*

Theorem 3 *Suppose that $\sigma = \{\pi, \pi^{'}\}$ for a set of primes π. Let G be a group. Then all Schmidt subgroups of G are σ-subnormal in G if and only if the following statements hold:*

1. *every* $S_{<r,s>}$*-subgroup of* G *is contained in* $\mathrm{O}_{\pi}(G)$*, for all* $r, s \in \pi$*;*
2. *every* $S_{<r,s>}$*-subgroup of* G *is contained in* $\mathrm{O}_{\pi'}(G)$*, for all* $r, s \in \pi^{'}$*;*
3. $G/Z_{\sigma}(G)$ *has the structure described in Theorem 2.*

We shall adhere to the notation and terminology of [1] and [2].

2 Preliminaries

Our first lemma collects some basic properties of σ-subnormal subgroups which are very useful in induction arguments.

Lemma 2 *([9]) Let* H*,* K *and* N *be subgroups of a group* G*. Suppose that* H *is* σ*-subnormal in* G *and* N *is normal in* G*. Then the following statements hold:*

1. $H \cap K$ *is* σ*-subnormal in* K*.*
2. *If* K *is a* σ*-subnormal subgroup of* H*, then* K *is* σ*-subnormal in* G*.*
3. HN/N *is* σ*-subnormal in* G/N*.*
4. *If* $N \subseteq K$ *and* K/N *is* σ*-subnormal in* G/N*, then* K *is* σ*-subnormal in* G*.*

Lemma 3 *Let* G *be a group and* S *a subgroup of* G*. Then* S *is* σ*-subnormal in* $S\,\mathrm{Z}_{\sigma}(G)$*.*

Proof Assume the result is not true and let G be a counterexample with $|G| + |G : S|$ minimal. Let $X = S\,\mathrm{Z}_{\sigma}(G)$. By [5, Proposition 2.5], $\mathrm{Z}_{\sigma}(G) \subseteq \mathrm{Z}_{\sigma}(X)$ and so $X = S\,\mathrm{Z}_{\sigma}(X)$. The minimal choice of G implies that $G = X$. Let M be a maximal subgroup of G containing S. Then $M = S(M \cap \mathrm{Z}_{\sigma}(G))$. By [5, Proposition 2.5], $M \cap \mathrm{Z}_{\sigma}(G) \subseteq \mathrm{Z}_{\sigma}(M)$. Thus $M = S\,\mathrm{Z}_{\sigma}(M)$. By minimality of G, S is σ-subnormal in M. If S were a proper subgroup of M, we would have that M would be σ-subnormal in $G = M\,\mathrm{Z}_{\sigma}(G)$ by the choice of the pair (G, S). By Lemma 2(2), S would be σ-subnormal in G, contradicting our supposition. Hence, S is a maximal subgroup of G. Assume that $\mathrm{Core}_G(S) \neq 1$ and let A be a minimal normal subgroup of G contained in $\mathrm{Core}_G(S)$. By [5, Proposition 2.5], $\mathrm{Z}_{\sigma}(G)A/A \subseteq \mathrm{Z}_{\sigma}(G/A)$ and so $G/A = (S/A)\,\mathrm{Z}_{\sigma}(G/A)$. By minimality of G, S/A is σ-subnormal in G/A. Applying Lemma 2(5), S is σ-subnormal in G. This contradiction yields $\mathrm{Core}_G(S) = 1$ and G is a primitive group. Let N be a minimal normal subgroup of G contained in $\mathrm{Z}_{\sigma}(G)$. Then $G = SN$ and $[N](G/\mathrm{C}_G(N))$ is σ-primary. If N is abelian, then $G \cong [N](G/\mathrm{C}_G(N))$ and if N is non-abelian, $\mathrm{C}_G(N) = 1$. In both cases, G is σ-primary and so S is σ-subnormal in G. This final contradiction proves the lemma.

□

The proof of Theorem 2 depends on the main result of [10] that is presented in the following lemma.

Lemma 4 *Every Schmidt subgroup of a group* G *is subnormal in* G *if and only if* $G/Z_{\infty}(G) \in \{1, G_1 \times G_2 \times ... \times G_n\}$*, where* G_i *is a* $F_{<p_i,d_i>}$*-group, and* $(d_i, d_j) = 1$ *for all* $i \neq j \in \{1, 2, ..., n\}$*.*

Lemma 5 *Assume that* $\sigma = \{\pi, \pi^{'}\}$*, where* π *is a set of primes. Assume that* G *is a group with a normal Hall* π*-subgroup. If* S *is a* σ*-subnormal* $S_{<p,q>}$*-subgroup of* G

with $p \in \pi$ and $q \in \pi'$, then $S = O^{\pi}(SO_{\pi}(G))$. In particular, S is a normal subgroup of $S\,O_{\pi}(G)$.

Proof Lemma 2(1), S is σ-subnormal in $S\,O_{\pi}(G)$. Hence, there exists a chain of subgroups

$$S = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n = S\,O_{\pi}(G)$$

such that $S_i/\operatorname{Core}_{S_i}(S_{i-1})$ is a π-group for all $i = 1, 2, \ldots, n$. Therefore, $O^{\pi}(S_i) \subseteq S_{i-1}$.

Thus, we have a subnormal chain

$$O^{\pi}(S_0) \subseteq O^{\pi}(S_1) \subseteq \ldots \subseteq O^{\pi}(S_{n-1}) \subseteq O^{\pi}(S_n) = O^{\pi}(S\,O_{\pi}(G))$$

such that $O^{\pi}(S_i)/O^{\pi}(S_{i-1})$ is a π-group for all $i = 1, 2, \ldots, n$. By [7, Lemma 3.1.7],

$$O^{\pi}(S) = O^{\pi}(S\,O_{\pi}(G)).$$

Assume that $O^{\pi}(S) \subset S$. Since S is a Schmidt group, $O^{\pi}(S)$ is nilpotent. Then the Sylow q-subgroup S_q of S is normal in S, which contradicts Statement (2) of Lemma 1. Hence, $S = O^{\pi}(S\,O_{\pi}(G))$, and $S \trianglelefteq S\,O_{\pi}(G)$, as desired.

□

There is a close relationship between σ-subnormal subgroups and direct decompositions of a group. In fact, the class of groups in which every subgroup is σ-subnormal is just the class of all groups that are direct products of Hall σ_i-subgroups, for every $\sigma_i \in \sigma$. These groups are called σ-*nilpotent* and the corresponding class is denoted by $\mathcal{N}_{\sigma}$. It is clear that if σ is the minimal partition, then $\mathcal{N}_{\sigma}$ is just the class of all nilpotent groups.

The class $\mathcal{N}_{\sigma}$ is a subgroup-closed saturated Fitting formation ([9, Corollary 2.4 and Lemma 2.5]), and the σ-subnormal subgroups of a group G are precisely the $K-\mathcal{N}_{\sigma}$-subnormal subgroups of G, and so they are a sublattice of the subgroup lattice of G (see [1, Chap. 6]).

Applying [1, Lemma 6.1.9 and Proposition 6.1.10]), we have:

Lemma 6 *If X is σ-subnormal in G, then the $\mathcal{N}_{\supset}$-residual $X^{\mathcal{N}_{\supset}}$ of X, that is, the smallest normal subgroup of X with σ-nilpotent quotient, is subnormal in G.*

The $\mathcal{N}_{\sigma}$-radical of a group G, that is, the largest normal σ-nilpotent subgroup of G, is denoted by $F_{\sigma}(G)$ and it is called the σ-*Fitting subgroup* of G. If $\sigma = \{\pi, \pi'\}$, then $F_{\sigma}(G) = O_{\pi}(G) \times O_{\pi'}(G)$.

The following lemma shows that $F_{\sigma}(G)$ contains every σ-nilpotent σ-subnormal subgroup of G.

Lemma 7 *([9]) If H is a σ-nilpotent σ-subnormal subgroup of G, then H is contained in $F_{\sigma}(G)$.*

The role of the σ-Fitting subgroup in the proof of Theorem 2 is determined by the main result of [11].

Lemma 8 *Suppose that G is a non-nilpotent group. If every Schmidt subgroup of G is σ-subnormal in G, then $G/\mathrm{F}_\sigma(G)$ is a cyclic group.*

3 Proofs of the Main Theorems

Proof of Theorem 1 Suppose that every Schmidt subgroup of G is σ-subnormal in G. Let $i \neq j \in \{1, 2, ..., k\}$ and let S be a Schmidt $(\sigma_i \cup \sigma_j)$-subgroup of G. Then S is $\{(\sigma_i \cup \sigma_j), (\sigma_i \cup \sigma_j)^{'}\}$-subnormal in G. By Lemma 7, $S \subseteq \mathrm{O}_{\sigma_i \cup \sigma_j}(G)$.

Let D be a Schmidt subgroup of $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$. Then, by Lemma 2 (1), D is σ-subnormal in $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$. Since $\pi(\mathrm{O}_{\sigma_i \cup \sigma_j}(G)) \subseteq \sigma_i \cup \sigma_j$, it follows that D is $\{\sigma_i, \sigma_i^{'}\}$-subnormal in $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$.

Assume now that the group G satisfies Statements (1) and (2) of the theorem. Let S be an $S_{<p,q>}$-subgroup of G. Then $p, q \in \sigma_i \cup \sigma_j$ for some $i \neq j \in \{1, 2, ..., k\}$. By Statement (1), $S \subseteq \mathrm{O}_{\sigma_i \cup \sigma_j}(G)$ and so S is $\{\sigma_i, \sigma_i^{'}\}$-subnormal in $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$ by Statement (2). Since $\pi(\mathrm{O}_{\sigma_i \cup \sigma_j}(G)) \subseteq \sigma_i \cup \sigma_j$, it follows that S is σ-subnormal in $\mathrm{O}_{\sigma_i \cup \sigma_j}(G)$. Applying Lemma 2(2), it follows that we have that S is σ-subnormal in G. □

Proof of Theorem 2 Suppose that G is a group with $Z_\sigma(G) = 1$, and every Schmidt subgroup in G is σ-subnormal. Since $Z_\sigma(G) = 1$, it follows that G is not nilpotent and so G has Schmidt subgroups. Moreover, $\pi(G) \cap \pi \neq \emptyset$ and $\pi(G) \cap \pi^{'} \neq \emptyset$. By Lemma 8, every chief factor of G is σ-primary, that is, G is σ-soluble. Applying [9, Proposition 2.2], we have that G has Hall π-subgroups and Hall π'-subgroups.

We show that G satisfies Statements (1)-(5) of the theorem. We split the proof into several steps.

Step 1. $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$. Then $\Phi(G)$ contains a minimal normal subgroup L of G and L is an elementary abelian r-subgroup of G for some prime r. Without loss of generality we may assume that $r \in \pi$. Since $Z_\sigma(G) = 1$, it follows that $[L](G/\mathrm{C}_G(L))$ is not a π-group. Hence $\mathrm{C}_G(L)$ does not contain any Hall $\pi^{'}$-subgroup of G. Let $G_{\pi^{'}}$ be a Hall $\pi^{'}$-subgroup of G. Then there exists a p-element $x \in G_{\pi^{'}} \setminus \mathrm{C}_G(L)$ for a prime p. Consequently $L < x >$ is a non-nilpotent $\{r, p\}$-group. Let V be a Schmidt subgroup of $L < x >$. Then V is an $S_{<r,p>}$-subgroup and $V_r \subseteq L$. By Lemma 2 (4), VL/L is σ-subnormal in G/L. Since VL/L is a p-subgroup of G/L, we can apply Lemma 7 to conclude that $VL/L \subseteq \mathrm{F}_\sigma(G/L) = \mathrm{O}_\pi(G/L) \times \mathrm{O}_{\pi^{'}}(G/L)$. Therefore, $VL/L \subseteq \mathrm{O}_{\pi^{'}}(G/L) = X/L$. By [2, Theorem A.11.3], there exists a Hall π'-subgroup R of X such that $X = LR$ and all of them are conjugate in X. By Frattini argument, $G = L\,\mathrm{N}_G(R) = \mathrm{N}_G(R)$ and so R is a normal subgroup of G. This means that V is nilpotent, contrary to assumption.

Step 2. $\mathrm{F}(G) = \mathrm{Soc}(G)$ *is abelian.*

Suppose that L is a non-abelian minimal normal subgroup of G. Since G is σ-soluble, it follows that L is either a π-group or a $\pi^{'}$-group. Without loss of generality, we may assume that L is a π-group. Let $G_{\pi'}$ be a Hall π'-subgroup of G. Then $G_{\pi'}$ of G is not contained in $C_G(L)$. As above, we can find a prime q and a q-element $z \in G_{\pi'} \setminus C_G(L)$. Then $L < z >$ is a non-nilpotent subgroup of G. Let S be a Schmidt subgroup of $L < z >$. Then S is an $S_{<p,q>}$-group with $p \in \pi$ and the Sylow p-subgroup of S is contained in L. By Lemma 2 (1), S is σ-subnormal in LS and so $1 \neq S^{\mathfrak{N}_\sigma}$ is subnormal in LS by Lemma 6. Since $S^{\mathfrak{N}_\sigma} \subseteq L$, we have $O_p(L) \neq 1$, which is impossible because L is non-abelian.

Consequently all minimal normal subgroups of G are abelian and so is $\mathrm{Soc}(G)$. Since $\Phi(G) = 1$, it follows that $F(G) = \mathrm{Soc}(G)$ by [2, Theorem A.10.6].

Step 3. G has a self-normalizing subgroup H such $G = F(G)H$ and $F(G) \cap H = 1$.

Let $F = F(G)$. By [2, Theorem A.10.6], G has a subgroup H such that $G = FH$ and $F \cap H = 1$. Let $K = N_G(H)$. Assume that $K \neq H$. Then $K = H(F \cap K) = H \times (F \cap K)$ and $F \cap K \neq 1$. Then $G = C_G(F \cap K)$, and so $1 \neq Z(G) \subseteq Z_\sigma(G)$, a contradiction. Consequently, $H = N_G(H)$.

Step 4. $H = O_\pi(H) \times O_{\pi'}(H)$.

Suppose that $H \neq O_\pi(H) \times O_{\pi'}(H)$. Then H is non-nilpotent and so H has an $S_{<p,q>}$-subgroup S such that either $p \in \pi$ and $q \in \pi'$ or $p \in \pi'$ and $q \in \pi$. Without loss of generality, we may assume that $p \in \pi$ and $q \in \pi'$. Then S is σ-subnormal in G and so $1 \neq S^{\mathfrak{N}_\sigma}$ is subnormal in G by Lemma 6. Since $S^{\mathfrak{N}_\sigma}$ is the Sylow p-subgroup of S, it follows that $S^{\mathfrak{N}_\sigma} \subseteq O_p(G)$. Hence, $H \cap F(G) \neq 1$, contradicting Step (3).

Consequently $H = O_\pi(H) \times O_{\pi'}(H)$ is a π-decomposable subgroup of G.

Step 5. If $r, s \in \pi$, then every $S_{<r,s>}$-subgroup of G is contained in $O_\pi(G)$. If $r, s \in \pi'$, then every $S_{<r,s>}$-subgroup of G is contained in $O_{\pi'}(G)$.

Note that in both cases, every $S_{<r,s>}$-subgroup of G is a σ-nilpotent σ-subnormal subgroup of G. The result follows from Lemma 7.

Step 6. $\mathrm{Soc}_\pi(G)H_{\pi'}/(O_{\pi'}(G) \cap H_{\pi'}) \in \{1, A_1 \times A_2 \times ... \times A_n\}$, where A_i is a $F_{<p_i,c_i>}$-group with $p_i \in \pi$ and $\pi(c_i) \subseteq \pi'$, and $(c_i, c_j) = 1$ for all $i \neq j \in \{1, 2, ..., n\}$.

Note that $\mathrm{Soc}_\pi(G)$ centralizes $O_{\pi'}(G)$. Thus, $O_{\pi'}(G) \cap H_{\pi'} \trianglelefteq \mathrm{Soc}_\pi(G)H_{\pi'}$. Without loss of generality, we may assume that $O_{\pi'}(G) \cap H_{\pi'} = 1$.

If $\mathrm{Soc}_\pi(G) = 1$, then $O_\pi(G) = 1$ and $F_\sigma(G) = O_{\pi'}(G)$. Thus, by Lemma 8 $G/O_{\pi'}(G)$ is a cyclic group. This implies that $G/O_{\pi'}(G)$ is a π-group and $O_{\pi'}(G)$ is the Hall π'-subgroup of G. Since $H_{\pi'}$ is contained in $O_{\pi'}(G)$, it follows that $\mathrm{Soc}_\pi(G)H_{\pi'} = 1$.

Assume that $\mathrm{Soc}_\pi(G) \neq 1$. Let S be an $S_{<p,q>}$-subgroup of $\mathrm{Soc}_\pi(G)H_{\pi'}$. Since $\mathrm{Soc}_\pi(G)$ is a has a normal abelian Hall π-subgroup of $\mathrm{Soc}_\pi(G)H_{\pi'}$ and $H_{\pi'} \cong H_{\pi'}F_\sigma(G)/F_\sigma(G)$ is cyclic by Lemma 8, it follows that $p \in \pi$ and $q \in \pi'$. By Lemma 5, S is normal in $S\,\mathrm{Soc}_\pi(G)$. Since $\mathrm{Soc}_\pi(G)H_{\pi'}/\mathrm{Soc}_\pi(G)$ is cyclic, it follows that $S\,\mathrm{Soc}_\pi(G)$ is a normal subgroup of $Soc_\pi(G)H_{\pi'}$. Consequently, S is subnormal in $\mathrm{Soc}_\pi(G)H_{\pi'}$.

Consequently, every Schimdt subgroup of $\mathrm{Soc}_\pi(G)H_{\pi'}$ is subnormal. Suppose that $Z_\infty(\mathrm{Soc}_\pi(G)H_{\pi'}) \neq 1$ and let L be a minimal normal subgroup of $\mathrm{Soc}_\pi(G)H_{\pi'}$ contained in $Z_\infty(\mathrm{Soc}_\pi(G)H_{\pi'})$. Then either L is a π-group or L is a π'-group since $\mathrm{Soc}_\pi(G)H_{\pi'}$ is σ-soluble. Assume that $L \subseteq H_{\pi'}$. By Step (4), H_π centralizes L. Then

L is a normal π'-subgroup of G and so $L \subseteq \mathrm{O}_{\pi'}(G) \cap H_{\pi'} = 1$, a contradiction. Then $L \subseteq \mathrm{Soc}_{\pi}(G)$ is a π-group and so $\mathrm{Soc}(G)H_{\pi'} \subseteq \mathrm{C}_G(L)$. Hence, $G = H_\pi \mathrm{C}_G(L)$. Note that if $x \in H_\pi$, then $\mathrm{Soc}(G)H_{\pi'} \subseteq \mathrm{C}_G(L^x)$. Therefore, $G = H_\pi \mathrm{C}_G(L^G)$, where L^G is the normal closure of L in G. In particular, $G/\mathrm{C}_G(L^G)$ is a π-group. Since L^G is a π-group, it follows that $L^G \subseteq \mathrm{Z}_\sigma(G)$, which contradicts our assumption. Consequently, $\mathrm{Z}_\infty(\mathrm{Soc}_\pi(G)H_{\pi'}) = 1$. By Lemma 4, we have

$$Soc_\pi(G)H_{\pi'} \cong A_1 \times A_2 \times \ldots \times A_n,$$

where A_i is a $F_{<p_i,c_i>}$-group with $p_i \in \pi$ and $\pi(c_i) \subseteq \pi'$, and $(c_i, c_j) = 1$ for all $i \neq j \in \{1, 2, \ldots, n\}$.

Exchanging π by π' in Step (6), we prove

Step 7. $\mathrm{Soc}_{\pi'}(G)H_\pi/(\mathrm{O}_\pi(G) \cap H_\pi) \in \{1, B_1 \times B_2 \times \ldots \times B_m\}$, *where* B_i *is a* $F_{<q_i,d_i>}$*-group with* $q_i \in \pi'$ *and* $\pi(d_i) \subseteq \pi$, *and* $(d_i, d_j) = 1$, *for all* $i \neq j \in \{1, 2, \ldots, m\}$.

Consequently, G satisfies Statements (1)-(5) of the theorem.

Conversely, assume that G satisfies Statements (1)-(5) of the theorem. We prove that every Schmidt subgroup of G is σ-subnormal.

Let S be an $S_{<p,q>}$-subgroup of G. Then one of the following cases holds:

1. $p, q \in \pi$;
2. $p, q \in \pi'$;
3. $p \in \pi, q \in \pi'$;
4. $q \in \pi, p \in \pi'$.

Assume that either $p, q \in \pi$ or $p, q \in \pi'$. By Statements (2) and (3), S is either contained in $\mathrm{O}_\pi(G)$ or in $\mathrm{O}_{\pi'}(G)$. By Lemma 2 (2), S is σ-subnormal in G.

Assume $p \in \pi$ and $q \in \pi'$. Then, by Lemma 1, $S = [S_p]S_q$, where S_p is a normal Sylow p-subgroup of S and $S_q = \langle a \rangle$ is a non-normal cyclic Sylow q-subgroup of S. Since $G = [\mathrm{Soc}(G)]H$ and H is π-decomposable, we may assume that S_q is contained in $H_{\pi'}\,\mathrm{O}_{\pi'}(G)$. Moreover, since S is not nilpotent, it follows that S_q is not contained in $\mathrm{O}_{\pi'}(G)$. Then $S_q \cap \mathrm{O}_{\pi'}(G) = \Phi(S_q) \cap \mathrm{O}_{\pi'}(G)$.

Suppose that S_p is not contained in $\mathrm{Soc}_\pi(G)$. Since $S_p/\Phi(S_p)$ is a chief factor of S, it follows that $S_p \cap \mathrm{Soc}_\pi(G) = \Phi(S_p) \cap \mathrm{Soc}_\pi(G)$. It allows us to conclude that $S\,\mathrm{Soc}_\pi(G)\,\mathrm{O}_{\pi'}(G)/\mathrm{Soc}_\pi(G)\,\mathrm{O}_{\pi'}(G)$ is an $S_{<p,q>}$-subgroup of $G/\mathrm{Soc}_\pi(G)\,\mathrm{O}_{\pi'}(G)$. But this is not possible because $G/\mathrm{Soc}_\pi(G)\,\mathrm{O}_{\pi'}(G) \cong H/H \cap \mathrm{Soc}_\pi(G)\,\mathrm{O}_{\pi'}(G)$ is π-decomposable by Statement (1).

Consequently $S_p \subseteq \mathrm{Soc}_\pi(G)$, and $S \subseteq \mathrm{Soc}_\pi(G)H_{\pi'}O_{\pi'}(G)$. Note that $\mathrm{Soc}_\pi(G)H_{\pi'} \cap \mathrm{O}_{\pi'}(G) = \mathrm{O}_{\pi'}(G) \cap H_{\pi'}$. Thus, $\mathrm{Soc}_\pi(G)H_{\pi'}\,\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G) \cong \mathrm{Soc}_\pi(G)H_{\pi'}/\mathrm{Soc}_\pi(G)H_{\pi'} \cap \mathrm{O}_{\pi'}(G) = \mathrm{Soc}_\pi(G)H_{\pi'}/H_{\pi'} \cap \mathrm{O}_{\pi'}(G)$. By Statement (4),

$$\mathrm{Soc}_\pi(G)H_{\pi'}\,\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G) \cong A_1 \times A_2 \times \ldots \times A_n,$$

where A_i is a $F_{<p_i,c_i>}$-group with $p_i \in \pi$, $\pi(c_i) \subseteq \pi'$ and $(c_i, c_j) = 1$ for all $i \neq j \in \{1, 2, \ldots, n\}$. Furthermore, $S\,\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G)$ is an $S_{<p,q>}$-subgroup of $\mathrm{Soc}_\pi(G)H_{\pi'}\,\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G)$. By Lemma 4, $S\,\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G)$ is subnormal in $\mathrm{Soc}_\pi(G)H_{\pi'}\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G)$. Hence, $S\mathrm{O}_{\pi'}(G)$ is subnormal in $\mathrm{Soc}_\pi(G)H_{\pi'}\mathrm{O}_{\pi'}(G)$. Moreover, $S_p = \mathrm{Soc}_\pi(G) \cap S\mathrm{O}_{\pi'}(G)$ is normal $S\mathrm{O}_{\pi'}(G)$ and $S\mathrm{O}_{\pi'}(G)/S_p$ is a π'-group. Thus, S/S_p is σ-subnormal in $S\mathrm{O}_{\pi'}(G)/S_p$. By Lemma 2 (5), S is σ-subnormal in $S\mathrm{O}_{\pi'}(G)$. By Lemma 2 (2), S is σ-subnormal in G.

Assume that $q \in \pi$, $p \in \pi'$. We can argue as above with the 3-tuple $(\mathrm{Soc}_{\pi'}(G), \mathrm{O}_\pi(G), H_\pi)$ and the structural information contained in Statement (3) to conclude that S is σ-subnormal in G. □

Proof of Theorem 3 Suppose that every Schmidt subgroup of G is σ-subnormal.

Assume that either $r, s \in \pi$ or $r, s \in \pi'$. In both cases, every $S_{<r,s>}$-subgroup of G is a σ-nilpotent σ-subnormal subgroup of G. Applying Lemma 7, we have that Statements (1) and (2) hold.

Let $Z = Z_\sigma(G)$. By [11, Lemma 5], every Schmidt subgroup of $\overline{G} = G/Z$ is σ-subnormal in $\overline{G}$. Since $Z_\sigma(\overline{G}) = 1$, it follows that $\overline{G}$ satisfies Statements (1)-(5) of Theorem 2 and therefore Statement (3) holds.

Conversely, suppose that G satisfies Statements (1)-(3). Write $Z = Z_\sigma(G)$.

Let S be an $S_{<p,q>}$-subgroup of G. Then one of the following cases holds:

1. $p, q \in \pi$;
2. $p, q \in \pi'$;
3. $p \in \pi, q \in \pi'$;
4. $q \in \pi, p \in \pi'$.

If $p, q \in \pi$, then S is contained in $\mathrm{O}_\pi(G)$ by Statement (1) and if $p, q \in \pi'$, then S is contained in $\mathrm{O}_{\pi'}(G)$. In both cases, S is σ-subnormal in G by Lemma 2 (2).

Assume that either $p \in \pi, q \in \pi'$ or $q \in \pi, p \in \pi'$. By Lemma 3, S is σ-subnormal in SZ. Moreover, SZ/Z is a $S_{<p,q>}$-subgroup of G/Z. By Theorem 2, SZ/Z is σ-subnormal in G/Z. Applying Lemma 2 (2), we conclude that S is σ-subnormal in G. □

Acknowledgements The first author is supported by the grant PGC2018-095140-B-I00 from the *Ministerio de Ciencia, Innovación y Universidades* and the *Agencia Estatal de Investigación*, Spain, and FEDER, European Union. Research of the second author was supported by Ministry of Education of the Republic of Belarus (Grant 20211779 "Convergence-2025").

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature

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